Valuing Multiple Natural Capital Stocks Under Correlated Volatility

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Abstract

Bioeconomic models can be used to value single and multiple coupled natural capital stocks as assets under real-world management conditions for the purposes of measuring accounting prices in context change-in-wealth based sustainability assessment. In this paper we extend prior work to consider the valuation of assets linked through deterministic relationships (i.e. biophysical coupling or shared management) to assets with stochastic dynamics including when there are multiple stock with correlated stochastic processes. We derive asset prices for natural capital stocks governed by correlated diffusions and show how function approximation techniques can be used to approximate these shadow prices across the domain of capital stocks. Using single examples, we develop intuition for the role of stochasticity on value changes in stocks of natural assets. We show that stochasticity is generally of second-order importance for a large class of natural assets. Therefore, concerns about stochasticity should not be used to hold back progress on change-in-wealth based sustainability assessments and scarce effort may be better focused on addressing the nuances of economic programs, spatial scale, and local institutions.

Keywords: Natural capital, Stochasticity, Risk, Sustainability, Wealth Accounting, Green Accounting

1. Introduction

Change-in-wealth based measures of sustainability (inclusive, comprehensive, or genuine wealth) that are grounded in economic theory (e.g., Dasgupta, 2001; Dasgupta and Mäler, 2000; Hamilton and Clemens, 1999), have gained substantial acceptance and credibility beyond economists (e.g., Matson, Clark, and Andersson, 2016). These approaches are employed regularly by the United Nations Environment Programme and the World Bank for the sustainability assessment of nation states (UNU-IHDP and UNEP, 2014), and individual countries are starting to produce their own reports.[1] Furthermore, change-in-wealth based approaches have been used to assess the sustainability of bounded systems such as cities (Dovern, Quaas, and Rickels, 2014), hydrological catchments (Pearson et al., 2013) and as an indicator of sustainable management for ecosystems (Yun et al., 2017).

The lack of defensible, theoretically and empirically grounded accounting prices for natural capital was once bemoaned as the "Achilles' heel" of the wealth-based approach to sustainability (Smulders, 2012). Many natural capital stocks provide service flows that are non-excludable, non-rivalrous, and managed in demonstrably inefficient, 'kakatopic' ways. These factors together limit the usefulness of the (scant) market data for pricing natural assets and undercut the validity of shadow prices from optimized bioeconomic models as a realistic guide for sustainability assessment. Fortunately, substantial theoretical, and some empirical, progress has been made in recent years. Fenichel and Abbott (2014) provide a theoretical foundation for pricing of natural assets by deriving the revealed shadow price or accounting price of natural capital under general, non-optimized forms of management, and link their derivation to foundational contributions in economic capital theory (Jorgenson, 1963).[2] In this and subsequent work with coauthors, they demonstrate the necessary com-

[1] For example, Canada contracted for a Comprehensive Wealth report in 2018 https://www.iisd.org/library/comprehensive-wealth-canada-2018-measuring-what-matters-long-term and the U.K. has developed a 25 year plan focused on natural capital, https://www.gov.uk/government/groups/natural-capital-committee.

[2] See Fenichel, Abbott, and Yun (2018) for a detailed development of natural capital pricing. This approach has subsequently been expanded to allow for the valuation of a portfolio of capital stocks whose

ponents of an accounting price for natural assets and develop and implement computational approaches to measure accounting prices.

One shortcoming of the Fenichel-Abbott approach to pricing natural assets, as well as much of the work that precedes it, is that it abstracts from stochasticity and uncertainty, which are a critical part of the sustainability question (Baumgärtner and Quaas, 2010). Valuing capital is about the future, but the future is inherently uncertain. It is therefore important that the theory for pricing natural assets incorporate risk explicitly – in order to understand when and to what extent stochastic effects are critical for ongoing shadow pricing efforts.

The most consistent way to incorporate risk is through a theoretically-grounded risk adjustment to the price itself. The ultimate objective of the shadow pricing endeavor is to put natural capital on the same conceptual and empirical ground as 'real' capital assets (i.e. reproducible capital). The latter are also subject to considerable uncertainty, and yet asset markets resolve the beliefs about uncertainty at a given moment into prices that reflect the collective assessment of risk and its valuation. The change-in-wealth approach to sustainability is about tracking changes in a societal balance sheet. This means that once prices and quantities are measured, measuring sustainability becomes an accounting problem. Accountants rarely include 'error bars' to account for uncertainty in the valuation of assets. Instead, prices for real and financial assets are taken as given by the market and already reflecting an appropriate risk adjustment.[3]

This paper contributes to the literature by generalizing the natural asset pricing approach to explicitly include stochastic dynamics, placing change-in-wealth based metrics for

dynamics may be interlinked through physical or biological processes or via human behavior (Yun et al., 2017). These methods have been used to value a range of natural capital stocks, from fish in single-species fisheries (Fenichel and Abbott, 2014), groundwater (Fenichel et al., 2016), coastal habitat (Bond, 2017), and an assemblage of interacting fish stocks (Yun et al., 2017).

[3]An alternative approach is to layer Monte Carlo simulation on a fundamentally deterministic valuation approach to compute error bounds on change-in-wealth metrics. However, this muddles the aggregation of accounting prices for natural capital and reproducible capital in ways that are unlikely to be acceptable in sustainability accounting and are also not fully theoretically grounded.

sustainability on a broader theoretical footing. We show how these realized shadow prices or accounting prices can be approximated, given a full bioeconomic model and specification of the diffusion process, through an extension of the functional approximation technique employed in (Yun et al., 2017). As a stepping stone to understanding the valuation of multiple, linked stochastic assets, we focus a significant portion of our efforts focus on the single-asset case and examine the implications of stochasticity in the dynamics of natural capital. Model-based shadow prices are inherently dependent on the underlying specification of the model of natural capital dynamics. However, there is often significant uncertainty with regard to these models. Our understanding of many natural processes is at best incomplete, with the result that the actual evolution of natural capital could deviate significantly from any deterministic specific model. The valuation of such an inherently risky asset may differ significantly from one where the capital dynamics are deterministic and known with certainty.

Despite these concerns, we show that stochasticity may be a second-order concern in the valuation of a sizable class of natural assets. Instead, managerial *responses* to stochasticity — as expressed in the degree of precaution reflected in the feedback control rules we adopt for managing natural capital stocks — have a far greater impact. One repercussion of this finding is that concerns about stochasticity may be of little consequence for the valuation of many natural assets—therefore offering little impediment to the development of wealth accounts using the capital asset pricing for nature approach (Yun et al., 2017; Fenichel, Abbott, and Yun, 2018). Nevertheless, stochasticity remains relevant in benefit-cost analyses intended to help choose the economic program.

Secondarily, we consider the case of linked natural capital assets. Many natural capital stocks in a given system are differentially vulnerable to a wide array of systemic and idiosyncratic shocks, with the result that changes in their stocks may be correlated – even in the absence of fundamental interactions in their dynamics. Since sustainability requires maintaining the wealth contained in a *portfolio* of capital stocks, it can be important to sustainable management to understand how the properties of the covariance structure of stocks in the 'ecosystem fund' influences the overall value of the portfolio and how this correlated volatility interacts with the mechanistic interactions between capital stocks and the portfolio balancing decisions embodied in management policies. We extend the theory of Yun et al. (2017) to consider the case of assets with diffusions linked through their 'drift' terms and through correlations in the noise terms of the diffusion.

The following section derives the shadow price formulas for the single- and multi-stock cases. Section 3 demonstrates the valuation approach for a single-stock, stochastic control problem where the optimal co-state (i.e. accounting price) is available in closed form. This allows us to validate our approach and also allows us to isolate the effects of stochasticity from the effects of the choice of a sub-optimal control rule (economic programs) that may stem from heuristics for coping with stochasticity. Section 4 extends our approach to a stochastic version of the Gulf of Mexico reef fish case study examined in Fenichel and Abbott (2014). This case allows us to consider the impact of natural stochasticity in a real-world, non-optimal setting. Section 5 concludes the paper.

2. Derivation of shadow pricing formula

2.1. The single asset case

Let $s(t)$ represent the known stock of a scalar asset at time t.[4] Suppose the dynamics of s are represented by a diffusion (also known as an Ito) process and stationary infinitesimal parameters $\mu\left(s, x\left(s\right)\right)$ and $\sigma(s)$. The diffusion process is written as

$$ds(t) = \mu\left(s(t), x\left(s(t)\right)\right) dt + \sigma(s(t))dZ(t) \tag{1}$$

where $dZ(t)$ is an increment of a Wiener process (Stokey, 2009). The drift of the diffusion $\mu\left(s, x\left(s\right)\right)$ is specified as a function of the current capital stock and as a function of the feedback control rule, known as the economic program or resource allocation mechanism, $x(s)$.

