How Automation that Substitutes for Labor Affects Production Networks, Growth, and Income Inequality

Matthew O. Jackson and Zafer Kanik*

October 2019

(The latest version can be found here)

Abstract

We study the impact of technological change on GDP growth, income inequality, and the interconnectedness of the economy. Technological advances in goods that complement labor increase productivity but do not change the interdependencies across sectors nor the relative wages between high-skilled and low-skilled labor. In contrast, technological advances that (directly or indirectly) substitute for labor (e.g., robots, AI) change both and have impacts that depend on the state of the economy. As automation becomes more productive, wages drop for workers employed in automatable tasks to slow their displacement. With less productive alternative opportunities for labor, there is a greater drop in the wages of replaceable workers (raising inequality), and hence less automation, and a lower growth in overall productivity. The growth effects of technological advances in automation emerge gradually, and propagate both downstream and upstream (due to wage effects). In addition, as automation progresses, the production network becomes denser, increasing the centralities of automation good producers and their (direct and indirect) suppliers. Our findings provide additional insights into what makes today's automation different from the previous ones.

Keywords: Automation, AI, Growth, Input-Output Analysis, Inequality, Productivity, Production Networks, Reallocation of Labor, Labor Displacement, Technology.

JEL Classification Numbers: D85, E23, E24, E32, F43, J31, O33, O41.

*Jackson: Department of Economics, Stanford University, Stanford, California 94305-6072 USA, and external faculty member at the Santa Fe Institute. Kanik: Department of Economics, Massachusetts Institute of Technology. Emails: jacksonm@stanford.edu and zkanik@mit.edu. We gratefully acknowledge support under NSF grant SES-1629446 and from Microsoft Research New England. We thank Daron Acemoglu, Susanto Basu, and Emmanuel Farhi for helpful conversations. We thank participants of the NBER Economics of AI Conference 2019 and the seminar series at Harvard Kennedy School Growth Lab.

1 Introduction

The production of goods and services has become increasingly complex and networked. Many involve multiple tasks or parts, some even hundreds or hundreds of thousands. Moreover, some goods and services used in production substitute for labor. We have seen this historically, as technological advances reduced the use of labor in agriculture and manufacturing, and are seeing at present as automation technologies are displacing labor in the production of an increasingly wide variety of goods and services. For instance, robots and AI substitute for labor in manufacturing (e.g., assembly lines), distribution (e.g., warehouses and drivers), and services (e.g., computer-based markets and apps and data collection systems). Similarly, autonomous vehicles are expected to lead to a large-scale labor displacement as driving and delivery are some of the largest professions in the world.

In addition, automation technologies transform the economy by altering supply chains. Such large-scale increases in intermediate good usage make today's automation technologies fundamentally different than a generic increase in the capital stock. For instance, different forms of autonomous vehicles (e.g., self-driving cars, autonomous delivery vehicles, drones) use different AI technologies that in turn increase the production and use of computer systems and related services. Similarly, various types of electric batteries, cameras, lasers, radars, and vehicle-specific computers are used in production of these autonomous vehicles. As a result, the labor displacement effects of automation are also accompanied by changes in the input-output structure of the economy.

In this paper, we study the effects of technological advances in such a world that is extensively networked and in which goods not only complement each other in production, but can also substitute for other inputs; and, in particular, for labor. Despite the complexity of such networked production processes, we show that there are tractable formulas that describe the impacts of various technological advances. Our analysis shows that the effects of technological advances in the presence of substitution effects differ fundamentally from the case of pure complements. Thus, traditional input-output analysis, which has focused on the case of complements and not accounted for how substitution works, offers an erroneous view of the basic effects of technological improvements on the production network, as well as overall consumption and relative wages.

A main feature of our analysis is a general equilibrium effect of changes in wages in reaction to technological advances. These effects counteract the impact of technological advances that substitute for labor. The extent to which they mitigate such advances depends on where the labor displaced by new technologies can be re-employed, which depends on the full production network in a way that we characterize.[1]

With this intuition in hand, we build an input-output model where intermediate goods that are complements for, or substitutes to, labor are produced within the economy. Then, in a general equilibrium framework, we study the underlying forces behind automation and

[1] We focus on the displacement of labor, but the analysis applies directly to the displacement of other productive inputs as well.

its overall impact on the economy. There are three main contributions of our study. First, we show how technological advances translate into increased level of automation and analyze how long this "transition to automation" takes in an economy. Second, we show how the macroeconomic consequences of automation, and resulting increases in income inequality, depend on the alternative uses of labor in the economy. Third, we use the model to analyze the evolution of the interconnectedness of the economy. Accordingly, by using a network centrality measure of the impact of productivity changes in different sectors, we discuss how network centralities evolve with technological progress via changes in the supply chain.

To analyze the substitutability of labor and intermediate goods, we consider two different types of labor: high- and low-skilled. The difference between these types of labor is that low-skilled labor performs routine and repetitive tasks that can be substituted by "automation products" (e.g., robots, software, driverless trucks, drones...), while high-skilled labor instead complements all other goods used in the production process. Dividing labor into these two classes[2] allows us to characterize how different types of labor are affected by technological changes, and also to identify new types of network effects in how change ripples through the economy.

After we define our general input-output model, we provide a result on the existence and the uniqueness of equilibrium. Then, we move to the analysis of a three-sector model that consists of a final good sector; an intermediate good sector that is a complement to labor in production of the final good - a "resource sector"; and an intermediate good sector that can substitute for labor in the production of the final good - an "automation sector". In the three-sector model part, we provide the main results of our study, which we later extend and generalize to an n-sector economy. We first study the implications of small (infinitesimal) technological changes and, later compare them with the implications of non-small (discrete) technological changes that involve the full reallocation effects due to automation. Our first two results fully characterize how small improvements in productivities affect total consumption, low- and high-skilled labor wages, and the relative wage (income inequality); as well as how these depend on the extent to which substitutable labor is still partly used or whether that substitution is already complete.

During the substitution (or transition) phase, low-skilled labor is displaced by automation. In this phase, the demand for low-skilled labor, and hence low-skilled wages, decrease as the productivity of the automation sector rises. At the same time, the demand for high-skilled labor can even rise, if automation sector needs high-skilled labor in production. As a result, the high-skilled to low-skilled wage ratio/gap rises following a technology improvement in the automation sector during the substitution phase. The substitution phase is not abrupt, but can be prolonged since the wage adjusts and so the adoption of automation is continuous in the change in its productivity, and this adjustment depends on the alternative uses of labor in different sectors. The impact of improvements in productivity of the automa-

[2]The term "high" and "low" skilled are artificial, as what is really relevant is whether a particular form of labor is substituted for or complemented by a change in some good. We use the terms since it is frequently the case that this corresponds to skill level.

tion sector on total consumption gradually increases as more labor is replaced, since then there is less labor left to be displaced and there is less attenuation of productivity growth due to wage adjustments.

Eventually, the substitution phase is complete and low-skilled labor is no longer used in this particular production process. Further technological improvements then have a classical input-output effect (i.e., wages and consumption rise) and have no impact on the relative wage.

Following these results, we discuss how the new employment of displaced workers depends on the labor productivities in each sector, which thus affect the welfare gains from automation. In a general equilibrium setting, we show that welfare gains from automation and how long the "transition to automation" takes depend on labor's alternative uses. With less attractive alternatives for low-skilled labor, the substitution phase is more gradual and greater technological advances are required to produce the same impact on the economy. Accordingly, we show how the implications of small vs non-small changes in productivities differ due to changes in labor reallocation. The results provide insight into what makes recent the implications of recent automation technologies different from previous ones, as alternative uses of low-skilled labor are lower than with previous displacements.

Following the three-sector model, we extend our analysis to an n-sector economy. Here, we consider arbitrary substitutable and non-substitutable tasks in each sector. In the general model, substitution can be quite indirect. For example, a technological advance in the production of a material like Kevlar can replace metal, which then makes robots lighter and more efficient, and thus spurs their use in warehouses. So, any good in a long supply chain can end up affecting the substitution. In this part, we investigate how these direct and indirect effects end up having overall effects on total consumption, income inequality and the interconnectedness of the economy.

In the analysis of the general model, we first discuss the indirect effects of how improvements in the productivity of some good in the supply chain of automation can have cascading effects, as well as automation that involves multiple technologies. Next, we discuss how Hulten's Theorem [47] relates to our setup. Hulten's Theorem states that the impact of a small technological change in a given sector on net-output is summarized by

$$\frac{\partial logC}{\partial logA_i} = \frac{p_i Y_i}{GDP},$$

where $\frac{p_i Y_i}{GDP}$ is known as the Domar [33] weight of sector i: the ratio of total sales of sector i to GDP. Our first result in this part shows that a modified version of Hulten's Theorem extends to our setting, and the impact of a shock to any sector, including the automation sectors, on GDP is summarized by its Domar weight. The key difference is that the Domar weights in our model depend on which of the phases the economy is in. During the substitution phase, the Domar weights change. Given that Hulten's Theorem provides a first-order approximation [3], it becomes inaccurate for non-infinitesimal changes since relative wages change

[3] See Baqaee and Farhi [18] for the importance of second order effects in a different model.

and thus so do the Domar weights as labor is being replaced. Our last result characterizes changes in the network influences of sectors due to automation. We show that an increase in level of automation increases the network influence of automation goods as well as both their direct and indirect suppliers, but that sectors that are not in the (upstream) supply chain of automation sectors do not have increased network influences. As a result, as productivities of the producers of automation products and/or their direct or indirect suppliers rise, the production network becomes denser and the weighted interconnectedness between sectors gets stronger until the substitution of labor is complete. These changes affect the absolute and relative impacts of future technological changes. For instance, a technological advancement in an automation good that increase its usage enhances the impact of the future technological changes in the same good; but can reduce the impact of technological advances in other goods due to the wage drop it causes, which in turn mitigate automation in other tasks.

Lastly, we discuss the overall impact of non-small technological changes while capturing the reallocation of labor in the n-sector economy, which provides insights into general equilibrium effects in the general n-sector model. The network effects with substitution differ from those identified by classical input-output studies that involve only complements (e.g., Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi [3]). Our results imply that the propagation of supply-side shocks is not limited to downstream (direct and indirect customer) sectors. Technological changes affect also the *relative* demand for different types of labor, and, therefore, ultimately affect relative wages, which changes the allocation of different types of labor even for the sectors completely separate from the supply chains experiencing technological changes.

1.1 Relation to the Literature

Understanding how technological changes can ripple through an economy is more important than ever, and has been an area of renewed research.[4] This has been studied both theoretically and empirically. For example, Carvalho, Nirei, Saito, and Tahbaz-Salehi [30] show how the supply chain disruptions in Japan after the Great East Japan Earthquake of 2011 led to wider disruption, both downstream and upstream. Similarly, Barrot and Sauvagnat [20] and Acemoglu, Akcigit, and Kerr [1] show evidence of network-based propagation of idiosyncratic shocks. Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3] showed how id-

[4]Following Leontief [52], the early (e.g., Hulten [47], Long and Plosser [53], Basu [22], Dupor [33], Horvath [45, 46], Basu and Fernald [23], and Shea [57]) and recent literatures (e.g., [38], Carvalho and Gabaix [32], Jones [48, 49], Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3], Acemoglu, Ozdaglar and Tahbaz-Salehi [4], Boehm, Flaaen and Pandalai-Nayar [26], Atalay [11], Bartleme and Gorodnichenko [21], Bigio and La'O [25], Baqaee [16], Fadinger, Ghiglino and Teteryatnikova [35], Baqaee and Farhi [17, 18], Bernard, Dhyne, Magerman, Manova, and Moxnes [24]) on the macroeconomic consequences of interconnectedness have made it clear that the productivity changes in one part of an economy can ripple through the economy and have a wide impact, and that idiosyncratic shocks do not all cancel out, but some can be magnified via the network (e.g., Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3] and Acemoglu, Ozdaglar and Tahbaz-Salehi [4]).

iosyncratic shocks can actually become amplified through a production network.

Our advance, as mentioned above, is to extend an input-output analysis to include substitutes and general equilibrium effects, while remaining tractable. We show how there are countervailing wage-adjustment forces that slow the impact of technological advances on the economy. We also show how these depend on the alternative uses for labor, while also showing how a version of Hulten's Theorem and expressions for the Domar weights extend to the substitute setting, but now vary with the level of automation.

In addition, we provide expressions for how inequality grows in response to technological changes, as well as how growth is attenuated by wage adjustments. The same technological change can have very different impacts depending on the rest of the economy and production network.

The role of complements versus substitutes in production has been discussed in important studies from Griliches [42] and Stokey [58] to Krusell, Ohanian, Rios Rull, and Violante [51], Autor and Dorn [12], and Hémous and Olsen [44]. For example, Krusell et al. [51] show how this difference can help explain the growth in income inequality and the growing skill premium observed over past decades.

Recently, Acemoglu and Restrepo [5, 6, 7] have studied the implications of automation[5] empirically and theoretically. The main difference between their modeling of automation and ours lies in the source of automation. Acemoglu and Restrepo analyze an increase in automation by considering an increase in the set of different production processes in which capital substitutes for labor. In our model, we model explicit goods (e.g., robots, software, etc.) substituting for labor rather than general capital, which is important in our study of supply chain effects. Also, in our model the automation occurs because those goods become cheaper to produce, due to some technological advance, rather than having some production process simply become automatable by capital. Therefore, differently from Acemoglu and Restrepo [5, 6, 7], we study how technological advances translate into automation in tasks that were previously performed by labor (e.g., routine tasks, repetitive tasks). Most importantly, this modeling difference leads our results to be different, and complementary to those from Acemoglu and Restrepo [5, 6, 7], allowing us to answer questions not addressable with their formulation. In particular, we detail the automation phase and how wages, productivity, and welfare change during this phase; as well as how long the phase takes and how it depends on the broader production processes in the rest of the economy. None of these questions can be answered in the other models. In addition, our analysis provides a new and more general form of input-output analysis, and new insight into how Domar weights

[5]Beyond these papers, there is also a broader literature on the impact of automation that includes Zeira [59], Autor and Salomons [15], Graetz and Michaels [40], Brynjolfsson and McAfee [29], Brynjolfsson, Rock, and Syverson [28], Michaels, Natraj, and Van Reenen [43], Frank et al. [36], Peri and Sparber [56], Katz and Murphy [50], Goos, Manning, Salomons [39], Alabdulkareem et al. [9], Autor, Katz and Krueger [14], Aghion, Jones, and Jones [8]. In a recent study, Moll, Rachel, and Restrepo [54] analyze the impacts of automation on income inequality while considering the distributional effects of automation via changing returns to wealth.

change with automation.

Our model also helps shed light on the Solow Paradox and slow growth in response to technological advances that seemingly should have a large effect on the economy. For instance Brynjolfsson, Rock, and Syverson [27] detail the modern version of this paradox, and examine four explanations. A leading one that they emphasize is that it takes time to develop complementary technologies that can take advantage of new advances and inventions. Our model provides a fifth explanation that differs from the four they offer. Ours is that many recent technological advances substitute in some way for labor or other inputs, and the wage/price adjustments due to the general equilibrium effects attenuates the impact of a technological advance.