[4] t is suppressed when doing so does not cause confusion.

It is easiest to assume that stochasticity comes through the ecological production process, but stochasticity could also come through the economic program. Once the substitution for the economic program has been made, the drift is an explicit function of only s.

Define the intertemporal welfare function, evaluated along the economic program and along the stochastic capital trajectory given by (1), as

$$V\left(s\left(t\right)\right)=\mathbb{E}_{t}\left[\int_{t}^{\infty}e^{-\delta(\tau-t)}W\left(s\left(\tau\right),x\left(s\left(\tau\right)\right)\right)d\tau\right] \tag{2}$$

where $\mathbb{E}_t$ is the expectations operator. The marginal value of an investment in the capital stock in expectation is defined as $p(s)\equiv V_s$. To derive the properties of $p(s)$, start by differentiating (2) with respect to t.

$$\frac{dV}{dt}=\mathbb{E}_{t}\left[\delta\int_{t}^{\infty}e^{-\delta(\tau-t)}W\left(\cdot\right)d\tau-W\left(s\left(t\right),x\left(s\left(t\right)\right)\right)\right]=\delta V-W\left(s\left(t\right),x\left(s\left(t\right)\right)\right) \tag{3}$$

The first equality in (3) assumes that the derivative can be carried through the expectation operator, which is ensured by the stationarity of the infinitesimal parameters of (1). The second equality holds because the state of the system is known at $\tau=t$.

We know that $\frac{dV}{dt}=\frac{\mathbb{E}_t[dV]}{dt}$. By Ito's Lemma

$$dV=\left[\mu\left(s\right)V_{s}+\frac{1}{2}\sigma^{2}\left(s\right)V_{ss}\right]dt+\sigma\left(s\right)V_{s}dZ$$

Taking the expected value, and employing the property that all stochastic integrals are identically zero (Stokey, 2009):

$$\mathbb{E}_{t}[dV]=\left[\mu\left(s\right)V_{s}+\frac{1}{2}\sigma^{2}\left(s\right)V_{ss}\right]dt$$

so that

$$\frac{dV}{dt}=\frac{\mathbb{E}_{t}[dV]}{dt}=\mu\left(s\right)V_{s}+\frac{1}{2}\sigma^{2}\left(s\right)V_{ss} \tag{4}$$

Setting (3) equal to (4) we obtain the stochastic Hamilton-Jacobi-Bellman (HJB) equation:

$$\delta V(s) = W(s(t), x(s(t))) + \mu(s) V_s + \frac{1}{2}\sigma^2(s) V_{ss} \tag{5}$$

If we substitute $p(s) \equiv V_S$ into the HJB equation yielding:

$$\delta V(s) = W(s(t), x(s(t))) + p(s)\mu(s) + \frac{1}{2}\sigma^2(s) p_s(s) \tag{6}$$

The first two terms on the RHS are the traditional deterministic current-value Hamiltonian. The third term captures the effect of risk even if the deterministic rate of change in the capital stock $\mu(s) = 0$. The risk effect captures the effect of Jensen's inequality via the curvature of the intertemporal welfare function. If the shadow price function is downward sloping then $p_s < 0$ so that risk has a negative effect on the intertemporal welfare function. Suppressing functional dependency on s, and differentiate (6) with respect to s yields:

$$\delta p = W_s + \mu_s p + \mu p_s + \sigma\sigma_s p_s + \frac{1}{2}\sigma^2 p_{ss}$$

Isolating p on the left-hand side we obtain the asset pricing equation:

$$p(s) = \frac{W_s + [\mu(s) + \sigma(s)\sigma_s(s)]p_s + \frac{1}{2}\sigma^2(s)p_{ss}}{\delta - \mu_s(s)} \tag{7}$$

In the case where the variance of the noise in (1) does not depend on s then (2.1) reduces to:

$$p(s) = \frac{W_s + \mu(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}}{\delta - \mu_s(s)}$$

and if capital dynamics are deterministic then this further reduces to

$$p(s) = \frac{W_s + \mu(s)p_s}{\delta - \mu_s(s)}$$

which is the same as in (Fenichel and Abbott, 2014) who show that this equation is equivalent to Jorgenson (1963).

The general asset pricing equation equation (7) contains two additional numerator terms relative to Fenichel and Abbott's deterministic derivation. The first term enters in a way that is symmetric to capital gains in a deterministic system and depends on the extent of "risk aversion" embodied in the curvature of the intertemporal welfare function (since $p_s \equiv V_{ss}$) and the extent to which the standard deviation of the diffusion is elastic with respect to s. If increasing investment in s increases the size of shock, and if the shadow price function is decreasing in the stock (analogous to risk aversion), then this results in a "capital loss." This term only matters if the variance depends on the capital stock, as in the case of geometric Brownian motion. Importantly, curvature of the intertemporal welfare function, which is defined over the domain of capital *stocks*, need not result from underlying curvature of the "social utility" or real income function for welfare flows $W(\cdot)$. Indeed, the nature of risk preferences over flows embodied in W (including risk neutrality) may have no direct mapping to the curvature of $V(s)$. Curvature of the intertemporal welfare function can be inherited from the underlying biophysical dynamics in (1) or from the economic program $x(s)$ - suggesting that the risk premia embodied in the numerator of (7) are endogenous to policy and may reflect actual existing levels of self-insurance and self-protection (Ehrlich and Becker, 1972). This first term pertains to how a marginal investment in the capital stock increases risk, holding the curvature of the intertemporal welfare function constant.

The second additional term in (7) is present with stochastic dynamics so long as the third derivative of the intertemporal welfare (or value) function is non-zero. There will be a premium if there is a positive third derivative (convex price function), while a negative third derivative (concave price function) yields a discount. If the value function is quadratic (i.e. zero derivatives above the second derivative), then this term is zero. Both additional terms in the numerator of (7) originate from differentiating $\frac{1}{2}\sigma^2(s)p_s$ term in (6). This second term can be interpreted as the affect of a marginal increase in the capital stock on risk aversion, holding risk constant or interpreted as "prudence," which is associated with precautionary savings (**?**)). If risk aversion or prudence is increased by the investment ($V_{sss} = p_{ss} < 0$) then the shadow price is decreased. In other words, the pricing of risk into the capital asset depends on how an investment affects the sensitivity to risk, given the biophysical dynamics and economic program in place, in addition to how the marginal investment affects the risk itself. This can be thought of as a "self insurance effect" because changes in the curvature of the intertemporal welfare function impact the consequences of stochastic events rather than their probability (Shogren and Crocker, 1999).

2.2. The multi-stock case

Let $\boldsymbol{s}(t) \in \mathbb{R}^S$ and $\boldsymbol{x}(\boldsymbol{s}(t)) : \mathbb{R}^S \to \mathbb{R}^X$and extend the diffusion in (1) to S distinct Ito processes

$$ds^i = \mu^i\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)dt + \sigma^i(\boldsymbol{s})dZ^i(t) \quad \text{for } i = 1, \ldots, S \tag{8}$$

The $dZ^i(t)$ can be correlated with a $S \times S$ correlation matrix $\boldsymbol{\rho}$ such that the covariance of the stochastic components of capital stocks i and j, which may differ from their observed covariance in-sample due to the presence of deterministic relations between the stocks in (8), is $\mathbb{E}_t\left[\sigma^i(\boldsymbol{s})dZ^i(t)\sigma^j(\boldsymbol{s})dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\mathbb{E}_t\left[dZ^i(t)dZ^j(t)\right] = \sigma^i(\boldsymbol{s})\sigma^j(\boldsymbol{s})\rho^{ij}dt$. If $i = j$, then the expression simplifies to $\sigma^i(\boldsymbol{s})^2dt$.

While the decomposition of the noise into a correlation matrix and standard deviations is intuitive and useful for model parameterization, we work directly with the covariance matrix to conserve on notation. Let $\Omega(\boldsymbol{s})$ be a $S \times S$ covariance matrix of the noise terms such that $\text{Cov}(ds^i, ds^j) = \Omega^{ij}(\boldsymbol{s})\,dt$. A Cholesky decomposition of the covariance matrix yields $\Omega(\boldsymbol{s}) = \omega(\boldsymbol{s})\omega(\boldsymbol{s})'$.[5]

Redefine the instantaneous return functions and intertemporal welfare functions in the multi-stock case as $W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right)$ and $V(\boldsymbol{s}(t))$. Once again, we know that $\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt}$.

[5]This approach generalizes (8) slightly by technically allowing for the *correlation* matrix - not just the standard deviations - to vary in the stock vector.

Applying Ito's Lemma (Dixit and Pindyck, 1994) yields:

$$dV(\boldsymbol{s}) = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] dt + \sum_{j=1}^{S} \sigma^j(\boldsymbol{s}) V_{s^j} dZ^j$$

Finding the expected value and dividing through by dt:

$$\frac{dV}{dt} = \frac{\mathbb{E}_t[dV]}{dt} = \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] \tag{9}$$

Setting (9) equal to the multidimensional generalization of (3) yields the HJB equation.

$$\delta V(\boldsymbol{s}) = W\left(\boldsymbol{s}\left(t\right), \boldsymbol{x}\left(\boldsymbol{s}\left(t\right)\right)\right) + \left[\sum_{j=1}^{S} \mu^j\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right) V_{s^j} + \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} \Omega^{jk}(\boldsymbol{s}) V_{s^j s^k}\right] \tag{10}$$

Partial differentiation of (10) yields the following expression for the shadow price of s^i

$$p^i(\boldsymbol{s}) = \frac{W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S} \frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^{S} p^j \mu^j_{s^i} + \frac{1}{2}\sum_{j}^{S}\sum_{k}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)}{\delta - \mu^i_{s^i}}$$

Factoring the final numerator term yields the final asset pricing equation.

$$\begin{aligned} p^i(\boldsymbol{s}) = &\left[W_{s^i} + \left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S} \frac{\partial p^j}{\partial s^i}\mu^j\right) + \sum_{j\neq i}^{S} p^j \mu^j_{s^i} + \frac{1}{2}\sum_{j=1}^{S}\left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j} + \sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right)\right. \\ &\left. + \frac{1}{2}\sum_{j=1}^{S}\sum_{k\neq j}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k} + \Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)\right] \Big/ \left(\delta - \mu^i_{s^i}\right) \end{aligned} \tag{11}$$

The first numerator term in (11) has the same interpretation as in the single-asset case. The next two terms in the numerator are present in the deterministic multi-asset case (Yun et al., 2017) and are forms of "capital gains." The second numerator term $\left(\frac{\partial p^i}{\partial s^i}\mu^i + \sum_{j\neq i}^{S} \frac{\partial p^j}{\partial s^i}\mu^j\right)$ reflects the effects of investment in s^i on the shadow price of stock i due to its prices of all assets in the portfolio (i.e. "price effects"). The third numerator term $\sum_{j\neq i}^{S} p^j \mu^j_{s^i}$ captures the deterministic effects of investment in stock i on the physical growth rates of all other stocks ("cross-stock effects"), which can stem from system ecology or production interactions within the economic program.

The additional numerator terms in (11) only exist in the stochastic case. The third term $\frac{1}{2}\sum_{j=1}^{S}\left(\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j}+\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}\right)$ operates solely through the individual variances of each asset and captures the "risk sensitivity" effect of an investment in asset i on the variance of each asset, $\sigma^{2j}_{s^i}\frac{\partial p^j}{\partial s^j}$. This part of the term reflects how substitution and complementarity relationships can provide "self-protection" through "portfolio diversification," which is the endogenous risk concept. Importantly, the $\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}$ term represents prudence and accounts for the fact that investments in i also affects the *sensitivity* to risk for all S assets, $\sigma^{2j}\frac{\partial^2 p^j}{\partial s^j \partial s^i}$, even if the variance for these other assets remains unchanged by the investment. This means that this term influences the consequences of stochastic events, and can be thought of as a self-insurance term. Together, these terms mirror the numerator terms, $\sigma(s)p_s + \frac{1}{2}\sigma^2(s)p_{ss}$, in (7).

The final term in the numerator of (11), $\frac{1}{2}\sum_{j=1}^{S}\sum_{k\neq j}^{S}\left(\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k}+\Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}\right)$, reflects the risk-related effects of investing in asset i that are mediated through the *covariances* of assets in the portfolio. This term is zero in the case that natural capital stocks are uncorrelated regardless of the vector of capital stocks. $\Omega^{jk}_{s^i}\frac{\partial p^j}{\partial s^k}$ is the effect of an investment in i on the covariances between other assets j, k as valued through the first cross-partial between these assets (i.e. the 2nd cross-partial of the intertemporal welfare function). If the covariances between asset stocks are invariant to capital stocks then this term is zero. $\Omega^{jk}\frac{\partial^2 p^j}{\partial s^k \partial s^i}$ reflects the fact that investing in i may itself affect the curvature of the intertemporal welfare function in the direction of k and i (i.e. $\frac{\partial^2 p^j}{\partial s^k \partial s^i} = \frac{\partial}{\partial s^i}V_{s^j s^k}$). If the effect of increasing asset i is to increase the concavity in the direction of increases in j and k ($\frac{\partial^2 p^j}{\partial s^k \partial s^i} < 0$) then the existence of positive correlation between the latter two assets results in a compensating reduction in the asset price. This creates addition "self insurance" opportunities from portfolio diversification.

Some insight on the numerator terms involving covariances can be gleaned by realizing that the covariance between innovations in s^j (the residual of changes in s^j after the deterministic drift $\mu^j(\boldsymbol{s}, \boldsymbol{x}(\boldsymbol{s}))$ is differenced away) and innovations in s^k can be viewed as their rescaled relationship in expectation. Specifically, if the conditional expectation of s^j and s^k is linear[6] $\mathbb{E}[ds^j|ds^k] = \beta ds^k$, then it is well known that $\beta = \frac{\Omega^{jk}}{\sigma^{2k}}$. In other words, the covariance terms in (11) reflect the expected marginal effect of ds^k on ds^j such that the risk terms in the multivariate asset case account for systematic (linear) cross-effects between perturbations in stocks in a way that is analogous to how the previous cross-terms in the numerator account for capital gains through deterministic relationships via price and cross-stock effects.

Finally, it is noteworthy that the effects of stochasticity disappear from (11) when two conditions hold: 1) when all second moments are constant regardless of the stock levels, and 2) the intertemporal welfare function, V, is quadratic such that investments have no effect on its curvature. However, since the intertemporal welfare function inherits the properties of the instantaneous benefits function, the economic program, and biophysical dynamics in a complex manner, the latter property is difficult to verify *ex ante*.

The numerical approximation of the shadow price function is carried out using "value function approximation" and is detailed in Appendix A. As detailed in Fenichel, Abbott, and Yun (2018) for the deterministic case and employed in Yun et al. (2017), this approach uses a Chebyshev polynomial basis to approximate the intertemporal welfare function using the HJB equation. We then differentiate the HJB equation to obtain estimates of the shadow prices.

3. An optimized single-stock example

The asset pricing approach presented in the previous section is valid regardless of whether the economic program maximizes intertemporal welfare or not (i.e. is the optimal feedback control rule). Nevertheless, given the substantial literature focusing on optimal economies, it useful to build intuition for realized shadow prices from an optimized economy model.

[6] Linearity of the conditional mean follows directly from the joint normality assumption for Ito processes.

Simple optimized models may also confer the benefit of a closed form solution for the co-state, thereby allowing for a direct validation of the numerical approximation approach.[7]

To provide this example, we draw upon a case explored in Pindyck (1984). In this seminal contribution, Pindyck extends the canonical infinite horizon, continuous-time renewable resource model for a single stock to allow for a stochastically evolving resource stock. The focus of the modeling is on revealing how the 'golden rule' of resource management is augmented by a risk premium term. He then explores how the biological and economic parameterization interacts with increases in risk to influence the extraction rate and the stochastic steady state distribution.

Our model draws directly on Pindyck's example 1 (p. 296), which is also explored in Miranda and Fackler (2004, p. 330). The objective is to maximize the infinite horizon expected net present value of the combined consumer and producer surplus from harvest q of the fish stock s.[8] The demand function is isoelastic, $q(p) = bp^{-\eta}$, and the marginal cost of harvest is $cs^{-\gamma}$. The resource dynamics evolve according to a diffusion characterized by a logistic drift function with stochasticity that follows a geometric Brownian motion process: $ds = rs\,(1 - s/K) - q + \sigma s dZ$.

In general, this model must be solved numerically. However, Pindyck (1984) demonstrates that a closed form solution to the HJB equation exists when $\eta = 1/2$ and $\gamma = 2$. Specifically, the optimized co-state (or rent) is:

$$V_s = \phi/s^2 \tag{12}$$

and the optimized economic program (feedback control rule) is:

$$x(s) = q^*(s) = \frac{b}{(\phi + c)^{1/2}} s \tag{13}$$

[7] Fenichel and Abbott (2014) follow a similar process in the deterministic case by evaluating how the natural asset pricing approach works on a simulated optimal program.

[8] Pindyck uses x for the state variable, we have changed this to s to avoid confusion and align with notation within this paper.