In our model, we use a Cobb-Douglas production technology, with the twist that some tasks involve the possibility of substituting one good (e.g., technology) for another (e.g., low-skilled labor). This sort of perfect substitutability of machines (or automation) for low-skilled labor and the unit elasticity of substitution between these two and another type of labor is used by Autor, Levy, and Murnane [13], who provide a detailed discussion of this specific modeling choice by giving examples on the characteristics of tasks (i.e. routine vs non-routine tasks). In a setup capturing different elasticities of substitution between factors, Krusell et al. [51] find that the key elasticity of substitution between low-skilled workers and capital is higher than the elasticity of substitution between high-skilled workers and capital.

In addition to the papers already mentioned above, our paper is also related to Baqaee and Farhi [17]. Both papers focus on the impacts of technological changes that can be divided into two main categories: a pure technology effect and a reallocative effect. However, our model can be thought of as a generalization that provides explanations for how given technological changes have different implications depending on the primitives of the economy. In other words, we investigate the factors that places one type of technological change into one category, and another technological change into another category. For instance, our model provides explanations for how the technological improvements that result in exactly the same productivity change in a resource good instead of the automation sector might have different implications. In addition, our model allows us to characterize the changes in the Domar weights as well as the reallocation effects. A more technical difference is that, differently from Baqaee and Farhi [17], (and also Baqaee [16], Grassi [41], and Bigio and La'O [25]), we focus on competitive rather than imperfectly competitive equilibrium. [6]

Lastly, our study is also loosely related to the literature on the endogenous formation of production networks, such as Acemoglu and Azar [2], Carvalho and Voigtländer [31], and Oberfield [55]. In our model, firms do not choose their set of suppliers, yet the production network changes following technological advances, and so there is a form of endogenous network. For instance, as the productivity of an automation sector rises, it can start to supply goods to other sectors, forming new links and increasing the interconnectedness of

[6]One could extend our analysis to an imperfectly competitive model or with other sorts of production functions, but both topics are beyond the scope of the present paper.

the economy.

The remainder of the paper is organized as follows. In Section 2, we provide a preview of labor-substitution in a general equilibrium framework. In Section 3, we introduce our production network model and discuss the network interactions. In Section 4, we study the impact of technological changes on aggregate welfare and income inequality, while incorporating the reallocation of labor into our analyses. In Section 5, we extend our analysis to an n-sector model and characterize the implications of technological change. In Section 6, we conclude.

2 A Preview of Labor-Substitution in a General Equilibrium Framework

Let us briefly provide some intuition behind our results concerning labor-substitution before more fully describing our equilibrium analysis.

Consider the production of some good, Y, that uses high-skilled labor, H, low-skilled labor, L, and an input good that can substitute for low-skilled labor, X. Suppose that the production function takes the form

$$Y = (L + AX)^{\alpha} H^{1-\alpha}, \tag{1}$$

where $\alpha \in (0, 1)$ tracks the relative shares of the high and low-skilled inputs. The good X (e.g., a robot or AI) substitutes for low-skilled labor at a rate A. A change in A reflects a technological advance in the input X that makes it a better substitute for labor, for instance a faster robot, or enhanced abilities of some software. Let us examine the effect of a change in A on the value of production Y:[7]

$$\frac{\partial Y}{\partial A} = \alpha X (L + AX)^{\alpha-1} H^{1-\alpha} = \alpha X \frac{Y}{L + AX}.$$

This expression is decreasing in L and increasing in X. Early in the substitution/automation process, L is high and X is low, and so the impact of an advance in A is low. As substitution takes place, L falls and X increases, and thus the derivative increases as well. This shows the basic force at work: the impact changes depending on the stage of automation and the values of L and X. As substitution is just beginning $X = 0$ and $L > 0$, and so the deriviative is 0 and there is no effect. As substitution continues, L drops and X increases, and so does $\frac{\partial Y}{\partial A}$ (as a function of the level of Y, which is also increasing). Eventually, when substitution is complete and $L = 0$, then $\frac{\partial Y}{\partial A} = \alpha \frac{Y}{A}$, which then looks like a standard complementary input.

The impact of a change in A, thus depends on the levels of L and X. The second key insight is that the levels of L and X depend on how productive L is in other sectors of the

[7] L, H, and X adjust with A, but by the Envelope Theorem, their effect washes out of the derivative.

economy. This is where the general equilibrium analysis is vital. If L is relatively productive elsewhere, then as A increases, L decreases rapidly in the production of this good and moves to the production other goods, and correspondingly the use of good X rises rapidly.[8] If instead, L is not so productive elsewhere, then the main change is a drop in the wage and only a slight decrease in L and increase in X. Thus, the changes of L and X in response to a technological advance depend critically on the overall production network, and therefore so does the relative level of the derivative $\frac{\partial Y}{\partial A}$.

Although how Y is impacted by a change in the productivity A of the X input has general equilibrium effects also for complements, there is a big difference in how this works with substitutes. If X is a complement to L, then when L has worse alternative uses in the economy, the derivative of Y with respect to A increases as it becomes easier to attract L in the current production process. In contrast, in the case of substitutes, as L is less productive elsewhere, this slows the movement of L to its alternative uses and hence the adoption of X and which decreases the derivative - at any given level of Y - producing a counter-acting force. So, the general equilibrium effects of the alternative uses of labor have opposite signs for complements and substitutes.

3 The Model

3.1 Production Processes

We consider a perfectly competitive economy consisting of a set of $N = \{1, ..., n\}$ sectors/firms, with a representative firm denoted by i.

We use the terms 'firm' and 'sector' interchangeably.[9]

We focus on the interactions of labor with other goods in production processes, but with a simple change of notation one could also allow these to involve substitution effects for capital.

In particular, a firm uses labor in two forms: high-skilled and low-skilled. We denote the amount of high and low skilled labor used by firm i by H_i and L_i, respectively. The difference is that low-skilled labor can be substituted for by the goods produced in automation sectors (i.e. robots, software, etc), while high-skilled labor does not have a direct substitute. One can simply think of defining high and low skills in this way - the words "high" and "low" have no other particular meaning in our model. We use the terminology since they often correspond to higher and lower skills in the data - as new technologies tend to enhance high skilled labor while replacing more routine tasks that are associated with lower skills. For instance, high skilled labor might include management, R&D, and some engineering,

[8] The effect is *not* discontinuous, as one might superficially expect given the linear substitution of X for L in equation (1), since the low-skilled wage drops as L shifts into sectors where it was initially marginally less productive. As we show below, that shift is continuous.

[9] We also abstract away from the use of capital in our analysis. It can be added but is of no particular consequence in our model.

while low-skilled labor would include warehouse workers, drivers, manufacturing line workers, various secretarial workers, customer service workers, and so forth. Moreover, there are tasks which are performed by low-skilled labor in each firm, but do not have a direct substitute.

We thus think of different inputs having different roles in the production process. In particular, some tasks that low-skilled labor perform can be replaced by some input good - e.g., a box packer can be replaced by a robot. While there are other input goods, such as the boxes, that are used in the production process but do not substitute for labor. We thus divide the inputs in the production by firm i by whether they can substitute for some low skilled labor, or whether they do not:

- $j \in a_i$: "automation" inputs, which can substitute for low-skilled labor in some tasks (e.g., software, industrial robots),
- $j \in n_i$: "non-automation" inputs, the goods from another sector that do not replace labor (e.g., electricity, raw materials).

The sets a_i and n_i are sector specific.

We let Y_i be the total production of each $i \in N$, and A_i^P be a productivity multiplier. We let X_{ij} denote the amount of input from $j \in a_i \cup n_i$ that i uses in production. We let L_{i0} denote the amount of low-skilled labor that firm i employs outside of automatable tasks, and L_{ij} denote the amount of low-skilled labor that can be replaced by the automation input j. In addition, A_j^Q represents the productivity (or quality) of good j. The production function of each (representative) firm $i \in N$ has the form:

$$Y_i = A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} [L_{ij} + A_j^Q X_{ij}]^{\alpha_{ij}^L} \right] \prod_{j \in n_i} (A_j^Q X_{ij})^{\alpha_{ij}^n}. \tag{2}$$

Production exhibits constant returns to scale and we take the exponents to be non-negative and to sum to 1:

$$\alpha_i^H + \alpha_{i0}^L + \left(\sum_{j \in a_i} \alpha_{ij}^L \right) + \left(\sum_{j \in n_i} \alpha_{ij}^n \right) = 1. \tag{3}$$

We further assume that there is always some use of low and/or high skilled labor in each sector. Thus, $\left(\sum_{j \in a_i} \alpha_{ij}^L \right) + \left(\sum_{j \in n_i} \alpha_{ij}^n \right) < 1$ holds, which implies that $\alpha_{i0}^L > 0$ and/or $\alpha_i^H > 0$ holds $\forall i \in N$.[10]

3.2 Labor Supply

Each type of labor is supplied perfectly inelastically. The total available supply of low-skilled and high-skilled labor are constant and denoted by L and H, respectively. In our model, we

[10]This assumption implies that all elements of the Leontief inverse of a given input-output matrix are non-negative.

abstract away from labor market dynamics such as changes in labor supply L and H via skill training in reaction to automation, or labor movement across tasks requiring different skill types. The analysis of such labor market reactions are left for future research.

3.3 Consumption

The good produced in any firm i is used as an intermediate good in other firms and/or for consumption by households.

Letting C_i denote the amount of production of firm i used for consumption, the total production of firm i satisfies:

$$Y_i = \sum_{j \in N} X_{ji} + C_i.$$

The consumption goods are evaluated by a utility function, or equivalently aggregated into a single final consumption good by an overall production function, that takes a Cobb-Douglas form:

$$C = \prod_{i \in N} \left(A_i^Q C_i \right)^{\beta_i},$$

where $\beta_i > 0$ for all $i \in N$ and $\sum_{i \in N} \beta_i = 1$.

C^L and C^H denote the consumption of the final good by low- and high-skilled labor, respectively. Thus, total consumption is given by:

$$C = C^L + C^H$$

3.4 Competitive Equilibrium

In a competitive equilibrium, the representative firm in each sector maximizes profit, and market clearing conditions hold for each good and each type of labor.

In particular:[11]

A *competitive equilibrium* is a set of prices $\{p_i\}_{i \in N}$, wages , w_L and w_H, and quantities $\left\{Y_i, H_i, \{L_{ij}\}_{j \in a_i \cup 0}, \{X_{ij}\}_{j \in N}, C_i^L, C_i^H, \right\}_{i \in N}$ such that

I. Firms maximize profits: For each $i \in N$, $\{L_{ij}\}_{j \in a_i \cup 0}, H_i, \{X_{ij}\}_{j \in N}$ solve

$$\max_{\{L_{ij}\}_{j \in a_i \cup 0}, H_i, \{X_{ij}\}_{j \in N}} \quad p_i \left(A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} \left(L_{ij} + A_j^Q X_{ij} \right)^{\alpha_{ij}^L} \right] \left[\prod_{j \in n_i} (A_j^Q X_{ij})^{\alpha_{ij}^n} \right] \right) - \left(\sum_{j \in a_i \cup 0} w_L L_{ij} + w_H H_i + \sum_{j \in N} p_j X_{ij} \right).$$

[11] With constant returns to scale, profit maximization in equilibrium implies that there are 0 profits, and so we do not specify who earns the profits, as those shares are irrelevant and would just add more notation.

II. $\{C_i^L\}$ (and similarly $\{C_i^H\}$) solve the utility maximization problem of the representative worker:

$$\max_{\{C_i^L\}:\sum_i p_i C_i^L \le w_L L} \prod_{i\in N} \left(A_i^Q C_i^L\right)^{\beta_i}.$$

III. Markets clear:

- goods: $Y_i = \sum_{j\in N} X_{ji} + C_i^L + C_i^H$,
- and labor markets: $L = \sum_{i\in N}\sum_{j\in a_i \cup 0} L_{ij}$ and $H = \sum_{i\in N} H_i$.

We remark that the utility maximization by the consumers (II) is exactly equivalent to having a representative firm in a perfectly competitive "final goods" market sell bundles of goods that solve

$$\max_{\{C_i\}} p_f \prod_{i\in N} \left(A_i^Q C_i\right)^{\beta_i} - \sum_{i\in N} p_i C_i,$$

and then having the low and high-skilled workers consume the bundled final good $C = \prod_{i\in N} \left(A_i^Q C_i\right)^{\beta_i}$ such that they exhaust their budgets: $p_f C^L = w_L L$ and $p_f C^H = w_H H$, and $C = C^L + C^H$.

This alternative formulation allows us to let the price of the final good C be the numeraire ($p_f = 1$), which enables us to highlight relative changes of the low-skilled labor wage, w_L, and high-skilled labor wage, w_H.

For Sections 4.3–5 we maintain the assumption that $\beta_i > 0$ for each i, while in the next section we allow for some 0's to simplify some examples.

3.5 The Equilibrium Level of Automation and the Input-Output Network

We define some notation that tracks the input-output network.

Let $t_{ij} \in [0, 1]$ denote the equilibrium share of expenditures on automation good $j \in a_i$ in sector $i \in N$, where

$$t_{ij} = \frac{p_j X_{ij}}{w_L L_{ij} + p_j X_{ij}}.$$

The equilibrium share of expenditures on labor in an automatable task $j \in a_i$ in sector $i \in N$ is then $1 - t_{ij}$. During the substitution phase, t_{ij} will vary from 0 up to 1.

We then define two different input-output matrices. One considers all of the possible structural relationships if automation were complete in the economy, and the other represents a current equilibrium (or actual) input-output network, as some automatable tasks might still have 0 automation at some point.

The "structural" (or most connected possible) $n \times n$ input-output network is denoted by

$$\Omega^S = \Omega^n + \Omega^L.$$

Ω^n summarizes the input-output linkages in the economy via the non-automatable tasks. The ij^{th} entry of the Ω^n is the weight of non-automatable task $j \in n_i$ in the production function of firm i, α_{ij}^n. Second, Ω^L summarizes the potential linkages via automatable tasks, where the ij^{th} entry of the Ω^L is α_{ij}^L.

The structural input-output network and the equilibrium input-output network might differ, depending on extent of automation. The equilibrium input-output network is denoted by

$$\Omega = \Omega^n + \Omega^a,$$

where Ω^a is determined by the equilibrium: $\Omega_{ij}^a = t_{ij}\alpha_{ij}^L$.

As a result, the structural input-output network is the extreme case network where all substitution is complete and only automation goods are used in each automatable task in the economy. On the other hand, the equilibrium level of interconnectedness is summarized by the Leontief inverse matrix:

$$(I - \Omega)^{-1} = (I - \Omega^n - \Omega^a)^{-1}.$$

The Leontief inverse matrix represents the dependencies across sectors at equilibrium. Broadly, if there exists a directed path between industry i and industry j at equilibrium, then the ij^{th} entry of the Leontief inverse matrix is positive, and it is zero otherwise. Thus, switching to automation in certain tasks increase the connectivity among industries and creates additional direct and indirect network effects.