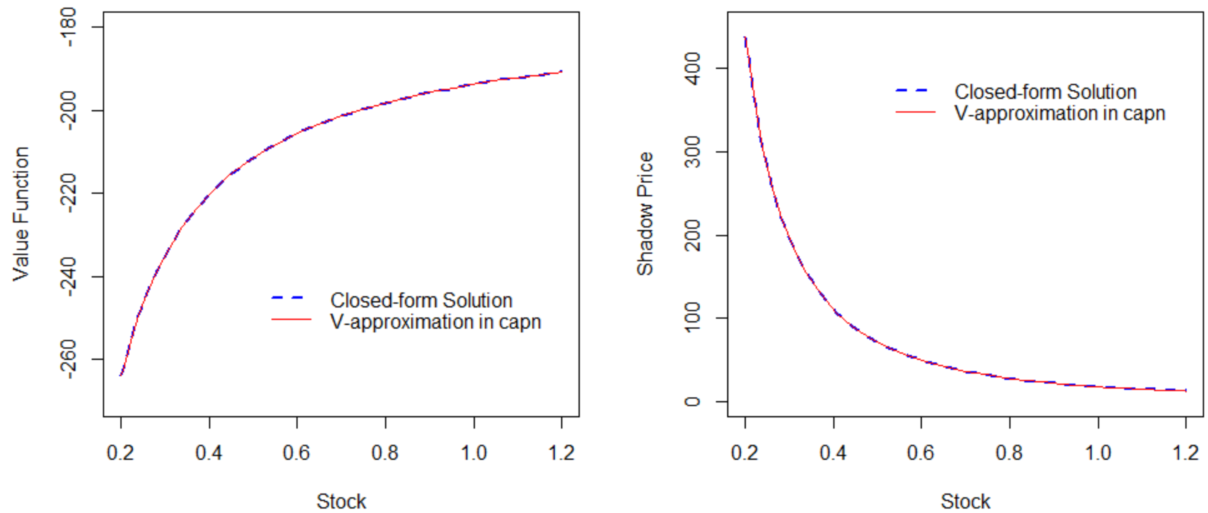


Figure 1: Illustration the the natural capital asset pricing approximation approach reproduces known value function and price curves for a stochastic system.

where

$$\phi = \frac{2b^2 + 2b[b^2 + c(r + \delta - \sigma^2)^2]^{1/2}}{(r + \delta - \sigma^2)^2} \tag{14}$$

The resulting economic program (13) is linearly increasing in the stock. Such rules imply a constant (per-capita) rate of fishing mortality (i.e. a "constant-F" rule) and are common in natural resource management. Although, the rate of harvest may not correspond to the optimal rate in many real world applications. Importantly, $\partial\phi/\partial\sigma^2 > 0$. This implies that $\partial q^*/\partial\sigma^2 < 0$ and $\partial V_s/\partial\sigma^2 > 0$, meaning that increasing stochasticity in this model always increases the accounting price of the stock, thereby *decreasing* the optimal rate of harvest at every stock level.

We approximate the value function using the approach detailed in Appendix A and using the optimal feedback rule (13) for the economic program.[9] Figure 1 shows that we are able to reproduce the analytical value function and shadow price to a high degree of accuracy.

Dynamic optimization in this example yields an economic program that reflects what

[9]We use parameter values of $\sigma = 0.1, \delta = 0.05$, $b = 1$, $r = 0.5$, and $K = 1$.

we might term "uniform precaution" (Figure 2, black line). Increases in stochasticity lead to a less aggressive harvest rate at *all* stock levels and therefore a larger 'target' steady state biomass. More generally, Pindyck shows that stock stochasticity has three competing effects that may lead to more or less aggressive (less precautionary) harvest relative to the deterministic case. The first, a variance reduction effect, encourages the manager to hold a lower stock due to the fact that the variance of stock increases in the stock size, and variance lowers the value function given its concavity. The second, a cost reduction effect, encourages the manager to hold a lower stock as variance increases due to the cost-increasing effects of stochastic fluctuations on expected harvest costs given the concavity of the harvest cost function – an implication of Jensen's inequality. The third, a growth rate effect, encourages managers to hold *more* stock as variance rises since stochasticity reduces the expected growth rate of the stock given the concavity of the growth function. He shows through a series of examples how the different effects can lead to more or less aggressive harvest under risk.

In practice, managers may choose to exercise more (or less) precaution than is optimal. We reflect these adjustments through two scalar shifts of the economic program, where harvests are either systematically lower (purple line, half the optimal harvest at every point) or greater (red line, 1.5 times the optimal harvest at every point) than the optimizing program (black line) (Figure 2).[10] These shifts lead to economic programs that are non-optimal everywhere and result in different stochastic "equilibria." These deviations from optimality are reflected in the intertemporal welfare functions and accounting price functions.

We also consider an economic program that deviates from the "constant-F" form (Figure 2, blue curve) by being a convex function of the stock. This program is 'adaptive' in its degree of precaution by being more conservative at low stocks and more aggressive at high stocks.[11] For the sake of comparison, we calibrate this control rule to have the same stochastic

[10] When the system is stochastic the catch curves representing the economic programs are slightly to left of the deterministic programs, though this difference is hardly noticeable when plotted so we have omitted the stochastic plots. Furthermore, applying deterministic program to the stochastic system in this setting has only a small effect.

[11] In fisheries management, this adaptivity is often accomplished in practice by distinct linear harvest

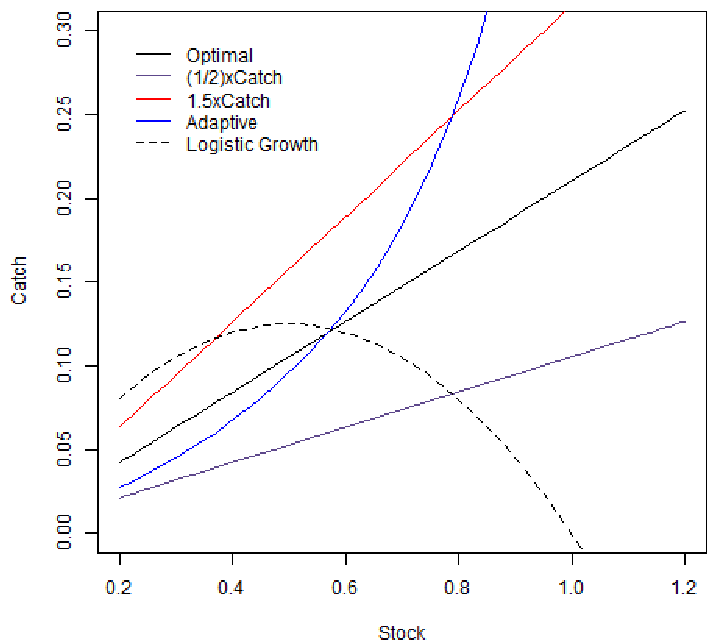


Figure 2: Stock-catch space showing the optimal harvest feedback rule and three alternative non-optimal economic programs

equilibrium as the optimal program. Therefore, the adaptive program is the optimal program if, and only if, the stock is at the stochastic equilibrium. Importantly, the strong convexity of the adaptive program reflects a managerial bias for system stability; the steady state probability distribution will have a lower variance than the optimal program. The shape of this control rule, but not its anchoring on the optimal steady state, is similar to the feedback process that Zhang and Smith (2011) estimate and Fenichel and Abbott (2014) use in their application to the Gulf of Mexico reef fish fishery.

Figure 3 compares the intertemporal welfare and price functions for the scalar transformations of the optimal harvest program for the stochastic ($\sigma = 0.1$, solid lines) and deterministic case ($\sigma = 0$, dotted lines). Importantly, the optimal and sub-optimal economic programs adjust for the value of σ according to the feedback rule in (13). The left-hand panel, showing the intertemporal welfare functions, shows that risk strictly reduces welfare,

control rules that are each applicable within different stock thresholds–in essence a linear spline function.

with risk having a similar effect across all three economic programs. Stochasticity appears to translate the intertemporal welfare functions down in a nearly constant manner (i.e. a location shift). This suggests that *changes* in welfare between stock levels – which are the relevant metrics for social benefit cost analysis and sustainability assessment – may be minimally affected by volatile stock dynamics. The first column of Table 1 considers the welfare change for a relatively large perturbation in stock from 0.37 to 0.57. Regardless of whether we consider the optimal or sub-optimal programs, we find that the change in welfare from a stock shift is 3 percent greater in the stochastic case relative to the deterministic case, despite substantial volatility. Therefore, ignoring stochasticity may systematically undervalue changes in natural capital. However, our example indicates that this bias may be small in some cases.[12] Indeed, we find that the changes in measured welfare across the three economic programs – holding stochasticity constant – are much more sizable than the effects of ignoring stochasticity. This suggests that the behavioral *responses* to stochasticity (i.e., excessive or inadequate precaution that push the system toward a sub-optimal equilibrium) may be more consequential for welfare than the effects of stochasticity itself.

Given the apparently near-vertical translations of the intertemporal welfare functions from introducing stochasticity, it is no surprise that risk has a muted effect on the accounting price functions (Figure 3). (Recall the price is the first derivative of the value function.) The effect of stochasticity on the shadow price is hardly noticeable. For the optimal program and its scalar multiples, price always increases in stochasticity. However, stochasticity has only a second-order effect on marginal values – hardly surprising given the small welfare effects of stochasticity for non-marginal stock changes (which integrate under the price curve) noted in the previous paragraph. Finally, stochasticity has no effect on the approximation error of welfare changes introduced by using a price index over the change multiplied by the change

[12] We find that employing the program associated with deterministic dynamics to a system with stochastic dynamics has a small effect. Using the “wrong” program can either lead to a larger or small assessment of the welfare change, relative to using the stochastic program. This is due to second-best nature of these feedback rules.

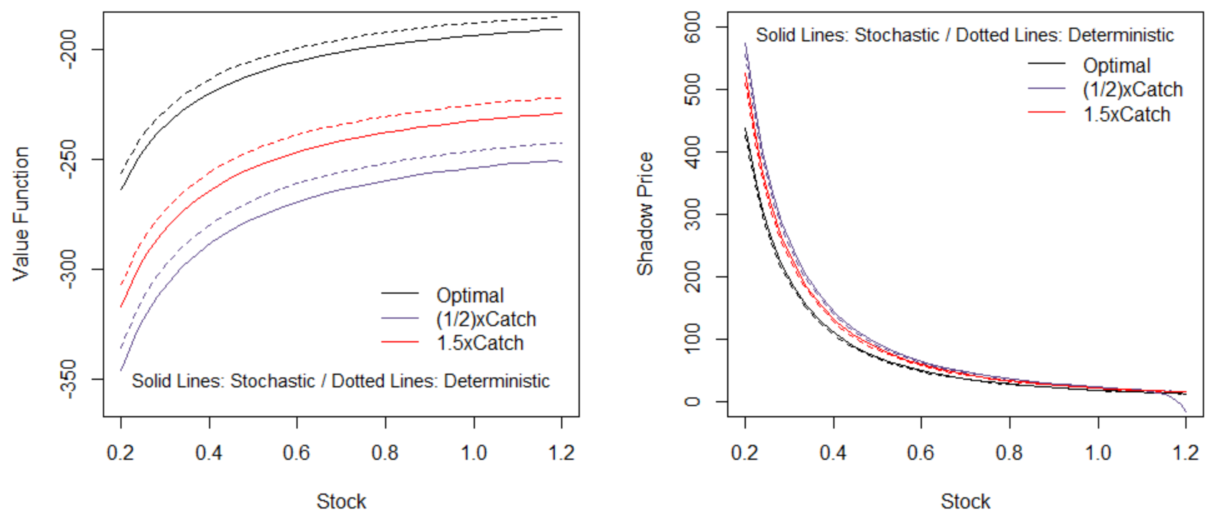


Figure 3: The intertemporal welfare (value) function and shadow price curves for optimal program and two scalar shifts of the optimal program with stochastic and deterministic dynamics.

in quantity (Table 1).[13]

Now consider the adaptive economic program (Figure 4), which mimics the asymmetric precaution observed in the management of many harvested resource systems and thereby ensures a greater degree of stability relative to linear feedback rules. Importantly, this rule has the same steady state as the optimal control rule, so all differences are due to the sub-optimal approach path and its potential interactions with stochasticity. As before, the most apparent effect of introducing stochasticity to the adaptive economic program is a downward shift in the intertemporal welfare function. However, there are subtle, but important differences relative to the linear control rule case.

In the deterministic case (dashed curves), the intertemporal welfare value in the region of the equilibrium is approximately the same under the adaptive and optimal programs (indeed, identical at the equilibrium itself). Therefore, small changes in the stock in this region result in near-identical welfare changes under either program. It is only as the system

[13] The use of price indexes is likely necessary in applied wealth accounting approaches for sustainability assessment. The Fisher Ideal price Index is the geometric mean of prices. The Mean price Index is the arithmetic mean of prices.

Table 1: Comparison of the change in welfare and the change in wealth using two different index number approaches. The Fisher Ideal index is the geometric mean of prices, and the Mean price index is the arithmetic mean of prices. The price index is multiplied by the change in quantity.

Program	Change in Welfare	Fisher Ideal Index	%error	Mean Price Index	%error
Optimal rule with deterministic dynamics	15.920	15.920	0.000	17.373	0.091
Optimal rule with stochastic dynamics	16.405	16.405	0.000	17.902	0.091
Adaptive rule with deterministic dynamics	17.110	16.901	-0.012	18.873	0.103
Adaptive rule with stochastic dynamics	17.730	17.517	-0.012	19.553	0.103
Scalar rule, 0.5 of the optimum, with deterministic dynamics	20.855	20.855	0.000	22.758	0.091
Scalar rule, 0.5 of the optimum, with stochastic dynamics	21.497	21.497	0.000	23.459	0.091
Scalar rule, 1.5 of the optimum, with deterministic dynamics	19.081	19.081	0.000	20.822	0.091
Scalar rule, 1.5 of the optimum, with stochastic dynamics	19.692	19.692	0.000	21.489	0.091

moves significantly from the equilibrium that there is a meaningful divergence between the intertemporal welfare functions in a deterministic system. By contrast, when the system is stochastic, the intertemporal welfare function under the adaptive program is always below that of the optimal program – even at the stochastic steady state biomass. This occurs because even at the equilibrium point there is an expectation of a shock that will move the system to a region where the adaptive program is meaningfully sub-optimal.

These features are reflected in the shadow price curves (Figure 4, right panel). The shadow price curves cross at the equilibrium. This must be the case in the deterministic model for the adaptive program to be sub-optimal everywhere except at the equilibrium; it is required for the intertemporal welfare function of the adaptive program to "bow in"

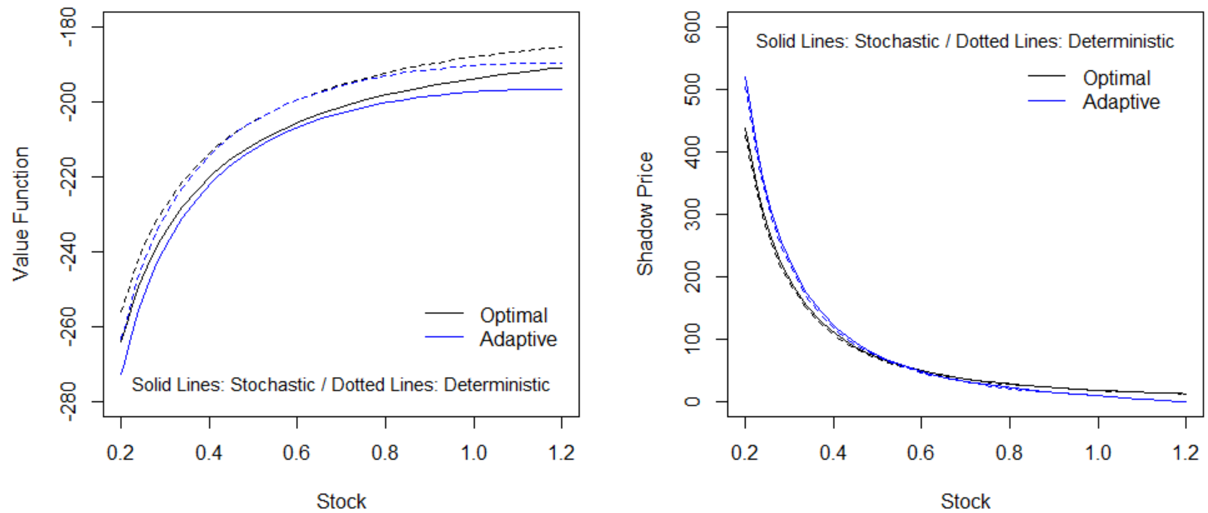


Figure 4: The intertemporal welfare (value) function and shadow price curves for the optimal program and a non-optimal adaptive "precautionary" economic program that preserves the stochastic equilibrium under stochastic and deterministic dynamics.

relative to the optimal program's value function. This feature is inherited in the stochastic setting as well.

Despite these subtleties, once again the shadow price curves for the deterministic and stochastic dynamics for the adaptive precautionary economic program are remarkably similar. Once again, the first order effects for valuation derive from the choice of the economic program – not from the introduction of stochasticity.