In summary, the equilibrium input-output network has the following properties:

i) $\Omega_{ij} = t_{ij}\alpha_{ij}^L \in [0, \alpha_{ij}^L]$ for all ij s.t. $j \in a_i$.

ii) $\Omega_{ij} = \alpha_{ij}^n$ for all ij s.t. $j \in n_i$.

iii) $\Omega_{ij} = 0$ for all ij s.t. $j \in N \setminus a_i \cup n_i$.

3.6 Equilibrium Characterization

We note that there exists unique equilibrium of prices, $\{p_i\}_{i\in N}$, wages, w_L and w_H, and quantities, $\{C_i\}_{i\in N}$. The uniqueness of the wages, prices and consumption is non-trivial because of the substitutability of inputs in automatable tasks.

Theorem 1 *There exists unique equilibrium prices, $\{p_i\}_{i\in N}$, wages, w_L and w_H, and quantities, $\{C_i\}_{i\in N}$, and a generically unique equilibrium set of quantities $\{Y_i, \{L_{ij}\}_{j\in K\cup 0}, H_i, \{X_{ij}\}_{j\in N}\}_{i\in N}$.*

4 Technological Changes, Total Consumption, and Income Inequality in a Three-Sector Economy

We now study how technological changes affect automation decisions of firms, labor allocation, wages, income inequality, and total consumption in a three-sector economy.

The three sectors are resource sector, an automation sector, and a final good sector; denoted by n, a, and f, respectively. In particular, the good produced in the automation sector is a substitute for the low-skilled labor in final good production, while the non-automation (resource) good is not.

The Cobb-Douglas production functions are:

$$Y_a = A_a^P (L_{a0})^{\alpha_{a0}^L} (H_a)^{\alpha_a^H} \tag{4}$$

$$Y_n = A_n^P (L_{n0})^{\alpha_{n0}^L} (H_n)^{\alpha_n^H} \tag{5}$$

$$Y_f = A_f^P (L_{f0})^{\alpha_{f0}^L} (H_f)^{\alpha_f^H} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n}. \tag{6}$$

In this three-sector setting, we simplify things and set $\beta_n = \beta_a = 0$ while $\beta_f = 1$, and so the only good that is directly consumed is the "final good". We also normalize $A_f^Q = 1$, and hence $Y_f = C = C^L + C^H$.

4.1 Technological Changes and Total Consumption

First, we analyze the implications of technological improvements on total consumption, contrasting the impact of improvements in the non-automation sector with the automation sector. We start with Example 1.

Example 1 *We set the productivity of the final good sector to $A_f^P = 1$, and weight of tasks in each sector to $\frac{1}{2}$. We also set $L = H = 1$. The production functions are as follows:*
$Y_n = A_n^P L_n^{0.5} H_n^{0.5}$,
$Y_a = A_a^P L_a^{0.5} H_a^{0.5}$,
$Y_f = (L_f + A_a^Q X_{fa})^{0.5} \left(A_n^Q X_{fn}\right)^{0.5}$.

The first thing that we examine is how automation progresses as a function of the productivity of sector a.

Figure 1 summarizes the transition to automation in sector f and depicts how the use of the automation input, denoted by t_{fa}, changes as the automation sector's productivity improves. In this case, what matters in terms of the productivity of sector a is the product of the two parameters: $A_a^P A_a^Q$.

As shown in Figure 1, for sufficiently small productivity values of the automation sector ($A_a^P A_a^Q$), there is no automation in sector f and sector a produces zero output at equilibrium. Once the productivity of the automation sector reaches a sufficiently high level ($A_a^P A_a^Q \geq A^*$),

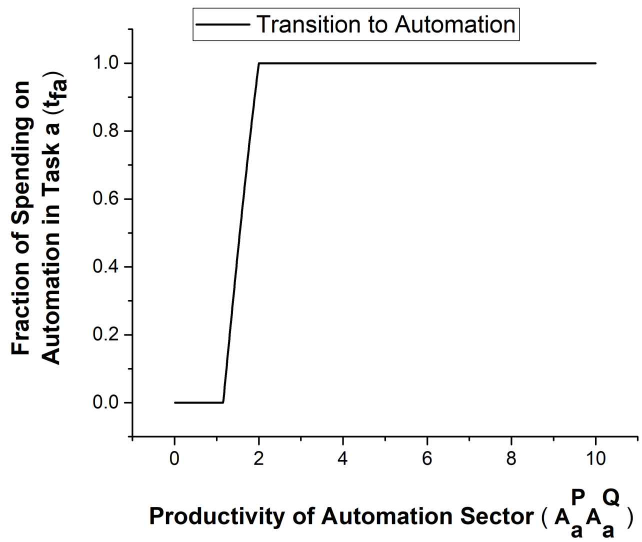

Figure 1: Technological change and the level of automation in the final good sector in Example 1

sector f starts to use automation good and firm a starts to produce positive amounts of output.

Given the Cobb-Douglas production function with constant returns to scale and zero profit conditions, the final good producer's total spending for task a at equilibrium is always equal to $\alpha_{fa}^L Y_f$. Accordingly, for the intermediate levels of productivity of automation sector (for $A^* < A_a^P A_a^Q < A^{**}$), sector f spends a fraction of $0 < t_{fa} < 1$ of its total cost for task a on automation and $(1 - t_{fa})$ fraction of its total spending for task a on low-skilled labor. In this intermediate range of productivity, as the automation sector becomes more productive, sector f's demand for the automation good rises and its demand for low-skilled labor falls. This is partly offset by a falling low-skilled wage, and a rising high-skilled wage, which makes this transition continuous. As the productivity increases further and reaches $A_a^P A_a^Q \geq A^{**}$, sector f eventually is fully automated. Note that although the fraction of automation expenditures in the final good sector is increasing gradually in response to improvements in productivity of automation sector and replacing some low skilled labor expenses, the fraction of expenses on the resource good is constant at α_{fn}^n.

Next, we examine the impact of changes in productivity in Example 1 on overall production/consumption, where we hold the automation sector and resource sector to have identical skill dependencies ($\alpha_{a0}^L = \alpha_{n0}^L$ and $\alpha_a^H = \alpha_n^H$). Figure 2[12] illustrates the following:

- in the pre-automation phase, a technological change in automation sector has no impact on total consumption, whereas a technological change in the non-automation good

[12]In Figure 2 panel b, we consider three different levels of productivity of the automation sector that represent the three phases. We keep the same values for the resource sector in Figure 2 panel a.

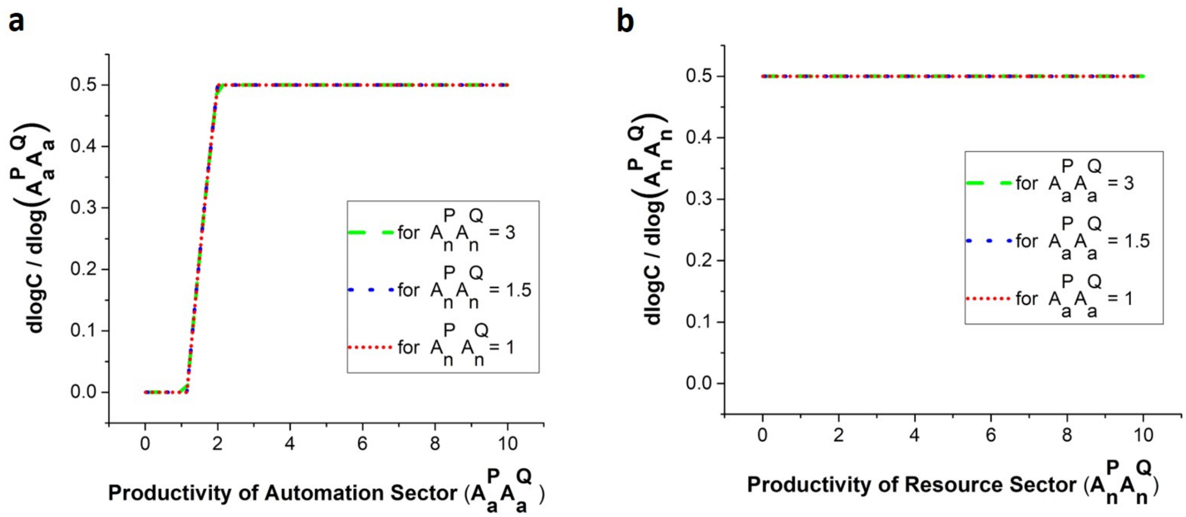

Figure 2: The changes in total consumption in response to technological changes in automation sector and resource sector in Example 1.

sector increases total consumption,

- during automation, a technological change in automation sector has an increasing impact on total consumption, but a smaller impact than that of a technological change in the non-automation good sector,
- in the post-automation phase, technological changes in either sector leads to the same increase in total consumption.

Example 1 illustrates that exactly the same changes in productivities in different sectors have different implications on total consumption, depending on whether the goods are complements or substitutes for labor, and how much labor is being used in production.

We now describe how this extends to the more general three-sector model, beyond the specific parameters of Example 1.

PROPOSITION 1 *In a three-sector economy, there exist two threshold levels of productivity of automation sector A^* and A^{**} such that there is no automation in final good sector if $A_a^P A_a^Q \leq A^*$; the level of automation gradually increases in between A^* and A^{**}; and the automation replaces all low-skilled labor employed in the automatable task in final good sector if $A_a^P A_a^Q \geq A^{**}$. The impacts of small technological changes on total consumption during the pre-automation, automation, and post-automation phases are approximately:*

$$\mathrm{dlog} C =: \begin{cases} \Gamma & \text{if } A_a^P A_a^Q < A^* \\ \Gamma + \left(\dfrac{\frac{H}{L}\left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H \left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)} - \dfrac{(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{\alpha_a^H} \right) \mathrm{dlog}(A_a^P A_a^Q) & \text{if } A^* < A_a^P A_a^Q < A^{**} \\ \Gamma + \alpha_{fa}^L \mathrm{dlog}(A_a^P A_a^Q) & \text{if } A_a^P A_a^Q > A^{**} \end{cases}$$

where $\Gamma = \text{dlog}(A_f^P) + \alpha_{fn}^n \text{dlog}(A_n^P A_n^Q)$.

Proposition 1 shows that the macroeconomic impact of technological changes depends on both the sources of the technological changes and the phase of the economy. During the automation phase, the change in total consumption in response to technological changes in automation sector is a function of the labor supply for each type of worker, initial productivity level in the automation sector, and the weights of low- and high- skilled labor (in all sectors). In contrast, the change in total consumption in response to technological changes in the resource sector is constant and equal to its weight in final good production (α_{fn}^n).

4.2 Technological Changes, Wage Adjustments, and Inequality

Next, we analyze how technological improvements lead to general equilibrium wage adjustments that provide for a continuous and prolonged transition despite the linear substitution specification; and also increase wage inequality along the way.

We start with an example with just two sectors to make things transparent: so we are dropping n for now, so that the final good sector uses only the automation good as an intermediate input.

Example **2** $\alpha_{a0}^L = \alpha_a^H = \alpha_{fa}^L = \alpha_f^H = 0.5$, *and* $A_f^P = L = H = 1$, *and the production functions are as follows:*

$Y_a = A_a^P L_a^{0.5} H_a^{0.5}$

$Y_f = H_f^{0.5}(L_{fa} + A_a^Q X_{fa})^{0.5}$.

As we discussed previously, there are essentially two key phases of automation (beyond a degenerate one where the automation good is so inefficient not to be used in the automatable task). The first key phase is when automation takes place and the final good producer uses both the automation input and low-skilled labor in combination. As this phase progresses, the demand for low-skilled labor decreases and the productivity gains that arise due to automation are captured by high-skilled labor. Eventually, the economy is fully automated, and then any technological change has only the classical input-output effect – all wages and consumption rise – and there is no impact on relative wages.

Figure 3 panel a summarizes the transition to automation in sector f. Figure 3 panels b,c,d depict how absolute and relative wages change in Example 2 as we change the productivity in the automation sector.

Table 1 shows the threshold levels of productivities and the wages in Example 2.

Phases	w_L	w_H	$\frac{w_H}{w_L}$
$A_a^P A_a^Q \leq 2$ (pre-automation)	$\frac{1}{2}$	$\frac{1}{2}$	1
$2 < A_a^P A_a^Q < 2\sqrt{3}$ (automation phase)	$\frac{1}{A_a^P A_a^Q}$	$\frac{A_a^P A_a^Q}{4}$	$\frac{\left(A_a^P A_a^Q\right)^2}{4}$
$A_a^P A_a^Q \geq 2\sqrt{3}$ (post-automation phase)	1	3	3

Table 1. The changes in automation, wages and income inequality in Example 2

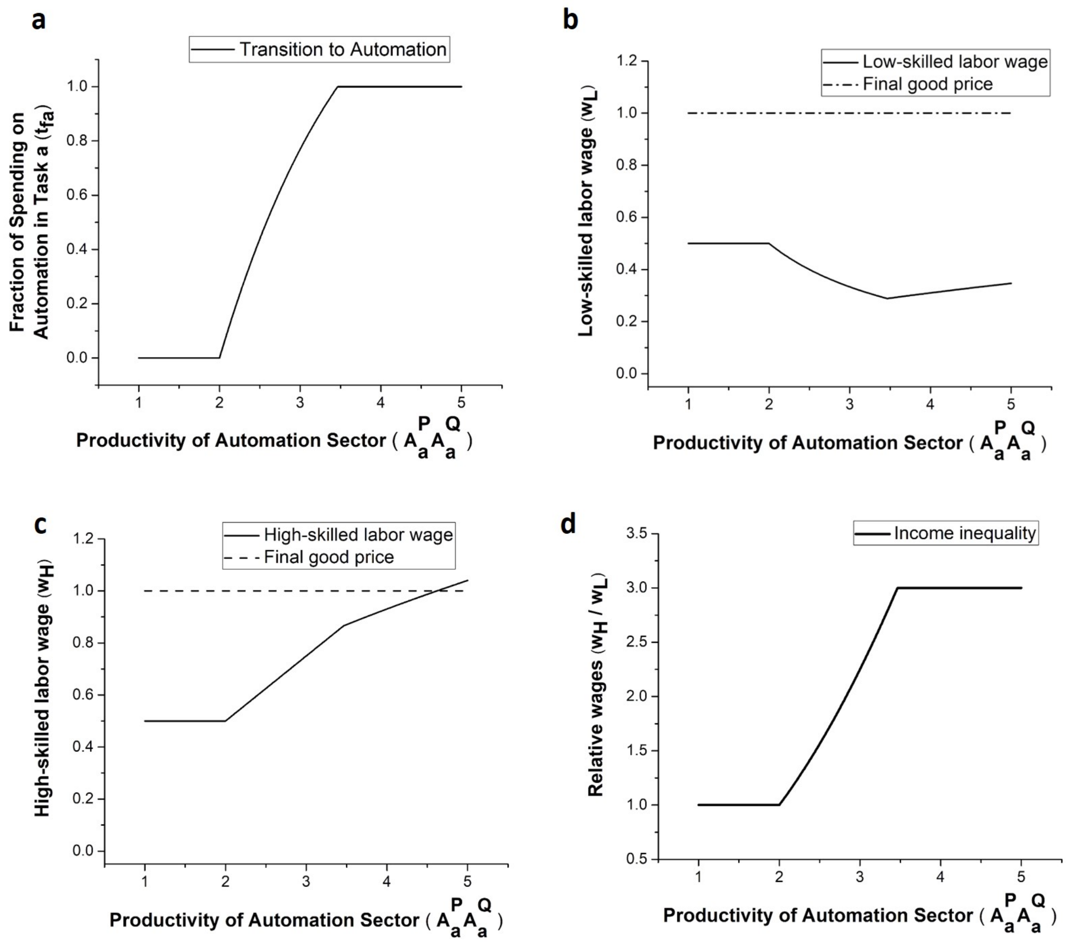

Figure 3: Automation in final good sector and its impact on wages in Example 2

Next, Proposition 2 shows how small improvements in productivity of an automation process affect wages and inequality, that formalize the numerical example above in the more general case.