The analysis of Pindyck's model suggests that stochasticity may be, at most, a second-order concern for social benefit cost analysis or sustainability assessment.[14] This appears in sharp contrast to much of the literature's broader concern with stochasticity, risk, and uncertainty. We conjecture that a key feature of the Pindyck model that supports this result is the existence of a single stochastic equilibrium. There is a substantial literature on multiple equilibria (reviewed by Fenichel et al. (2015)), but this literature largely focuses on

[14]Lest the reader think we are cherry-picking an extreme example to minimize stochasticity, Pindyck's example 2 in the same paper replaces the logistic growth function with a Gompertz growth function to show that risk has *no* effect on shadow prices, and hence *no effect on changes in welfare*, in the optimal management case.

deterministic models to examine how the optimal pursuit of alternative long-run equilibria depends on initial conditions. Fenichel, Abbott, and Yun (2018) argue that the difficulties caused by multiple equilibria for valuation purposes (where the economic program is typically pre-determined, and the relevant basin(s) of attraction are therefore known) are lessened compared to optimal control. However, stochasticity could complicate matters by shocking the system into a different basin. An important question is whether real world economic programs are robust to these shocks. Nevertheless, we suspect that the findings from the Pindyck model may serve as a reasonable qualitative metaphor for a number of real-world systems. In the next section, we investigate a real-world system that that has similar features to the Pindyck model and show that second-order nature of stochasticity persists for a real world calibration.

4. Gulf of Mexico Reef Fish

The Gulf of Mexico reef fish example presented in Fenichel and Abbott (2014) has many of the same properties as the Pindyck (1984) model. Zhang and Smith (2011) estimated a logistic growth equation for the stock, and Zhang (2011) estimated an empirically-grounded feedback rule with similar properties as the adaptive rule illustrated in prior section – though Zhang's rule is not calibrated to bisect the optimal equilibrium (Figure 5).[15]

We extend this deterministic model to the stochastic case. As in Pindyck, we augment the logistic stock dynamics with an additive geometric Brownian motion (GBM) noise term. Geometric Brownian motion is consistent with the assumptions of log-normal disturbances frequently used in population dynamic modeling and fisheries stock assessment. Utilizing the assessed biomass data from the fishery we calibrate $\sigma = 0.067$; therefore the standard deviation from the deterministic drift given by the logistic growth equation with harvest is

[15] Indeed, the subsequent rebuilding of many stocks that has occurred, with support of the GOM fleet, under rights-based management suggests that the former economic program, as approximated by Zhang, under-invested in the stock relative to the economic optimum.

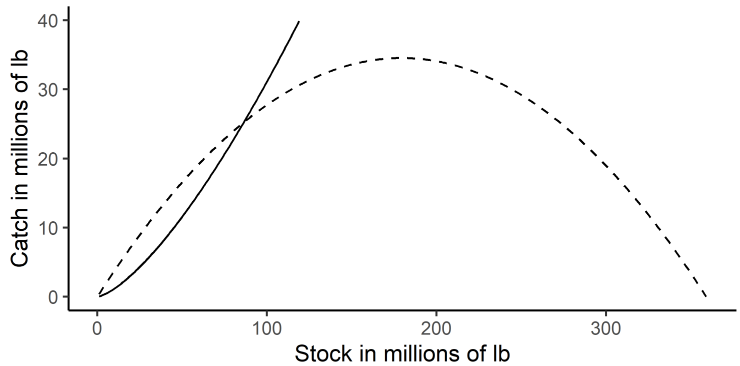


Figure 5: The growth function and economic program for the Gulf of Mexico model.

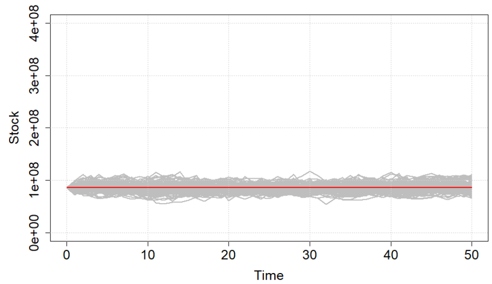
approximately 6.7 percent of the stock level. The stock dynamics are

$$ds = \left(0.3847s(t)\left(1 - \frac{s(t)}{3.59 \times 10^8}\right) - h\left(x\left(s\left(t\right)\right), s\left(t\right)\right)\right)dt + 0.067s(t)dZ(t) \tag{15}$$

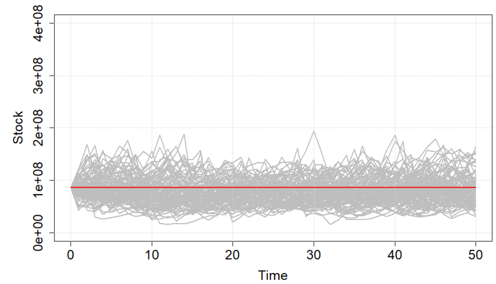
The economic program, the feedback relationship linking stock status (in pounds (lbs)) and effort (in crew-days) in the fishery, is provided by a power rule, $x(s) = ys^{\gamma}$, where $\gamma = 0.7882$ and $y = 0.157$. We assume that the valuation of income flows in the fishery is directly expressed in terms of monetary profits, with price-taking firms and costs that are linear in effort: $W = mh - cx$, with $m = \$2.70/\text{lb.}$, $c = \$153/\text{crew-day}$. The production function for harvests is of a generalized Schaefer form $h = qsx\left(s\right)^{\alpha}$, with $q = 3.17 \times 10^{-4}$ and $\alpha = 0.544$. $W(s)$ is a strictly *convex* function of the stock once the endogenous feedback from the stock level to harvest behavior $x(s)$ is incorporated, despite the linearity of harvests and costs for a fixed allocation of effort x. Abstracting from stochasticity, Figure 5 shows the dynamics of the system are similar to the Pindyck model with the adaptive control rule.

4.1. The effects of risk, σ

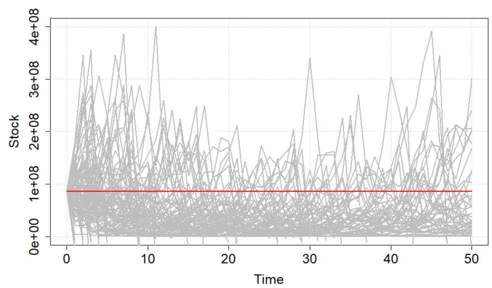
Figure 6 illustrates stochastic simulations of stock paths originating from the steady state biomass and harvest under four levels of stochasticity. The level of noise introduced by stochasticity in the base case (Fig. 6a) is already substantial and reminiscent of the noise seen in many ecological systems. While extinction is technically impossible in continuous



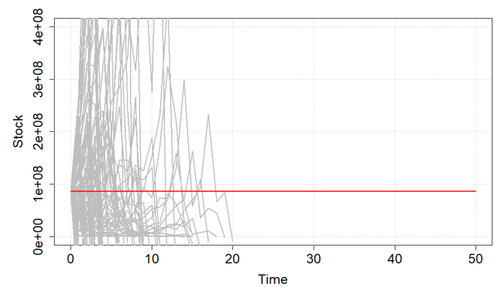
(a) $\sigma = .067$



(b) $\sigma = .2$



(c) $\sigma = .5$



(d) $\sigma = 1$

Figure 6: Stochastic simulations of stock dynamics over a range of values for σ. Note that the values for $\sigma = 1$ exceed the range of the graph on a number of runs.

time, using the current economic program and geometric Brownian motion, our numerical simulations nevertheless show that the number of paths that tend to a numerically zero level increase dramatically with increases in σ. Indeed, all paths reach numerical extinction within 20 periods when $\sigma = 1$. This suggests that levels of σ of 0.5 or 1 are likely inconsistent with the dynamics of most real-world species. Dixit and Pindyck (1994) provide a similar example where they argue that volatility can only be so high given a reasonable probability of observing the stock at all.[16]

Unsurprisingly, increasing levels of volatility reduce the intertemporal welfare shown by the graph of the value function (Figure 7). Following Eq. (6) and the Pindyck example, the value function in the stochastic case includes an additional risk term that serves, in part, to shift the value function downward with increasing stochasticity. In the current

[16]We thank Martin Quaas for bring this to our attention.

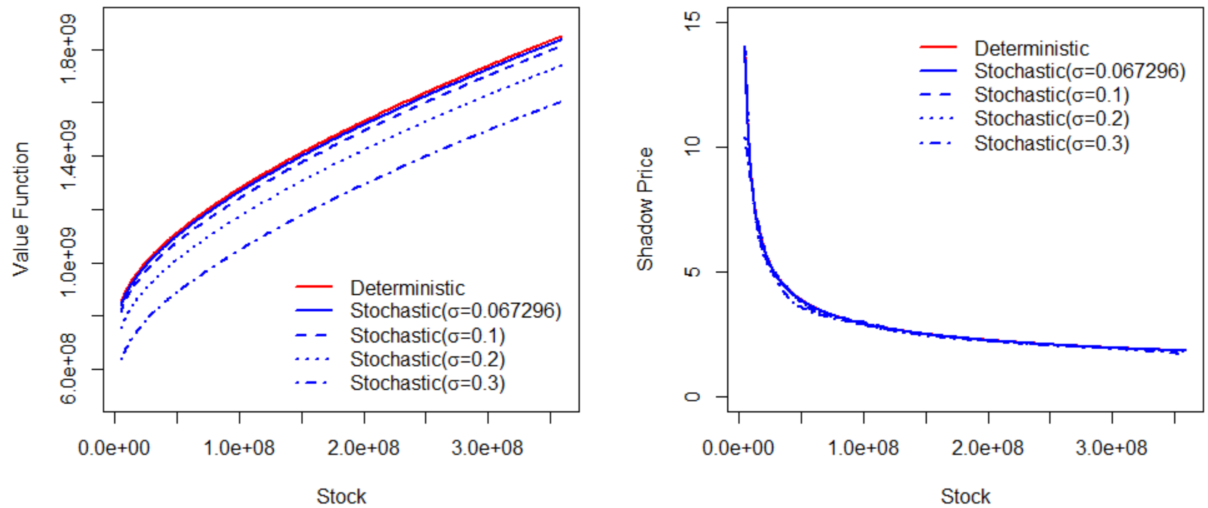


Figure 7: The intertemporal welfare or value function and shadow price or account price function of the Gulf of Mexico reef fish example with four different values of σ

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case, the value function is concave in s (i.e. the shadow price curve is downward-sloping) so that increasing σ has the effect of reducing the expected net present value at any given stock level. This adjustment is small for the empirically-justified level of stochasticity in our system ($\sigma = .067$) – suggesting that the economic program is fairly robust to the level of stochasticity in the system by maintaining stock levels in a relatively insensitive range of the profit function. However, higher levels of stochasticity lead to much less controlled systems, resulting in devaluations of the 'ecosystem portfolio.'

Higher-order effects on the shape of the value function with increases in σ exist, but are small. Changes in the shadow prices (i.e. the derivative of the value function) are hardly noticeable (Fig. 7, right panel) and suggest the volatility is creating a nearly vertical shift in the value function. Thus, while risk devalues the stock in total (albeit mildly), stochasticity has no appreciable effect on its *marginal* valuation.

Examining the Gulf of Mexico case reinforces the intuition developed by the Pindyck example in the context of a real world, well calibrated system. Risk appears to be a decidedly second-order feature. Yet, not all of the intuition from the Pindyck examples carries through.

Consider the value of a change from the observed equilibrium to the stock level supporting maximum sustained yield or half of carrying capacity. The change in the value function for the deterministic case is $244 million, whereas in the stochastic case the value is $243 million.[17] In this case, use of the deterministic system as a proxy for the stochastic system appears to overvalue the change in welfare or wealth slightly. This reinforces an insight from Pindyck under optimal management to the general case – that the effects of risk on the shadow price are contingent on bioeconomic parameters. However, these errors remain small.

5. Conclusion

The implications of uncertainty for decision-making and valuation are a longstanding concern in natural resource economics and real-world resource management. There is a large literature applying stochastic optimal control theory to the optimal management of resources subject to stochastic shocks (e.g., Sethi et al., 2005; LaRiviere et al., 2017). There is also a growing literature applying modern portfolio theory to the design optimal portfolios of harvested species or portfolios of spatial conservation across landscapes or seascapes according to the social planner's risk-return preferences (e.g., Ando and Mallory, 2012). Meanwhile, decision makers are increasingly influenced by a wave of thought, loosely organized under the heading of the "precautionary principle," urging less aggressive action, or delay of irreversible actions, under conditions of risk or Knightian uncertainty. This mode of thinking is provided some qualified economic support by the literature on option value (Arrow and Fisher, 1974; Dixit and Pindyck, 1994; Gollier, 2003). This cautious approach to uncertainty is countered by the literature on adaptive management (e.g., Walters, 1986), which urges active learning in the presence of risk and uncertainty. Given these divergent approaches, and their often conflicting advice, it is of little surprise that resource governance has struggled with how to

[17] Using the Fisher Ideal index the change in value for the deterministic and stochastic cases are both $245 million and using a mean price index both are $250 million. In both cases there are difference of less than $1 million.

incorporate risk and uncertainty into decision making.

The challenges posed by risk and uncertainty for sustainability assessment and natural capital valuation likewise appear formidable. There are several relevant uncertainties to consider, including stochasticity in resource dynamics, measurement error of the stocks themselves, implementation error in policy (i.e., a stochastic economic program), and profound uncertainty about the current and future substitutability of the services provided by different capital stocks (Gollier, 2019). How should these risks enter sustainability assessment and natural capital accounts?

We address this question for one form of risk, process error in natural capital dynamics, under real-world, non-optimized conditions. We show how the stochastic dynamics of natural resources can be incorporated into a single revealed shadow (accounting) price for natural assets at risk. This finding parallels financial markets that yield a price for traded assets conditional on the information-contingent forecasts in the minds of traders – even as the flow of dividends, which may be dependent on physical or managerial processes, from these assets is uncertain.

In the case of single assets, we find that risk enters into the marginal valuation of natural capital in two ways. The first, an "endogenous risk" effect, reflects how capital investments influence the extent of volatility itself. This effect is valued through the curvature of the value function. It reflects the degree of self-protection in the economic program. The second, an "endogenous risk aversion" effect, reflects how these same investments affect the *valuation* of the risk by moving from regions of the value function with different degrees of curvature, which suggest a self-insurance feature. This second feature is likely to affect the accuracy of price indexes in the presence of stochasticity, since the accuracy of a price index depends on its ability to adjust for curvature in the price function – to impose a linear index on a fundamentally non-linear welfare valuation.

The revealed value of risk therefore depends on the second and third derivatives of the value function, which depend on the totality of the properties of the utility function valuing income flows from capital stocks, the shape of the growth functions of capital stocks, and the feedback rules between capital stocks and human behavior embodied in the economic program. In other words, the extent of "intertemporal risk aversion" and prudence – the curvature and change in curvature of the value function with respect to *stocks* of capital – is not "baked in" solely through the curvature of the welfare function evaluating income flows (i.e., ecosystem services) from natural capital. Rather, it is a global property of the coupled human-natural system in question, including its management. Even in the case of a single natural asset, the feedback rule employed to respond to fluctuations in the resource stock can affect the level of objective risk faced and the sensitivity of intertemporal welfare to that risk. In other words, resource management plays a significant role in shaping the marginal value of an asset. Indeed, this is the logic behind the endogenous risk framework (Shogren and Crocker, 1999) and the broader literature on self-protection, self-insurance, and market based insurance (Ehrlich and Becker, 1972).

The logic from the single-asset case carries over to the multi-asset case, but in an even richer form. Investments in a given capital stock have the potential to affect the variances and the covariances of other capital stocks in the portfolio and the multi-dimensional curvature of the value function. These "portfolio effects" further elevate the role of the economic program. The feedback rule between the vector of capital stocks and human actions on these stocks serves as a portfolio rebalancing rule that influences the overall value of the portfolio. The valuation approach we have outlined provides a metric for understanding how alternative portfolio management strategies influence the valuation of individual capital stocks and the sustainability of management itself.

Despite the rich manner in which risk *theoretically* influences the valuation of natural assets, our investigation of the single-stock, logistic model with GBM shocks found that stochasticity has only a minor impact on measures of changes in wealth for marginal and non-marginal perturbations to capital stocks. This result is robust across optimized and non-optimized settings and for quite high (arguably unrealistic) levels of volatility. One explanation of this result is inherent in Pindcyk's analysis. He notes that risk acts in subtle and countervailing ways on shadow values (and hence the extraction rate) so that the qualitative effect of risk is unclear *a priori*. It is possible that in a number of cases that these effects may approximately cancel out. Our results suggest that for a large and important class of natural capital assets this is the case – risk is truly a second-order concern. Therefore, we argue that the lack of risk-adjustment in accounting prices is a poor reason to avoid or delay tracking changes in societal wealth to measure progress on sustainability. Deterministic estimates of shadow prices seem to be able to capture most of the change in value. Errors introduced by standard measurement error and index number error likely introduce errors of similar or greater magnitude.