Proposition **2** *In a three-sector economy, the impacts of small (infinitesimal) technological changes on wages and income inequality (or relative wage) are:*

- *(pre-automation) for* $A_a^P A_a^Q < A^*$*, low-skilled labor wage and high-skilled labor wage change at the same rate, and hence the income inequality* ($\frac{w_H}{w_L}$) *remains constant,*
- *(transition to automation) for* $A^* < A_a^P A_a^Q < A^{**}$*, high-skilled labor wage rises at a higher rate than the low-skilled labor wage, and hence the income inequality increases,*
- *(post-automation) for* $A_a^P A_a^Q > A^{**}$*, low-skilled labor wage and high-skilled labor wage change at the same rate, and hence the income inequality remains constant.*

In particular:

$$\mathrm{dlog} w_L =: \begin{cases} \Gamma & \textit{if } A_a^P A_a^Q < A^* \\ \Gamma - \left(\frac{\alpha_f^H + \alpha_{fn}^n \alpha_n^H}{\alpha_a^H}\right) \mathrm{dlog}\left(A_a^P A_a^Q\right) & \textit{if } A^* < A_a^P A_a^Q < A^{**} \\ \Gamma + \alpha_{fa}^L \mathrm{dlog}\left(A_a^P A_a^Q\right) & \textit{if } A_a^P A_a^Q > A^{**} \end{cases}$$

$$\mathrm{dlog} w_H =: \begin{cases} \Gamma & \textit{if } A_a^P A_a^Q < A^* \\ \Gamma + \left(\frac{1 - \alpha_f^H - \alpha_{fn}^n \alpha_n^H}{\alpha_a^H}\right) \mathrm{dlog}\left(A_a^P A_a^Q\right) & \textit{if } A^* < A_a^P A_a^Q < A^{**} \\ \Gamma + \alpha_{fa}^L \mathrm{dlog}\left(A_a^P A_a^Q\right) & \textit{if } A_a^P A_a^Q > A^{**} \end{cases}$$

$$\mathrm{dlog}\left(\frac{w_H}{w_L}\right) =: \begin{cases} 0 & \textit{if } A_a^P A_a^Q < A^* \\ \frac{\mathrm{dlog}\left(A_a^P A_a^Q\right)}{\alpha_a^H} & \textit{if } A^* < A_a^P A_a^Q < A^{**} \\ 0 & \textit{if } A_a^P A_a^Q > A^{**} \end{cases}$$

where $\Gamma = \mathrm{dlog} A_f^P + \alpha_{fn}^n \mathrm{dlog}\left(A_n^P A_n^Q\right)$.

First, as Proposition 2 shows, wage inequality is constant during the pre-automation and post-automation phases; which follows since technological change in these stages does not substitute for labor. In contrast, income inequality rises in the automation phase. In that phase the high-skilled labor wage increases, while the low-skilled labor wage might increase or decrease depending on whether the productivity effect or substitution effect dominates, and also on whether there are technological improvements in other sectors. Regardless, the low-skilled wage continues to fall behind the increase in the high-skilled wage. In response to small technological changes, a key parameter determining the change in wage gap is the high-skilled labor dependency of the automation sector. As shown in Proposition 2, for higher values of α_a^H, the growth in relative wages in response to small improvements in technology is lower. However, for higher values of α_a^H, the ultimate change in the wage gap (once automation is complete) is greater since the good that replaces low-skilled labor is more high-skilled intensive. This is discussed in more detail in Section 4.4. Second, in response to the same technological changes, the (constant) rate of change in wages within the productivity range $A_a^P A_a^Q > A^{**}$ is weakly higher than the (constant) rate of change in wages within the productivity range $A_a^P A_a^Q < A^*$. Thus, productivity gains from technological advances increase as the automation becomes more extensive. Lastly, the low-skilled dependency of automation sector and low-skilled dependency of other sectors have reversed effects on the low-skilled labor wage. For higher values of α_a^H (equivalently for lower values of α_{a0}^L) automation happens more slowly, and with a slower change in the wage gap, $\text{dlog}\left(\frac{w_H}{w_L}\right)$.

In Sections 4.3 and 4.4, we revisit Propositions 1 and 2 for non-small changes. As wages adjust in equilibrium, the overall effects on total output change, and so the derivatives are constantly adjusting. The large effects are still tractable, and we compare them to the local approximations.

4.3 Alternative Uses of Labor and the Duration of Automation

As we have seen, wages adjust as automation improves which attenuates the impact of technological improvements in automation. The extent to which that happens depends on how labor can be reallocated, which depends on its productivity elsewhere in the economy. We begin with Example 3, which illustrates one aspect of this.

Example **3** *Again, $L = H = 1$, and now production functions are:*

$$Y_n = A_n L_n^{\alpha_n^L} H_n^{1-\alpha_n^L}$$
$$Y_a = A_a$$
$$Y_f = A_f (L_f + X_{fa})^{\alpha_{fa}^L} X_{fn}^{1-\alpha_{fa}^L}$$

In this example the low-skilled labor used in the final good production is

$$\begin{cases} L_f = \frac{\alpha_{fa}^L - A_a(\alpha_n^L(1-\alpha_{fa}^L))}{\alpha_{fa}^L + \alpha_n^L(1-\alpha_{fa}^L)} & 0 < A_a < \frac{\alpha_{fa}^L}{\alpha_n^L(1-\alpha_{fa}^L)} \\ L_f = 0 & A_a \geq \frac{\alpha_{fa}^L}{\alpha_n^L(1-\alpha_{fa}^L)} \end{cases}$$

and the corresponding final good production is

$$\begin{cases} Y_f = A_f A_n^{1-\alpha_{fa}^L} \left(\frac{1+A_a}{\alpha_{fa}^L + \alpha_n^L(1-\alpha_{fa}^L)}\right)^{\alpha_{fa}^L + \alpha_n^L(1-\alpha_{fa}^L)} (\alpha_{fa}^L)^{\alpha_{fa}^L} [\alpha_n^L(1-\alpha_{fa}^L)]^{\alpha_n^L(1-\alpha_{fa}^L)} & \text{if } 0 < A_a < \frac{\alpha_{fa}^L}{\alpha_n^L(1-\alpha_{fa}^L)} \\ Y_f = (A_a)^{\alpha_{fa}^L} (A_n)^{1-\alpha_{fa}^L} & \text{if } A_a \geq \frac{\alpha_{fa}^L}{\alpha_n^L(1-\alpha_{fa}^L)}. \end{cases}$$

We can see from the expressions for the labor used in final good production, the threshold level of A_a is lower as α_n^L increases: the more useful low-skilled labor in the resource sector, the faster it is substituted for by automation. This then leads to a greater increase in final good production as well, as we see in the first expression for Y_f.

This is then illustrated in Figure 4 which shows how the low-skilled labor dependency of the resource sector, α_n^L, plays a key role in determining the change in total consumption in

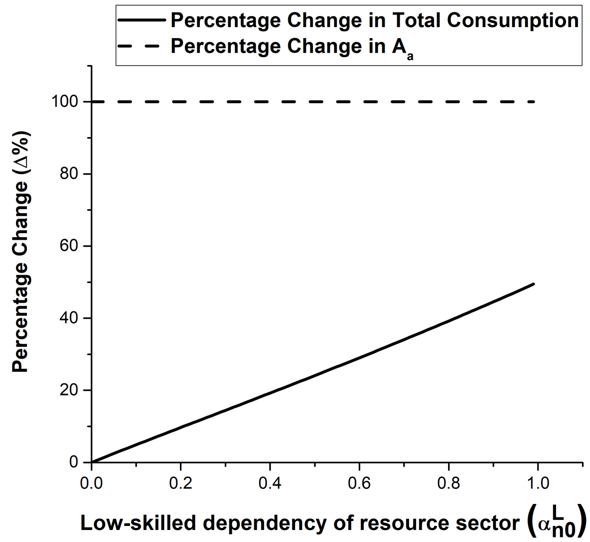

Figure 4: The change in total consumption in response to a change in productivity of the automation sector for different values of low-skilled labor dependency of the resource sector in Example 3

response to technological improvements in automation. As shown in Figure 4, the impact of a technological change in automation on total consumption is increasing in α_n^L, because labor becomes more productive in its alternative uses and the displacement is faster.

Another way to see the interaction between automation and the uses of labor elsewhere in the economy is to examine how the thresholds A^* and A^{**} that define when automation starts and stops displacing labor as a function of improvements in automation.

First, we revisit Example 1 and consider two different values for $\frac{L}{H}$. As shown in Figure 5, the threshold levels of productivity in the automation sector to start and stop displacing labor depend on the ratio of $\frac{L}{H}$ as well as how important low-skilled labor is in the resource sector. For instance, as $\frac{L}{H}$ increases, the threshold levels A^* and A^{**} both increase, so that automation only happens at much higher levels of productivity. For higher levels of $\frac{L}{H}$, there is much more low-skilled labor available and so it becomes relatively cheap and thus is harder to replace ($\frac{w_L}{p_a}$ is smaller and so $A_a^P A_a^Q$ needs to be larger to trigger sector f to switch to automation). This is depicted in Figure 5 panel b. A similar interpretation also holds for the skill dependencies in resource sector, as shown in Figure 5 panel a: in which the threshold levels A^* and A^{**} are decreasing in the low-skilled labor dependency of the resource sector, α_n^L. As α_n^L rises, the low-skilled labor wage rises and it is demanded more in the resource sector, and low-skilled labor is more easily displaced. Table 5 in the Appendix provides the threshold levels of technology for different levels of labor supply.

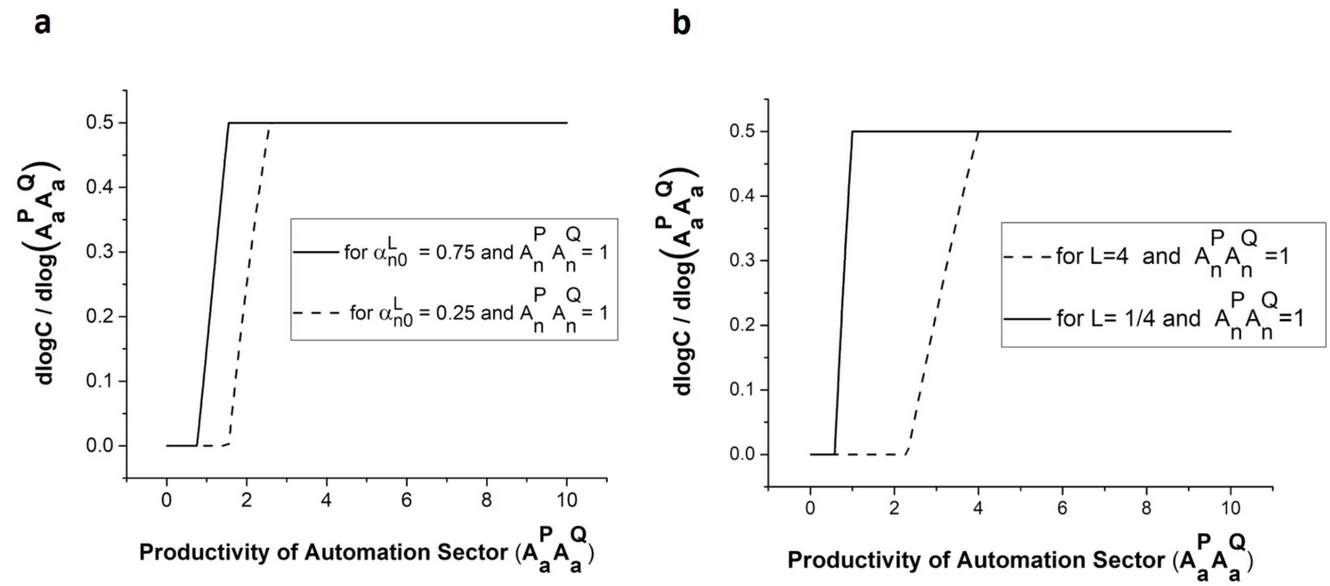

Figure 5: Transition to automation and changes in total consumption in response to technological changes for different levels of $\frac{L}{H}$ or skill dependencies in Example 1.

More generally, the threshold levels A^* and A^{**} are as follows:

$$A^* = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1 - (\alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{\alpha_a^H}$$

and

$$A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1 - (\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{\alpha_a^H}.$$

Proposition 3 provides the corresponding comparative statics for the general three-sector model.

Proposition 3

- *A^* and A^{**} are increasing in $\frac{L}{H}$,*
- *A^* and A^{**} are decreasing in α_n^L (and increasing in α_n^H),*
- *For constant α_{fn}^n and α_{fa}^L, A^* and A^{**} are decreasing in α_{f0}^L (and increasing in α_f^H),*
- *There exists an $(\alpha_a^L)' \in (0,1)$ such that A^* is decreasing in α_a^L for $(\alpha_a^L)' < \alpha_a^L < 1$ and A^* is increasing in α_a^L for $0 < \alpha_a^L < (\alpha_a^L)'$, and $\frac{A^{**}}{A^*}$ is decreasing in α_{a0}^L for any $0 \le \alpha_{a0}^L \le 1$.*

We already discussed the first two parts of Proposition 3. Lastly, as shown in Figure 6, the threshold level A^* has the maximum value at an interior level of α_a^L. The reason is

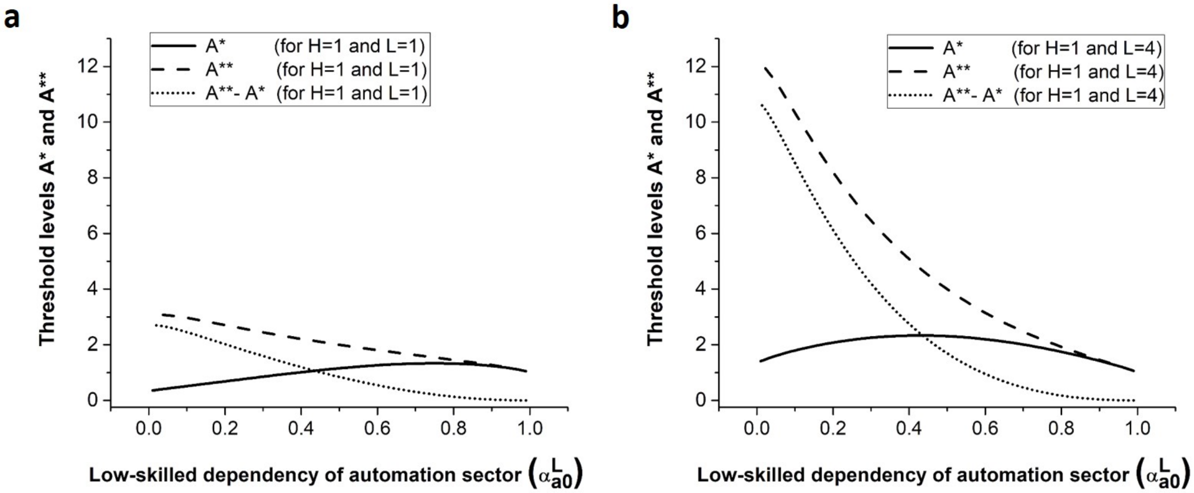

Figure 6: The impact of labor supply and skill dependencies in automation sector on the threshold technology levels in Example 1.

that the price of automation good is increasing in both wages and, thus, for given sectoral productivities, $\frac{w_L}{p_a}$ is minimized at an interior level of α_a^L. Figure 6 also shows how the interior level for α_a^L depends on the supply of each type of labor.