Notwithstanding this strong conclusion, there are many aspects of risk and uncertainty which remain to be considered. We focused on diffusions in continuous time, while many other stochastic processes are also possible. For example, there is the possibility of resource dynamics experiencing discontinuous Poisson shocks that transition the system into an alternative basin of attraction. In these cases the stochasticity of the shock may be best thought of as reflecting our uncertainty of where the "critical thresholds" lie in an otherwise deterministic system. While these forms of risk fall outside the class of correlated diffusions considered here, we conjecture that they can nevertheless be handled in a relatively straightforward manner through extensions of analogues in the literature (Walker et al., 2010; Reed and Heras, 1992). This provides a narrow, but perhaps important, window for stochasticity to be of first-order concern.

Including credible measures of changes in wealth from natural capital in accounting and social benefit cost analysis is imperative for structuring policy discussions about sustainable development and encouraging better decision making. However, it's not easy. Developing credible accounting price estimates for natural assets, which resist aggregation due to their dependence on localized features and institutions (Addicott and Fenichel, 2019), is a daunting task. It is therefore important to prioritize efforts. The preliminary analysis in this paper suggests that adjusting shadow prices and changes in intertemporal welfare for the effects of risk may be of secondary importance for accurate valuation.

To be clear, risk is often important for decision making – while the economic program is being "chosen" – but accounting prices "build in" the feedback rule of the given economic program. The treatment of risk is conditional on a *certain* management plan for how to respond to it. This suggests a potential hidden vulnerability in the treatment of risk if there are unknown structural breaks in the economic program in response to risk. For example, a political revolution may be facilitated by a large resource shock such as a famine or stock collapse; or, scarcity-induced innovation may lead to new technologies that alter the nature of substitutability between capital stocks. How risks of such discontinuous technological and institutional change capitalize into the valuation of natural assets may be tractable conceptually, but often necessarily rests upon a highly speculative empirical basis. It may be exactly such "known unknowns" and "unknown unknowns" that ultimately trouble the minds of decision makers – not the polite and well-behaved risks. The difficulty of accountant approaches to capture such Knightian uncertainty may be one more reason why (Dasgupta, 2001) calls non-declining wealth a necessary, but not sufficient, condition for sustainability.

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Appendix A. Numerical approximation

Given a complete deterministic bioeconomic model of a social-ecological system it is possible, at least in principle, to obtain approximate shadow values for a given stock at a given initial state vector by perturbing the desired natural capital stock and calculating the change in the net present value of benefits flows over the indefinite future. While straightforward, this approach is computationally intensive and cumbersome for forecasting or backcasting the wealth dynamics of a system and may be inappropriate in stochastic settings. Fenichel, Abbott, and Yun (2018) and Yun et al. (2017) describe how the HJB equation can be combined with functional approximation approaches frequently used in numerical dynamic programming to approximate the entire shadow price *function* over a closed domain of capital stocks. For the deterministic, multi-asset case they advocate approximating $V(\boldsymbol{s}(t))$ using the HJB equation (analogous to (10)), replacing $V(\boldsymbol{s}(t))$ on the LHS of the equation with a weighted sum of the tensor product of Chebyshev basis functions in the stock vector $\boldsymbol{s}(t)$ and replacing the partial derivatives of the value function on the RHS with the partial derivatives of this approximation. The coefficients that determine the weightings on the basis functions can be solved analytically and are chosen (in a system with as many approximation points as coefficients) to make the LHS and RHS of the approximated HJB equation hold with equality.[18]

This value (intertemporal welfare) function approximation technique can be adapted with relatively minor changes to the stochastic diffusion case. First, define the bounded approximation interval for each state variable. Then choose M evaluation points within this interval for each of the S capital stocks and then calculate $W\left(\boldsymbol{s}\left(t\right), x\left(\boldsymbol{s}\left(t\right)\right)\right)$, $\mu\left(\boldsymbol{s}, \boldsymbol{x}\left(\boldsymbol{s}\right)\right)$ and $\Omega(\boldsymbol{s})$ at each point.[19] The univariate node coordinates are then permuted to yield

[18]In some cases it may be desirable to utilize more approximation nodes than the number of coefficients - an over-determined system. In this case, the coefficients can be chosen to minimize the sum of squared deviations between the LHS and RHS of the approximation. The analytical expression for this solution is analogous to ordinary least squares (Fenichel, Abbott, and Yun, 2018).

[19]In many cases the evaluation nodes are found by finding the M roots of a unidimensional Chebyshev polynomial on the bounded approximation range for each state variable. However, care must be taken so

M^S grid points. We define ϕ^i as the $M \times (q^i + 1)$ basis matrix of q^ith degree for state variable i. This is a matrix of $q^i + 1$ basis functions - Chebyshev polynomials of ascending degree in our case - evaluated at the M evaluation points. To approximate over the bounded domain in $\mathbb{R}^S$ we find the tensor product across all dimensions (i.e. allow for full interactions across the univariate basis functions) to form an $M^S \times \prod_{i=1}^{S}(q^i + 1)$ basis matrix: $\boldsymbol{\Phi}(\boldsymbol{S}) = \phi^N \otimes \phi^{N-1} \otimes \ldots \otimes \phi^1$ where $\boldsymbol{S}$ is the $M^S \times S$ matrix of evaluation points (i.e. all grid nodes of M evaluation points for all S state variables). We can now define our approximation to the intertemporal welfare function $V(\boldsymbol{S}^m) \approx \boldsymbol{\Phi}^m(\boldsymbol{S})\boldsymbol{\beta}$ where m indexes the M^S distinct capital stock vectors (i.e. the individual evaluation points in the S-dimensional grid) and $\boldsymbol{S}^m$ is the mth row of $\boldsymbol{S}$. $\boldsymbol{\Phi}^m(\boldsymbol{S})$ is the mth row of $\boldsymbol{\Phi}(\boldsymbol{S})$, and $\boldsymbol{\beta}$ is a $\prod_{i=1}^{S}(q^i + 1) \times 1$ vector of unknown approximation coefficients. Using the fact that $\frac{\partial V(\boldsymbol{S}^m)}{\partial s^i} \approx \left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i}\right)\boldsymbol{\beta}$ and $\frac{\partial^2 V(\boldsymbol{S}^m)}{\partial s^i \partial s^j} \approx \left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^i \partial s^j}\right)\boldsymbol{\beta}$ we can replace the HJB equation in (10) with the following approximation:

$$\begin{aligned}\delta\boldsymbol{\Phi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m) + \Bigg[&\sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right)\boldsymbol{\beta} \\ &+ \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\boldsymbol{\beta}\Bigg]\end{aligned} \tag{A.1}$$

Collecting terms involving β yields:

$$\left[\delta\boldsymbol{\Phi}^m(\boldsymbol{S}) - \sum_{j=1}^{S} diag(\mu^j(\boldsymbol{S}^m))\left(\frac{\partial \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j}\right) - \frac{1}{2}\sum_{j=1}^{S}\sum_{k=1}^{S} diag(\Omega^{jk}(\boldsymbol{S}^m))\left(\frac{\partial^2 \boldsymbol{\Phi}^m(\boldsymbol{S})}{\partial s^j \partial s^k}\right)\right]\boldsymbol{\beta}$$

$$= \boldsymbol{\Psi}^m(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S}^m)$$

Stacking these M^S vector equations results in the equation $\boldsymbol{\Psi}(\boldsymbol{S})\boldsymbol{\beta} = W(\boldsymbol{S})$. If $M^S =$

that the nodes are laid out in a way that the system dynamics do not leave the approximating domain in expectation.

$\prod_{i=1}^{S}(q^i+1)$ (i.e. the number of approximation points equals the number of unknown approximation coefficients) then the approximation coefficients can be calculated in a straight foward way through matrix inversion. Alternatively, if $M^S > \prod_{i=1}^{S}(q^i+1)$ then the $\boldsymbol{\beta}$ can be found using least squares.

$$\boldsymbol{\beta} = \left(\boldsymbol{\Psi}(\boldsymbol{S})'\boldsymbol{\Psi}(\boldsymbol{S})\right)^{-1}\boldsymbol{\Psi}(\boldsymbol{S})'W(\boldsymbol{S}) \tag{A.2}$$

After obtaining the approximation $\boldsymbol{\Phi}(\boldsymbol{S})$ it is straightforward to find the shadow values of any given capital stock by taking its partial derivative.

Fenichel, Abbott, and Yun (2018) discuss the importance of determining the domain of approximation. They show that in multi-dimensional systems the system dynamics to can lead outside the approximation domain, which hinders the ability to recover shadow prices. They argue that it is important to make sure the approximation domain is sufficient to include dynamic from any stock size for which a shadow price is desired. In the single stock deterministic case this is never an issue so long as the system has attractors that are within the approximation domain. However, this property does not extend to stochastic dynamics. This is because a shock at the edge of the approximation domain could lead the system outside the approximation domain for a non-trivial period of time. Therefore, extra attention is needed to enlarge the approximation domain when system dynamics are stochastic.