As a result, in addition to the input-output network structure, the labor supply and skill dependencies of each sector play key roles in the level of automation (t_{fa}), and the completeness of the automation phase, which altogether determine the macroeconomic impact of technological changes. In a given economy with Cobb-Douglas production functions, the productivity parameters of sectors that do not cause any substitution effect have no implications for the threshold levels of technologies for automation. The change in such productivity parameters translate into a similar effect for low- and high-skilled labor, which is the classical input-output effect.

From the expressions for A^* and A^{**}, it follows that

$$\frac{A^{**}}{A^*} = \left(\frac{\alpha_{fa}^L \alpha_a^H + \zeta}{1 - \zeta - \alpha_{fa}^L \alpha_a^H} \frac{1-\zeta}{\zeta} \right)^{\alpha_a^H}$$

where $\zeta = \alpha_{fn}^n \alpha_n^H + \alpha_f^H$. Then, holding ζ and α_{fa}^L fixed, it follows that:

$$\frac{d(\frac{A^{**}}{A^*})}{d\alpha_a^H} > 0.$$

Thus, $\frac{A^{**}}{A^*}$ is increasing in the high-skilled labor dependency of the automation sector. On the other hand, holding α_{fa}^L and α_a^H fixed, it follows that:

$$\frac{d(\frac{A^{**}}{A^*})}{d\zeta} \geq 0 \text{ iff } 2\zeta + \alpha_{fa}^L \alpha_a^H \geq 1.$$

Next, we study the implications of non-small improvements in productivity taking into account reallocation effects.

4.4 Macroeconomic Impacts of Technological Changes under Reallocation Effects

Non-small technological changes in our setting work differently from just small changes, because adjustments in wages and relative prices change decisions of firms as technology advances. This is important to understand, since it helps understand speed of growth and when and why growth is slower.

To see this, we compare the changes in total consumption in response to small versus large technological changes in automation sector, drawing on Propositions 1 and 2.

The expression for consumption change from Proposition 1, due to small changes in the productivity of the automation sector $\text{dlog}(A_a^P A_a^Q)$, can be written as :

$$\text{dlog}C = \left(\frac{1-s_L}{\alpha_a^H}\right)\text{dlog}(A_a^P A_a^Q) - \left(\frac{\alpha_{fn}^n \alpha_n^H + \alpha_f^H}{\alpha_a^H}\right)\text{dlog}(A_a^P A_a^Q)$$

where $s_L = \frac{w_L L}{w_L L + w_H H}$ is the income share of low-skilled labor.

The impact of non-small (discrete) changes in the productivity of automation sector from Proposition 4 can be written as:

$$\Delta\text{log}C = \Delta\text{log}\left(\frac{L}{s_L}\right) - \left(\frac{\alpha_{fn}^n \alpha_n^H + \alpha_f^H}{\alpha_a^H}\right)\Delta\text{log}(A_a^P A_a^Q) \tag{7}$$

where $\Delta\text{log}C = \text{log}C^{after} - \text{log}C^{before}$ is the difference between log consumption after and before the technological change. As one can see from the expressions above, the change in total consumption for non-small technological changes reflects the changes in wage shares and, so, the reallocation effects. In contrast, the expression for small-changes in technology does not capture reallocation effects. In Section 7.7 in the Appendix, we provide further analysis of $\Delta\text{log}C$ under various reallocation effects.

Next, we characterize the impacts of non-small technological changes in each phase of the economy on wages. Here, it turns out that, in contrast to the effect on consumption, the impacts of small and large changes are captured by the same multiplier.

Proposition 4 *Let* $\Gamma = \Delta\text{log}(A_f^P) + \alpha_{fn}^n \Delta\text{log}\left(A_n^P A_n^Q\right)$.

Then, in a three-sector economy, the change in log consumption and log wages in response to (non-small) sectoral technological changes are described by:

i) in pre-automation:

$$\Delta\text{log}w_L = \Delta\text{log}w_H = \Delta\text{log}C = \Gamma$$

ii) during automation :

$$\Delta \log C = \Gamma + \Delta \log \left(L \left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L} \right)^{\frac{-\alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} + H \left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L} \right)^{\frac{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} \right)$$

$$\Delta \log w_L = \Gamma - \left(\frac{\alpha_f^H + \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

$$\Delta \log w_H = \Gamma + \left(\frac{1 - \alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

iii) post-automation:

$$\Delta \log w_L = \Delta \log w_H = \Delta \log C = \Gamma + \alpha_{fa}^a \Delta \log (A_a^Q A_a^P).$$

As shown in Proposition 4, the net-effect of technological changes during the automation phase depend on skill dependencies of each sector, supply of both types of labor, and the level of technology in the automation sector, which becomes especially important whenever there is an alternative use of low-skilled labor in automation sector. Following Proposition 4, the expressions for the wages can be rewritten as follows:

$$\mathrm{d}\log w_L = \Gamma - \left(\frac{\alpha_f^H + \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \mathrm{d}\log \left(A_a^P A_a^Q \right), \text{ and } \Delta \log w_L = \Gamma - \left(\frac{\alpha_f^H + \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

$$\mathrm{d}\log w_H = \Gamma + \left(\frac{1 - \alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \mathrm{d}\log \left(A_a^P A_a^Q \right), \text{ and } \Delta \log w_H = \Gamma + \left(\frac{1 - \alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

Thus, the changes in log wages in response to technological changes are determined by the skill dependencies of sectors. The skill dependencies in each sector determine the alternative usage of labor and hence, the productivity of labor whenever the reallocation occurs.

In Corollary 1, we rewrite the equations for wage adjustments to see how they depend on initial levels of shares of different types of skilled workers.

COROLLARY **1** *During the automation phase:*

$$\Delta \log w_L = - \left(\frac{s_H^{pre}}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

$$\Delta \log w_H = \left(\frac{s_L^{pre}}{\alpha_a^H} \right) \Delta \log \left(A_a^P A_a^Q \right)$$

implying that

$$\Delta\log(\frac{w_H}{w_L}) = \left(\frac{1}{\alpha_a^H}\right)\Delta\log\left(A_a^P A_a^Q\right),$$

where $s_H^{pre} = \frac{w_H H}{w_L L + w_H H}$ *and* $s_L^{pre} = \frac{w_L L}{w_L L + w_H H}$ *are the share of high- and low-skilled labor before any automation occurs, respectively.*

Corollary 1 allows us to compare the implications of automation in two different economies E and E' having different labor share compositions initially. Consider an economy E that has a smaller low-skilled labor share than the economy E'. Then, low-skilled labor wage falls more in the economy E where low-skilled labor has initially relatively smaller share, under certain symmetry conditions. Similarly, high-skilled labor wage rises less in economies that initially has relatively smaller low-skilled labor share. Moreover, the change in income inequality is captured by a single parameter: high-skilled dependency of the automation sector (α_a^H).

It is important to note that the implications on income inequality here do not capture the *length of the transition phase*. A transition can be steeper, but more abrupt, and so the overall effect depends not only on the derivative of change, but how long that change lasts. For this reason, it actually turns out that the increase in income inequality due to the full transition phase is *higher* for higher α_a^H. In particular, by using the equation for $\frac{A^{**}}{A^*}$, we can write the change in income inequality as follows:

$$\frac{\left(\frac{w_H}{w_L}\right)'}{\frac{w_H}{w_L}} = \left(\frac{A^{**}}{A^*}\right)^{\frac{1}{\alpha_a^H}} \tag{8}$$

where $\frac{w_H}{w_L}$ and $\left(\frac{w_H}{w_L}\right)'$ are the ratio of high-skilled labor wage to low-skilled labor wage before any automation happens and after the automation (in full) happens, respectively.

More specifically, in Proposition 2, we showed that the change in income inequality in response to small changes in technology is decreasing in α_a^H. However, when we consider a technological change from pre automation to full automation, then Equation (8) implies that the change in income inequality is actually increasing in α_a^H. Therefore, focusing only on local changes is misleading.

Proposition 5 summarizes how technological changes that lead to a switch from zero automation to full automation affect income inequality.

Proposition 5 *Consider two vectors of sector level productivities and technological changes such that we start with* $(A_i^P A_i^Q)^{start}$ *and end with* $(A_i^P A_i^Q)^{end}$*. And, consider the set of economies for which* before the technological changes, at $(A_i^P A_i^Q)^{start}$, the final good sector only uses labor in the automatable task, and after the technological changes, at $(A_i^P A_i^Q)^{end}$, the automatable task in the final good sector uses no labor. *Call this technological change from* $(A_i^P A_i^Q)^{start}$ *to* $(A_i^P A_i^Q)^{end}$ *as* Δ^{full}*. Then, everything else held constant:*

- $\Delta^{full}\log(\frac{w_H}{w_L})$ *is decreasing in* $\frac{\alpha^L_{n0}}{\alpha^H_n}$ *and* $\frac{\alpha^L_{f0}}{\alpha^H_f}$ *if* $s^{pre}_H + s^{post}_H > 1$, *but increasing in* $\frac{\alpha^L_{n0}}{\alpha^H_n}$ *and* $\frac{\alpha^L_{f0}}{\alpha^H_f}$ *if* $s^{pre}_H + s^{post}_H < 1$,
- $\Delta^{full}\log(\frac{w_H}{w_L})$ is decreasing in $\frac{\alpha^L_{a0}}{\alpha^H_a}$;

and note that in terms of primitives that $s^{pre}_H + s^{post}_H = 2\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right) + \alpha^L_{fa}\alpha^H_a$.

Proposition 5 shows how the ultimate change in income inequality depends on the initial level of income inequality as well as labor's role in different production processes.

The results in this section shed some light on what makes automation in the last few decades different from previous ones. One difference is the differences in the production processes of labor-substitution technologies. AI technologies, software, and industrial robots rely on different production processes and skills than production of machines that replaced labor historically, and thus substitute differently. In addition, the alternative uses of labor are different now than when automation displaced labor from agriculture, and later from manufacturing.

5 Technological Changes and Automation in an n-Sector Economy

With most of the basic insights in hand from the analysis of the three-sector model, we now extend our analysis to a full n-sector economy.

One added feature is that now improvements in automation can be triggered by an improvement in any input into the production of an automation good and so supply chains play a nontrivial role. Another important added feature is that now wage effects impact all of the production processes, and can have further feedback into production decisions.

In this general version of the model, arbitrary combinations of automatable and non-automatable tasks are admitted in each sector and the production function of each sector is of the form:

$$Y_i = A^P_i (L_{i0})^{\alpha^L_{i0}} (H_i)^{\alpha^H_i} \left[\prod_{j\in a_i} [L_{ij} + A^Q_j X_{ij}]^{\alpha^L_{ij}}\right] \prod_{j\in n_i} (X_{ij})^{\alpha^n_{ij}}$$

In what follows, we focus on changes in the basic productivity of various goods A^P_i's, and simply normalize the quality parameters, $A^Q_i = 1$, for all $i \in N$. The analysis of specific changes in A^Q_i is an easy extension, and the normalization saves on notation.

5.1 Indirect Automation

An increase in automation might occur via direct and/or indirect network effects. For instance, a productivity increase in some material that is used in the production of industrial robots can lead to switches to usage of industrial robots in some sectors. To illustrate this point, we provide an example. Differently from our previous three-sector analysis, now the automation sector uses the resource good:

EXAMPLE **4** $A_a^P = A_f^P = L = H = 1$, $\beta_f = 1$ *and* $\beta_a = \beta_n = 0$, *and the production functions are as follows.*

$Y_n = A_n^P L_n$

$Y_a = H_a^{0.5}(X_{an})^{0.5}$

$Y_f = H_f^{0.5}(L_f + X_{fa})^{0.5}$

In this example, following technological improvement in sector n, product n gets cheaper, and hence product a becomes cheaper as well. This causes ripple effects to sector f, which starts to use product a. More specifically, for $A_n^P < 4$, sector f uses no automation good, for $4 \leq A_n^P \leq 12$, good a becomes as cheap as the low-skilled labor and sector f starts to automate task a, and for $A_n^P > 12$ sector f is fully automated in task a.

5.2 Multiple Automation

In addition to the indirect effects of technological advances, another interesting consideration is that automation can happen in multiple tasks in multiple sectors. For a given automation product, the threshold levels for automation depend on the wage levels, which depend on the automation levels in other tasks. More specifically, low-skilled labor becomes relatively cheaper as the level of automation rises for a given automatable task, which then implies that a higher technology is required for switching to automation in other tasks compared to the case where there is no automation initially in that given task. These general equilibrium effects help us to understand how future technological changes together with labor market reactions shape the automation decisions of firms.

We provide a simple example to illustrate this point.

EXAMPLE **5** $L = H = 1$ *and the production functions are:*

$Y_a = A_a$

$Y_b = A_b$

$Y_f = A_f H_f (L_{f0})^{\alpha_{f0}^L} (L_{fa} + X_{fa})^{\alpha_{fa}^L} (L_{fb} + X_{fb})^{\alpha_{fb}^L}$

In this example, the two different automation technologies a and b are both used in the production of the final good, and enter in similar ways. In fact, if we set $\alpha_{fa}^L = \alpha_{fb}^L$, then they are fully identical in the way that they enter into the production of the final good.

Now, consider a case where both technologies a and b begin being unproductive. Let us suppose that one of them innovates first, so that it makes an advance and gets taken up in the production of the final good, so for instance b innovates first. Does that make it easier or harder for advances in a to become adopted?

To answer this, let us begin by setting $A_a = 0$ and finding the threshold levels for A_b for automation in task b. In this case, the threshold level for *completing* the automation in task b is given by

$$A_b^{**} = \frac{\alpha_{fb}^L}{\alpha_{f0}^L + \alpha_{fa}^L}.$$

Now, let us suppose that the innovation in b is sufficient so that $A_b \geq \frac{\alpha_{fb}^L}{\alpha_{f0}^L+\alpha_{fa}^L}$. Now let us consider the automation in task a. The threshold level for completing the automation in task a is then given by

$$A_a^{**} = \frac{\alpha_{fa}^L}{\alpha_{f0}^L}.$$

For $\alpha_{fa}^L = \alpha_{fb}^L$, it then follows that

$$A_a^{**} > A_b^{**}.$$

This example shows how earlier automation slows down the next automation. Here the reason for this is that as b becomes automated, the wages for low skilled labor drop, and so it becomes less attractive to automate task a. This shows the importance of the general equilibrium effects on wages, and how they change the impact of technological advancements.

5.3 Hulten's Theorem

A remarkably simple way to encapsulate all of the direct and indirect effects (in response to small technological changes) is via Hulten's Theorem. In particular, Hulten [47] shows that in competitive economies a total factor productivity (TFP) change for some producer i (a change in A_i):

$$\text{dlog}C = m_i \text{dlog}A_i,$$

where $C = \sum_{i \in N} C_i$ is the total net-output in the economy, and A_i is the TFP of producer i. The term m_i is the *Domar weight* of producer i; that is,

$$m_i = \frac{p_i Y_i}{\sum_{i \in N} p_i C_i},$$

where p_i is the price of good i, Y_i is the total production of sector i (so, $p_i Y_i$ is the total sales of sector i), and $\sum_{i \in N} p_i C_i$ is total GDP.

The key implication of Hulten's Theorem is that, to a first-order approximation (in logs), for small changes, one can ignore the full details of the network structure and use the observable sales shares of each firm/industry to derive the effects of technology changes on net-output.

If one looks at a competitive economy in which all production processes involve complements – for instance, a purely Cobb-Douglas version of our model with no substitution – then

Hulten's Theorem would extend to hold non-locally. That is, the relative spending shares would not change in response to a change in some A_i, and so the results of the theorem extend to non-small changes. As Baqaee and Farhi [17] show, however, there are things that make Hulten's theorem fail for larger changes, as then second order effects become large as the Domar weights change. Baqaee and Farhi [17] work in the context of imperfect competition in which there are various price adjustments that affect those weights. Here we stick with perfect competition, but show that there are other things that can also lead to changes in the Domar weights – in our cases the substitution of labor changes relative wages and spending in different sectors of the economy. Thus, an important effect in our setting is that (as one can also see from Eq. (36) in the Appendix) the Domar weights change as automation replaces labor. Therefore, the first order approximation that does not capture the labor market reactions is only locally valid and is otherwise misleading during the automation phase, while the theorem applies with no approximation error in other phases.

One can infer from Proposition 1 that a version of Hulten's Theorem extends to our setup even though the economy enters into a transition path with changing growth levels. For the n-sector economy, we start with Proposition 6, which states that a version of Hulten's Theorem extends to our general setup.

Proposition 6 *In the general n-sector economy, let $m_i = \frac{p_i Y_i}{C}$ be the equilibrium Domar weight of sector i: $[\vec{m}_i] = (I - \Omega')^{-1}\left[\vec{\beta}_i\right]$. Then, the impact of small (infinitesimal) technological changes on total consumption and wages are:*

$$\mathrm{dlog}C = \sum_{i\in N} m_i \mathrm{dlog}A_i^P,$$

$$\mathrm{dlog}w_H = \sum_{i\in N} m_i \mathrm{dlog}A_i^P + \mathrm{dlog}\left(\sum_{i\in N}\alpha_i^H m_i\right),$$

$$\mathrm{dlog}w_L = \sum_{i\in N} m_i \mathrm{dlog}A_i^P + \mathrm{dlog}\left(\sum_{i\in N}\alpha_i^L m_i\right),$$

where $\alpha_i^L = \alpha_{i0}^L + \sum_{j\in a_i}(1 - t_{ij})\,\alpha_{ij}^L$ is the equilibrium share of low-skilled labor in sector i.

Of course, the Domar weights m_i and all the equilibrium values depend on the full production network, which determines the levels of automation which are critical in determining how much of each input is being used where. Still, the implication of the theorem is that to see the impact of small productivity changes, one can simply look at the current equilibrium expenditure levels.

5.4 Automation, and the Evolution of the Input-Output Network

Our next result sheds light on how the Domar weights and the network influences change as the substitution occurs in the economy.

It is useful to normalize production processes to separate out their productivity, so let $F_i = \frac{Y_i}{A_i^P}$ denote the normalized production process of sector i.

The following partial order is useful.

Consider two economies $E = \left(\left\{A_i^P\right\}, \{F_i\}, L, H\right)$ and $E' = \left(\left\{A_i^P\right\}', \{F_i\}, L, H\right)$ that have identical $\{F_i\}$, L, and H. We say that economy E' is *weakly more automated* than economy E if the equilibrium share of expenditures on automation in every automatable task $j \in a_i$ in every sector $i \in N$ is weakly greater in E' than in E. And we say that it is *more automated* if in addition, the equilibrium share of expenditures on automation in some $j \in a_i$ for some $i \in N$ is strictly greater in E' than in E.

Let the *network influence* of sector i be defined as $\frac{\text{dlog}C}{\text{dlog}A_i^P}$.

The network influence of a sector measures the overall growth effect of a productivity change in that sector. Following Proposition 6, the network influence of any sector is equal to its Domar weight. Given that the Domar weights evolve during automation phase, we provide an analysis of the change in sectoral network influences, which can be obtained by ordering the Domar weights as an economy changes.

PROPOSITION **7** *If economy E' is more automated than economy E, then:*

- *the network influence of each sector i is weakly higher in the economy E' than in the economy E, and*
- *the network influence of sector i is strictly higher in the economy E' than in the economy E if and only if i is one of the more automated tasks or there exists a directed upstream (supplier) path from i to at least one of the more automated tasks $j \neq i$ (i is either direct or indirect supplier of at least one of the more automated tasks j) in E'.*

Proposition 7 shows how interconnectedness in the economy changes following automation substitution in the economy. As the substitution of labor by automation goods occurs, the size of the interactions in the economy get larger due to the increasing share of expenditures on automation goods. Proposition 7 shows that following an increase in the level of automation in a given set of tasks, the Domar weights of the producers of those automation goods and their direct and indirect suppliers rise. Given that the Domar weights represent the network influences of sectors, this result implies that the automation good producers and their direct and indirect suppliers experience a growing network influence over time due to the substitution effects that results in increased connectivity in the economy.

5.5 Reallocation Effects in an n-Sector Economy

Proposition 6 provides an expression for the (local) macroeconomic impacts of technological changes in an n-sector economy, and Proposition 7 shows how these impacts (network influences) change with automation in the economy. We thus close by examining the overall impact of non-small technological changes when capturing the reallocation effect.

With multiple automation goods, supply chains involving automation goods make the general equilibrium effects more complex. More specifically, decisions to automate depend on how technological advances propagate in the economy through supply chains as well as how wages are determined and indirectly affect other sectors. Nonetheless, we can still develop expressions for these effects.

First of all, similar to the three-sector economy, for parameter regions in which there is no automation in any sector or each automatable task is fully automated in all sectors, then productivity changes translate into gains by both types of workers with constant relative wages: since the input-output network remains fixed for each sector, $\frac{w_H}{w_L}$ also remains constant. Therefore, if the economy is in the pre-automation or post-automation phase, then:

$$\Delta \text{log} C = \Delta \text{log} w_L = \Delta \text{log} w_H = \sum_{i \in N} m_i (\Delta \text{log} A_i^P)$$

Next, consider an economy for which some automation good j is in the transition phase. Note that $p_j = w_L$. Therefore, any sector i that has this automatable task is indifferent between using the automation good and low-skilled labor. The equilibrium levels of $\frac{p_i}{w_L}$ are thus described by:

$$log \left(\vec{\frac{p_i}{w_L}} \right) = (I - \Omega)^{-1} \left[log B_i + \vec{\alpha_i^H} log \left(\frac{w_H}{w_L} \right) \right]$$

where

$$B_i = \left((A_i^P)(\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L} \left[\prod_{j \in N} (\alpha_{ij}^L)^{\alpha_{ij}^L} \right] \left[\prod_{j \in N} (\alpha_{ij}^n)^{\alpha_{ij}^n} \right] \right)^{-1}.$$

For such an automation good j, $log \left(\frac{p_j}{w_L} \right) = 0$ and therefore the j^{th} entry of the vector of $(I - \Omega)^{-1} \left[log B_i + \vec{\alpha_i^H} log \left(\frac{w_H}{w_L} \right) \right]$ is equal to zero. As one can see from the equation above, that entry being zero depends on the task dependencies and productivity parameters of each sector, as well as the actual automation levels for different automation goods.

Section 7.8 in the Appendix, shows how changes in wages and the overall consumption in response to technological changes during a transition phase capture the reallocation effects, and how those depend on the skill dependencies in production processes.

6 Concluding Remarks

We have analyzed the impact of technological change on an economy in which both complement and substitute inputs are present. Our results show that when there exists different types of labor and intermediate goods that can substitute for labor, then the input-output structure, the skill-dependencies and sector level productivities play key roles in determining the impacts of technological changes on aggregate welfare and income inequality, since these factors all together determine the allocation of labor and wages, prices of goods and services, and the usage of substitutable intermediate goods (low-skilled labor).

Besides the fact that a local version of Hulten's Theorem extends to our setting, our model allows us to quantify the changes in the Domar weights following technological changes. Our model also enables us to provide predictions as to conditions under which the final good sector will switch to automation, how long the transition to automation phase will last, and how the overall impact of technological advancements depends on alternative uses for labor.

Our results shed light on productivity paradoxes and wage inequality, and suggest that understanding the impact of technological change must account for substitution in production processes.

References

[1] Acemoglu, Daron, Ufuk Akcigit, and William Kerr. "Networks and the macroeconomy: An empirical exploration." NBER Macroeconomics Annual 30, no. 1 (2016): 273-335.

[2] Acemoglu, Daron, and Pablo D. Azar. "Endogenous production networks." No. w24116. National Bureau of Economic Research, 2017.

[3] Acemoglu, Daron, Vasco M. Carvalho, Asuman E. Ozdaglar, and Alireza Tahbaz-Salehi. 2011. "The Network Origins of Aggregate Fluctuations." Econometrica 80 (5): 1977–2016.

[4] Acemoglu, Daron, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. 2017. "Microeconomic Origins of Macroeconomic Tail Risks." American Economic Review 107 (1): 54–108.

[5] Acemoglu, Daron, and Pascual Restrepo. "The race between machine and man: Implications of technology for growth, factor shares and employment." No. w22252. National Bureau of Economic Research, 2016.

[6] Acemoglu, Daron, and Pascual Restrepo. "Robots and jobs: Evidence from US labor markets." (2017).

[7] Acemoglu, Daron, and Pascual Restrepo. Artificial Intelligence, Automation and Work. No. w24196. National Bureau of Economic Research, 2018.

[8] Aghion, Philippe, Benjamin F. Jones, and Charles I. Jones. "Artificial Intelligence and Economic Growth." No. w23928. National Bureau of Economic Research, 2017.

[9] Alabdulkareem, Ahmad, Morgan R. Frank, Lijun Sun, Bedoor AlShebli, César Hidalgo, and Iyad Rahwan. 2018. "Unpacking the Polarization of Workplace Skills." Science Advances 4 (7): eaao6030.

[10] Arrow, Kenneth J., and Gerard Debreu. "Existence of an Equilibrium for a Competitive Economy." Econometrica 22, no. 3 (1954): 265-90.

[11] Atalay, Enghin. "How important are sectoral shocks?." American Economic Journal: Macroeconomics 9, no. 4 (2017): 254-80.

[12] Autor, David H., and David Dorn. 2013. "The Growth of Low-Skill Service Jobs and the Polarization of the US Labor Market." American Economic Review, 103 (5): 1553-97.

[13] Autor, David H., Frank Levy, and Richard J. Murnane. "The skill content of recent technological change: An empirical exploration." The Quarterly journal of economics 118, no. 4 (2003): 1279-1333.

[14] Autor, David H., Lawrence F. Katz, and Alan B. Krueger. "Computing inequality: have computers changed the labor market?." The Quarterly Journal of Economics 113, no. 4 (1998): 1169-1213.

[15] Autor, David, and Anna Salomons, "Is automation labour-displacing? Productivity growth, employment, and the labour share," Brookings Papers on Economic Activity, Spring 2018, 1-63

[16] Baqaee, David Rezza, "Cascading Failures in Production Networks (May 2016)." CEPR Discussion Paper No. DP12922. Available at SSRN: https://ssrn.com/abstract=3178092

[17] Baqaee, David Rezza, and Emmanuel Farhi. Productivity and Misallocation in General Equilibrium. No. w24007. National Bureau of Economic Research, 2017a.

[18] Baqaee, David Rezza, and Emmanuel Farhi. "The macroeconomic impact of microeconomic shocks: beyond Hulten's Theorem." No. w23145. National Bureau of Economic Research, 2017b.

[19] Baqaee, David Rezza, and Emmanuel Farhi. "The Microeconomic Foundations of Aggregate Production Functions," 2018.

[20] Barrot, Jean-Noel, and Julien Sauvagnat. 2016. "Input Specificity And The Propagation Of Idiosyncratic Shocks In Production Networks." The Quarterly Journal of Economics, 1543–92.

[21] Bartelme, Dominick, and Yuriy Gorodnichenko. "Linkages and economic development." No. w21251. National Bureau of Economic Research, 2015.

[22] Basu, Susanto. 1995. "Goods and Business Cycles: Implications for Productivity and Welfare." The American Economic Review 85 (3): 512–31.

[23] Basu, Susanto, and John G. Fernald. 2002. "Aggregate Productivity and Aggregate Technology." European Economic Review 46 (6): 963–91.

[24] Bernard, Andrew B., Emmanuel Dhyne, Glenn Magerman, Kalina Manova, and Andreas Moxnes. 2017. "The origins of firm heterogeneity: a production network approach." Tuck School of Business at Dartmouth

[25] Bigio, Saki, and Jennifer La'O "Financial frictions in production networks." No. w22212. National Bureau of Economic Research, 2016.

[26] Boehm, C., A. Flaaen, and N. Pandalai-Nayar (2014). "Input linkages and the transmission of shocks: Firm-level evidence from the 2011 tōhoku earthquake." 15th Jacques Polak Annual Research Conference.

[27] Brynjolfsson, Erik, Daniel Rock, and Chad Syverson. "Artificial intelligence and the modern productivity paradox: A clash of expectations and statistics." The Economics of Artificial Intelligence: An Agenda. University of Chicago Press, 2018.

[28] Brynjolfsson, Erik, Daniel Rock, and Chad Syverson. The productivity J-curve: How intangibles complement general purpose technologies. No. w25148. National Bureau of Economic Research, 2018.

[29] Brynjolfsson, Erik and Andrew McAfee (2014) The Second Machine Age: Work, Progress, and Prosperity in a Time of Brilliant Technologies, W. W. Norton & Company

[30] Carvalho, Vasco M., Makoto Nirei, Yukiko Saito, and Alireza Tahbaz-Salehi. 2017. "Supply Chain Disruptions: Evidence from the Great East Japan Earthquake."

[31] Carvalho, Vasco M., and Nico Voigtländer. Input diffusion and the evolution of production networks. No. w20025. National Bureau of Economic Research, 2014.

[32] Carvalho, Vasco M, and Xavier Gabaix. 2013. "The Great Diversification and Its Undoing." American Economic Review 103 (5): 1697–1727.

[33] Domar, Evsey D. "On the measurement of technological change." The Economic Journal 71.284 (1961): 709-729.

[34] Dupor, Bill. 1999. "Aggregation and Irrelevance in Multi-Sector Models." Journal of Monetary Economics 43 (2): 391–409.

[35] Fadinger, Harald, Christian Ghiglino, and Mariya Teteryatnikova. "Income differences and input-output structure." (2016).

[36] Frank, Morgan R., Sun Lijun, Cebrian Manuel, Youn Hyejin, and Rahwan Iyad. 2018. "Small Cities Face Greater Impact from Automation." Journal of The Royal Society Interface 15 (139): 20170946.

[37] Frey, Carl B. and Michael A. Osborne (2013) "The Future of Employment: How Susceptible are Jobs to Computerisation?" Mimeo. Oxford Martin School.

[38] Gabaix, Xavier. 2011. “The Granular Origins of Aggregate Fluctuations.” Econometrica 79 (3): 733–72.

[39] Goos, Maarten, Alan Manning, and Anna Salomons. 2014. “Explaining Job Polarization: Routine-Biased Technological Change and Offshoring.” American Economic Review 104 (8): 2509–26.

[40] Graetz, Georg, and Guy Michaels. ”Robots at work.” Review of Economics and Statistics 100, no. 5 (2018): 753-768.

[41] Grassi, Basile. “IO in IO: Competition and volatility in input-output networks.” Unpublished Manuscript, Bocconi University (2017).

[42] Griliches, Zvi. 1969 “Capital-skill complementarity.” The review of Economics and Statistics, 465-468.

[43] Michaels, Guy, Ashwini Natraj and John Van Reenen (2014) “Has ICT Polarized Skill Demand? Evidence from Eleven Countries over Twenty-Five Years,”Review of Economics and Statistics, 96(1): 60–77

[44] Hémous, David, and Morten Olsen. “The rise of the machines: Automation, horizontal innovation and income inequality.” (2014).

[45] Horvath, Michael. 1998. “Cyclicality and Sectoral Linkages: Aggregate Fluctuations from Independent Sectoral Shocks.” Review of Economic Dynamics 1 (4): 781–808.

[46] Horvath, Michael. 2000. “Sectoral Shocks and Aggregate Fluctuations.” Journal of Monetary Economics 45 (1): 69–106.

[47] Hulten, Charles R. 1978. “Growth Accounting with Intermediate Inputs.” The Review of Economic Studies 45 (3): 511–18.

[48] Jones, Charles I. Misallocation, economic growth, and input-output economics. No. w16742. National bureau of economic research, (2011a).

[49] Jones, Charles I. (2011). Intermediate goods and weak links in the theory of economic development. American Economic Journal: Macroeconomics, 1–28.

[50] Katz, Lawrence F., and Kevin M. Murphy. 1992. “Changes in Relative Wages, 1963–1987: Supply and Demand Factors.” Quarterly Journal of Economics 107 (1): 35–78.

[51] Krusell, Per, Lee E. Ohanian, Jose Victor Rios Rull, and Giovanni L. Violante. “Capital skill complementarity and inequality: A macroeconomic analysis.” Econometrica 68, no. 5 (2000): 1029–1053.

[52] Leontief, Wassily W. 1936. "Input and Output Relations in the Economic Systems of the United States." The Review of Economics and Statistics 18 (3): 105–25.

[53] Long, John B, and Charles Plosser. 1983. "Real Business Cycles." Journal of Political Economy 91 (1): 39–69.

[54] Moll, Benjamin, Lukasz Rachel, and Pascual Restrepo. "Uneven Growth: Automation's Impact on Income and Wealth Inequality." Manuscript, Princeton University (2019).

[55] Oberfield, Ezra. "A Theory of Input–Output Architecture." Econometrica 86, no. 2 (2018): 559-589.

[56] Peri, Giovanni, and Chad Sparber. 2009. "Task Specialization, Immigration, and Wages." American Economic Journal: Applied Economics 1 (3): 135–69.

[57] Shea, John. 2002. "Complementarities and Comovements." Journal of Money, Credit and Banking 34 (2): 412–33.

[58] Stokey, Nancy L. 1996 "Free trade, factor returns, and factor accumulation." Journal of Economic Growth 1.4 (1996): 421-447.

[59] Zeira, Joseph (1998) "Workers, Machines, and Economic Growth," Quarterly Journal of Economics, 113(4): 1091–1117.

7 APPENDIX

7.1 Equilibrium in the Three-Sector Model

The cost minimization problem for firm $i \in \{a, n\}$ is

$$\min_{L_{i0}, H_i} w_L L_{i0} + w_H H_i \text{ subject to } 1 = A_i^P (H_i)^{\alpha_i^H} (L_{i0})^{\alpha_{i0}^L}$$

The Lagrangian function is:

$$\mathcal{L} = w_L L_{i0} + w_H H_i - \lambda_i \left(A_i^P (H_i)^{\alpha_i^H} (L_{i0})^{\alpha_{i0}^L} - 1 \right)$$

The first order conditions are:

- $\frac{\partial \mathcal{L}}{\partial H_i} = w_H - \frac{\lambda_i^* \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{H_i^*} = 0$

  $H_i^* = \frac{\lambda_i^* \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{i0}} = w_L - \frac{\lambda_i^* \alpha_{i0}^L \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{L_{i0}} = 0$

  $L_{i0}^* = \frac{\lambda_i^* \alpha_{i0}^L \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 1 - A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L} = 0$

The FOCs above imply:

$$\frac{1}{\lambda_i^*} = A_i^P (\frac{\alpha_i^H}{w_H})^{\alpha_i^H} \left(\frac{\alpha_{i0}^L}{w_L}\right)^{\alpha_{i0}^L}$$

Then, the zero profit condition implies that $\lambda_i^* = w_H H_i^* + w_L L_{i0}^* = C_i(\mathbf{p}, w_L, w_H, 1) = p_i$.

The factor demands and prices can be written as follows:

$$L_{a0} = \frac{\alpha_a^H p_a Y_a}{w_L} \text{ and } H_a = \frac{\alpha_a^H p_a Y_a}{w_H}$$

$$L_{n0} = \frac{\alpha_i^H p_i Y_i}{w_L} \text{ and } H_n = \frac{\alpha_n^H p_n Y_n}{w_H}$$

$$p_a = \frac{1}{A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}} (w_L)^{\alpha_{a0}^L} (w_H)^{\alpha_a^H} \tag{9}$$

$$p_n = \frac{1}{A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}} (w_L)^{\alpha_{n0}^L} (w_H)^{\alpha_n^H} \tag{10}$$

Next, we solve for the cost minimization for firm f.

$$\begin{aligned} \underset{L_{f0}, L_{fa}, H_f, X_{fa}, X_{fn}}{min} & w_L(L_{f0} + L_{fa}) + w_H H_f + p_a X_{fa} + p_n X_{fn} \text{ subject to} \\ 1 = & A_f^P (H_f)^{\alpha_f^H} (L_{f0})^{\alpha_{f0}^L} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n} \end{aligned}$$

The Lagrangian function is:

$$\begin{aligned} \mathcal{L} = & w_L(L_{f0} + L_{fa}) + w_H H_f + p_a X_{fa} + p_n X_{fn} \\ & -\lambda \left(A_f^P (H_f)^{\alpha_f^H} (L_{f0})^{\alpha_{f0}^L} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n} - 1 \right) \end{aligned}$$

The FOCs imply:

- $\frac{\partial \mathcal{L}}{\partial \lambda} = A_f^P (H_f^*)^{\alpha_f^H} (L_{f0}^*)^{\alpha_{f0}^L} [L_{fa}^* + A_a^Q X_{fa}^*]^{\alpha_{fa}^L} (A_n^Q X_{fn}^*)^{\alpha_{fn}^n} = 1$

By plugging the equation above into the other FOCs, we get:

- $\frac{\partial \mathcal{L}}{\partial H_f} = w_H - \lambda_f^* \frac{\alpha_f^H}{H_f^*} = 0$

  $H_f^* = \lambda_f^* \frac{\alpha_f^H}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{f0}} = w_L - \lambda_f^* \frac{\alpha_{f0}^L}{L_{f0}^*} = 0$

  $L_{f0}^* = \lambda_f^* \frac{\alpha_{f0}^L}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial X_{fn}} = p_n - \lambda_f^* \frac{\alpha_{fn}^n}{X_{fn}^{n*}} = 0$

  $X_{fn}^* = \lambda_f^* \frac{\alpha_{fn}^n}{p_n}$

- $\frac{\partial \mathcal{L}}{\partial L_{fa}} = \left(w_L - \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) \geq 0$ and $L_{fa}^* \left(w_L - \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) = 0$

  $\frac{\partial \mathcal{L}}{\partial X_{fa}} = \left(p_a - A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) \geq 0$ and $X_{fa}^* \left(p_a - A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) = 0$

  Both L_{fa}^* and X_{fa}^* can not be zero, otherwise $Y_f = 0$. Then,

  Case 1. $w_L = \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa}^* = \lambda_f^* \frac{\alpha_{fa}^L}{w_L}$; and $p_a > A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = 0$. In this case, $\frac{w_L}{p_a} < \frac{1}{A_a^Q}$.

  Case 2. $w_L > \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa} = 0$; and $p_a = A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = \lambda_f^* \frac{\alpha_{fa}^L}{p_a}$. In this case, $\frac{w_L}{p_a} > \frac{1}{A_a^Q}$.

  Case 3. $w_L = \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa}^* = \lambda_f^* \frac{(1-t_{fa})\alpha_{fa}^L}{w_L}$; $p_a = A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = \lambda_f^* \frac{t_{fa}\alpha_{fa}^L}{p_a}$. In this case, $\frac{w_L}{p_a} = \frac{1}{A_a^Q}$.

$$\{L_{fa}^*, X_{fa}^*\} =: \begin{cases} \{\lambda_f^* \frac{\alpha_{fa}^L}{w_L}, 0\} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \{\lambda_f^* \frac{(1-t_{fa})\alpha_{fa}^L}{w_L}, \lambda_f^* \frac{t_{fa}\alpha_{fa}^L}{p_a} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \{0, \lambda_f^* \frac{\alpha_{fa}^L}{p_a}\} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

The FOCs above together imply:

$$\lambda_f^* = \begin{cases} \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[A_a^Q \frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

Under the zero profit conditions and the normalization of $p_f = 1$, it follows that:

$$\begin{cases} A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{A_a^Q\alpha_{fa}^L A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}{(w_L)^{\alpha_{a0}^L}(w_H)^{\alpha_a^H}}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

Lastly, by plugging p_a and p_n into the equation above, it follows that:

$$\begin{cases} A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} < 1 \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} = 1 \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{A_a^Q\alpha_{fa}^L A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}{(w_L)^{\alpha_{a0}^L}(w_H)^{\alpha_a^H}}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} > 1 \end{cases} \tag{11}$$

Then, the conditional factor demands are:

$$H_f =: \begin{cases} \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[A_a^Q\frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{12}$$

$$L_{f0} =: \begin{cases} \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[A_a^Q\frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{13}$$

$$X_{fn} =: \begin{cases} \dfrac{\alpha^L_{fn} Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[\frac{\alpha^L_{fa}}{w_L}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \dfrac{\alpha^L_{fn} Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[\frac{\alpha^L_{fa}}{w_L}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \dfrac{\alpha^L_{fn} Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[A_a^Q \frac{\alpha^L_{fa}}{p_a}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{14}$$

$$\{L_{fa}, X_{fa}\} =: \begin{cases} \{\dfrac{\alpha^L_{fa} Y_f}{w_L A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[\frac{\alpha^L_{fa}}{w_L}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}}, 0\} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \{\dfrac{(1-t_{fa})\alpha^L_{fa} Y_f}{w_L A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[\frac{\alpha^L_{fa}}{w_L}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}}, \dfrac{t_{fa}\alpha^L_{fa} Y_f}{p_a A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[\frac{\alpha^L_{fa}}{w_L}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}}\} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \{0, \dfrac{\alpha^L_{fa} Y_f}{p_a A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha^L_{f0}}{w_L}\right)^{\alpha^L_{f0}} \left[A_a^Q \frac{\alpha^L_{fa}}{p_a}\right]^{\alpha^L_{fa}} \left[A_n^Q \frac{\alpha^n_{fn}}{p_n}\right]^{\alpha^n_{fn}}}\} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{15}$$

7.2 Proofs of Proposition 1, 2, 3, and 4

Combining the budget constraint and FOCs of the utility maximization leads to:

$$C^L = w_L L \tag{16}$$

$$C^H = w_H H \tag{17}$$

$$Y_f = C = w_L L + w_H H \tag{18}$$

By combining the market clearing conditions, factor demands for low-skilled and high-skilled labor, we get:

$$L =: \begin{cases} \frac{p_n Y_n \alpha^L_{n0} + p_f Y_f \left(\alpha^L_{f0} + \alpha^L_{fa}\right)}{w_L} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha^L_{a0} + p_n Y_n \alpha^L_{n0} + p_f Y_f \left(\alpha^L_{f0} + (1-t_{fa})\alpha^L_{fa}\right)}{w_L} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha^L_{a0} + p_n Y_n \alpha^L_{n0} + p_f Y_f \alpha^L_{f0}}{w_L} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{19}$$

$$H =: \begin{cases} \frac{p_n Y_n \alpha^H_n + p_f Y_f \alpha^H_f}{w_H} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha^H_a + p_n Y_n \alpha^H_n + p_f Y_f \alpha^H_f}{w_H} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha^H_a + p_n Y_n \alpha^H_n + p_f Y_f \alpha^H_f}{w_H} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{20}$$

Market clearing for goods, factor demands, and $p_f = 1$ together imply:

$$p_a Y_a = p_a X_{fa} = t_{fa} \alpha_{fa}^L Y_f \tag{21}$$

where $t_{fa} = 1$ for $\frac{w_L}{p_a} > \frac{1}{A_a^Q}$, $t_{fa} = 0$ for $\frac{w_L}{p_a} < \frac{1}{A_a^Q}$, and $0 \le t_{fa} \le 1$ for $\frac{w_L}{p_a} = \frac{1}{A_a^Q}$.

$$p_n Y_n = p_n X_{fn} = \alpha_{fn}^n Y_f \tag{22}$$

Next, we write the condition for t_{fa} for further simplification.
Then, by using Eq. (18), Eq. (21), and Eq. (22), we can rewrite the market clearing for labor as follows:

$$L =: \begin{cases} \frac{\left(\alpha_{n0}^L \alpha_{fn}^n + \alpha_{f0}^L + \alpha_{fa}^L\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\left(t_{fa} \alpha_{fa}^L \alpha_{a0}^L + \alpha_{fn}^n \alpha_{n0}^L + \alpha_{f0}^L + (1 - t_{fa}) \alpha_{fa}^L\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\left(\alpha_{fa}^a \alpha_{a0}^L + \alpha_{fn}^n \alpha_{n0}^L + \alpha_{f0}^L\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{23}$$

$$H =: \begin{cases} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{24}$$

$$\frac{w_H}{w_L} =: \begin{cases} \frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1 - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{L}{H} \frac{\left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1 - \left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1 - \left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{25}$$

Then, we derive w_L by plugging Eq. (25) into the Eq. (11):

$$w_L =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{H}{L} \frac{1 - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{\alpha_f^H + \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P < A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{H}{L} \frac{1 - \left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{\alpha_f^H + \alpha_n^H \alpha_{fn}^n} & \text{if } A^* \le A_a^Q A_a^P \le A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{H}{L} \frac{1 - \left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{\alpha_f^H + \alpha_a^H \alpha_{fa}^L + \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{26}$$

where

$$
\begin{cases}
A^{*} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{\alpha_a^H} \\
A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{\alpha_a^H}
\end{cases}
\tag{27}
$$

By plugging Eq. (26) into the Eq. (25), we get:

$$
w_H =: \begin{cases}
A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H} \frac{(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P < A^{*} \\
A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H} \frac{(t_{fa}\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(t_{fa}\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n} & \text{if } A^{*} \leq A_a^Q A_a^P \leq A^{**} \\
A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{L}{H} \frac{(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)} \right)^{1-\alpha_f^H - \alpha_a^H \alpha_{fa}^L - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P > A^{**}
\end{cases}
\tag{28}
$$

Next, we derive the fraction t_{fa} from Eq. (28):

$$
t_{fa} =: \begin{cases}
0 & \text{if } A_a^Q A_a^P < A^{*} \\
\frac{\frac{H}{L}\left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} \left(1-\alpha_{fn}^n \alpha_n^H - \alpha_f^H\right) - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fa}^L \alpha_a^H\right)\left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)} & \text{if } A^{*} \leq A_a^Q A_a^P \leq A^{**} \\
1 & \text{if } A_a^Q A_a^P > A^{**}
\end{cases}
\tag{29}
$$

We can rewrite it as follows:

$$
t_{fa} =: \begin{cases}
0 & \text{if } A_a^Q A_a^P < A^{*} \\
\frac{\frac{H}{L}\left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\left(\alpha_{fa}^L \alpha_a^H\right)\left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)} - \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fa}^L \alpha_a^H\right)} & \text{if } A^{*} \leq A_a^Q A_a^P \leq A^{**} \\
1 & \text{if } A_a^Q A_a^P > A^{**}
\end{cases}
\tag{30}
$$

Then, we plug Eq.(29) into Eq. (26) and Eq. (28), and get:

$$
\frac{w_H}{w_L} =: \begin{cases}
\frac{L}{H} \frac{(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fn}^n \alpha_n^H + \alpha_f^H)} & \text{if } A_a^Q A_a^P < A^{*} \\
\left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} & \text{if } A^{*} \leq A_a^Q A_a^P \leq A^{**} \\
\frac{L}{H} \frac{(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)}{1-(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H)} & \text{if } A_a^Q A_a^P > A^{**}
\end{cases}
\tag{31}
$$

Then, for constant weights and labor supply, by taking the logs and the total derivatives of each side of Eq. (31) (excluding the values for $A_a^Q A_a^P$ where $\frac{w_H}{w_L}$ is non-differentiable), we get:

$$\text{dlog}\left(\frac{w_H}{w_L}\right) =: \begin{cases} 0 & \text{if } A_a^Q A_a^P < A^* \\ \frac{1}{\alpha_a^H}\text{dlog}\left(A_a^Q A_a^P\right) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ 0 & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{32}$$

Lastly, by plugging Eq. (29) into Eq. (26) and Eq. (28), we get:

$$w_L =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{H}{L}\frac{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_n^H\alpha_{fn}^n} & \text{if } A_a^Q A_a^P \leq A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{H}{L}\frac{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_a^H\alpha_{fa}^L+\alpha_n^H\alpha_{fn}^n} & \text{if } A_a^Q A_a^P \geq A^{**} \end{cases} \tag{33}$$

$$w_H =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n} & \text{if } A_a^Q A_a^P < A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} & \text{if } A^* \leq A_a^Q A_a^P \leq A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{L}{H}\frac{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{1-\alpha_f^H-\alpha_a^H\alpha_{fa}^L-\alpha_n^H\alpha_{fn}^n} & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{34}$$

Then, by taking the logs and the total derivatives of the equation above (we exclude the values for $A_a^Q A_a^P$ where w_L and w_H are non-differentiable), we get:

$$\text{dlog}w_L =: \begin{cases} \text{dlog}A_f^P + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) & \text{if } A_a^P A_a^Q < A^* \\ \text{dlog}A_f^P \; + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) \; - \left(\frac{\alpha_f^H+\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}\right)\text{dlog}\left(A_a^P A_a^Q\right) & \text{if } A^* < A_a^P A_a^Q < A^{**} \\ \text{dlog}A_f^P \; + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) + \alpha_{fa}^L \text{dlog}\left(A_a^P A_a^Q\right) & \text{if } A_a^P A_a^Q > A^{**} \end{cases}$$

$$\text{dlog}w_H =: \begin{cases} \text{dlog}A_f^P + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) & \text{if } A_a^P A_a^Q < A^* \\ \text{dlog}A_f^P \; + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) + \left(\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}\right)\text{dlog}\left(A_a^P A_a^Q\right) & \text{if } A^* < A_a^P A_a^Q < A^{**} \\ \text{dlog}A_f^P \; + \alpha_{fn}^n \text{dlog}\left(A_n^P A_n^Q\right) + \alpha_{fa}^L \text{dlog}\left(A_a^P A_a^Q\right) & \text{if } A_a^P A_a^Q > A^{**} \end{cases}$$

This is the end of the proof of Proposition 2.

Next, we derive $w_H H + w_L L = C$ by multiplying Eq. (26) by L and multiplying Eq. (28) by H and summing up these two, the consumption level in three phases are as follows with the ordering of pre-automation, automation, and post-automation phases:

$$C =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^a\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \frac{L}{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n} \left(\frac{H}{L}\frac{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_n^H\alpha_{fn}^n} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^a\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\right) \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \frac{L}{1-\left(\alpha_{fa}^a\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} \left(\frac{H}{L}\frac{1-\left(\alpha_{fa}^a\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fa}^a\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_a^H\alpha_{fa}^a+\alpha_n^H\alpha_{fn}^n} \end{cases} \tag{35}$$

Then, by taking the logs of and totally differentiating both sides, we get:

$$\mathrm{dlog}C = \begin{cases} \mathrm{dlog}A_f^P + \alpha_{fn}^n \mathrm{dlog}(A_n^Q A_n^P) & \text{if } A_a^Q A_a^P < A^* \\ \mathrm{dlog}A_f^P + \alpha_{fn}^n \mathrm{dlog}(A_n^Q A_n^P) + \mathrm{dlog}\left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\right) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ \mathrm{dlog}A_f^P + \alpha_{fn}^n \mathrm{dlog}(A_n^Q A_n^P) + \alpha_{fa}^L \mathrm{dlog}(A_a^P A_a^Q) & \text{if } A_a^Q A_a^P > A^{**} \end{cases}$$

Call $L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} = B$

Next, we derive $\frac{\mathrm{dlog}B}{\mathrm{dlog}(A_a^Q A_a^P)} = \frac{(A_a^Q A_a^P)}{B}\frac{dB}{d(A_a^Q A_a^P)}$

$$\frac{dB}{d(A_a^Q A_a^P)} = \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\left(H\left(1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\right)}{\alpha_a^H A_a^Q A_a^P}$$

Then,

$$\frac{\mathrm{dlog}B}{\mathrm{dlog}(A_a^Q A_a^P)} = \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\left(H\left(1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\right)}{\alpha_a^H\left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\right)}$$

$$= \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\left(H\left(1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\right)}{\alpha_a^H L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}\left(1+H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}$$

$$= \frac{\left(\frac{H}{L}\left(1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}-\left(\alpha_f^H+\alpha_n^H\alpha_{fn}^n\right)\right)}{\alpha_a^H\left(1+\frac{H}{L}\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}.$$

Eq. (29)implies that for $0 < t_{fa} < 1$:

$$t_{fa}\alpha_{fa}^L = \frac{\frac{H}{L}\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\left(1-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{\alpha_a^H\left(1+\frac{H}{L}\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}$$

Thus, $\text{dlog}B = t_{fa}\alpha_{fa}^L\text{dlog}(A_a^Q A_a^P)$ for $A^* < A_a^Q A_a^P < A^{**}$. Then, we get

$$\text{dlog}C = \begin{cases} \text{dlog}A_f^P + \alpha_{fn}^n\text{dlog}(A_n^Q A_n^P) & \text{if } A_a^Q A_a^P < A^* \\ \text{dlog}A_f^P + \alpha_{fn}^n\text{dlog}(A_n^Q A_n^P) + t_{fa}\alpha_{fa}^L\text{dlog}(A_a^Q A_a^P) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ \text{dlog}A_f^P + \alpha_{fn}^n\text{dlog}(A_n^Q A_n^P) + \alpha_{fa}^L\text{dlog}(A_a^P A_a^Q) & \text{if } A_a^Q A_a^P > A^{**} \end{cases}$$

This completes the proof of Proposition 1.

Lastly, the expressions in Proposition 4 also follows from Equation (35).

Proof of Proposition 3:

$$\begin{cases} A^* = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} \\ A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}\left(\frac{L}{H}\frac{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} \end{cases}$$

i) A^* and A^{**} are increasing in $\frac{L}{H}$

$$\frac{\partial A^*}{\partial\left(\frac{L}{H}\right)} = \left(\frac{L}{H}\right)^{\alpha_a^H-1}\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{1-\alpha_a^H}\left(\frac{\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} > 0$$

$$\frac{\partial A^{**}}{\partial\left(\frac{L}{H}\right)} = \left(\frac{L}{H}\right)^{\alpha_a^H-1}\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{1-\alpha_a^H}\left(\frac{\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} > 0$$

ii) A^* and A^{**} are increasing in α_n^H

$$\frac{\partial A^*}{\partial(\alpha_n^H)} = \frac{\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{\alpha_{a0}^L}\alpha_{fn}^n\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

$$\frac{\partial A^{**}}{\partial(\alpha_n^H)} = \frac{\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{\alpha_{a0}^L}\alpha_{fn}^n\left(\frac{L}{H}\frac{\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fa}^L\alpha_a^H-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

iii) For constant α_{fn}^n and α_{fa}^L, A^* and A^{**} are increasing in α_f^H (decreasing in α_{f0}^L).

$$\frac{\partial A^*}{\partial\left(\alpha_f^H\right)} = \frac{\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{\alpha_{a0}^L}\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

$$\frac{\partial A^{**}}{\partial\left(\alpha_f^H\right)} = \frac{\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{\alpha_{a0}^L}\left(\frac{L}{H}\frac{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fa}^L\alpha_a^H-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

Moreover, the Domar weight of firm a is given by $\alpha_{fa}^L t_{fa}$. Thus,

$$\frac{d^2 logC}{d^2 log(A_a^P A_a^Q)} = \frac{d(\alpha_{fa}^L t_{fa})}{\mathrm{dlog}(A_a^P A_a^Q)}$$
$$= \frac{HL\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H\left(H\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}+L\right)\left(H(\alpha_{fn}^n\alpha_n^H+\alpha_f^H-1)\left(A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}+L\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)\right)} \tag{36}$$

As one can see from the equation above, the second order term depends on the high-skilled and low-skilled labor supply, weights of tasks in each production process and the initial level of productivity in automation sector.

7.3 Proof of Proposition 5

The result follows from Equation (8). Taking the derivative of $\left(\frac{A^{**}}{A^*}\right)^{\frac{1}{\alpha_a^H}}$ w.r.t. related parameters gives the result. ∎

7.4 Derivations Behind the Examples

7.4.1 Example 2

$Y_a = A_a^P L_{a0}^{0.5} H_a^{0.5}$,

$Y_f = H_f^{0.5}(L_{fa} + A_a^Q X_{fa})^{0.5}$.

We plug the given parameters into Eq.(27) and Eq. (29), and get:

$$t_{fa} =: \begin{cases} 0 & \text{if} \quad A_1^P A_{21}^Q < 2 \\ \left(\frac{2(A_a^P A_a^Q)^2-8}{(A_a^P A_a^Q)^2+4}\right) & \text{if} \quad 2 \leq A_1^P A_{21}^Q \leq 2\sqrt{3} \\ 1 & \text{if} \quad A_1^P A_{21}^Q > 2\sqrt{3} \end{cases} \tag{37}$$

which can be rewritten as:

$$t_{fa} = min\left\{max\left\{0, \left(\frac{2(A_a^P A_a^Q)^2 - 8}{(A_a^P A_a^Q)^2 + 4}\right)\right\}, 1\right\}$$

By using the equation for t_{fa}, we derive the following equations:

$$p_f = 1$$

$$p_a = \begin{cases} \frac{1}{A_a^P} & \text{if} \quad A_a^P A_a^Q < 2 \\ \frac{1}{A_a^P} & \text{if} \quad 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^Q}{2\sqrt{3}A_a^P}} & \text{if} \quad A_a^P A_a^Q \geq 2\sqrt{3} \end{cases} \tag{38}$$

$$w_L =: \begin{cases} \frac{1}{2} & \text{if } A_a^P A_a^Q < 2 \\ \frac{1}{A_a^P A_a^Q} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^P A_a^Q}{2\sqrt{3}}}\left(\frac{1}{2\sqrt{3}}\right) & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{39}$$

$$w_H =: \begin{cases} \frac{1}{2} & \text{if } A_a^P A_a^Q < 2 \\ \frac{A_a^P A_a^Q}{4} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^P A_a^Q}{2\sqrt{3}}}\left(\frac{2\sqrt{3}}{4}\right) & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{40}$$

$$\frac{w_H}{w_L} =: \begin{cases} 1 & \text{if } A_a^P A_a^Q < 2 \\ \frac{\left(A_a^P A_a^Q\right)^2}{4} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ 3 & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{41}$$

Phases	t_{fa}	p_a	p_f	w_L	w_H	$\frac{w_H}{w_L}$	Y_a	$Y_f = C$
$A_a^P A_a^Q \leq 2$ (pre-automation phase)	0	$\frac{1}{A_a^P}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1
$2 < A_a^P A_a^Q < 2\sqrt{3}$ (automation phase)	$2\left(\frac{\left(A_a^P A_a^Q\right)^2 - 4}{\left(A_a^P A_a^Q\right)^2 + 4}\right)$	$\frac{1}{A_a^P}$	1	$\frac{1}{A_a^P A_a^Q}$	$\frac{A_a^P A_a^Q}{4}$	$\frac{\left(A_a^P A_a^Q\right)^2}{4}$	$A_a^P\left(\frac{\left(A_a^P A_a^Q\right)^2 - 4}{4\left(A_a^P A_a^Q\right)}\right)$	$\frac{4 + \left(A_a^P A_a^Q\right)^2}{4\left(A_a^P A_a^Q\right)}$
$A_a^P A_a^Q \geq 2\sqrt{3}$ (post-automation phase)	1	$\sqrt{\frac{A_a^Q}{A_a^P 2\sqrt{3}}}$	1	1	3	3	$\frac{A_a^P}{\sqrt{3}}$	$\sqrt{\frac{2A_a^P A_a^Q}{3\sqrt{3}}}$

7.4.2 Example 1

The derivations for Example 1 follow from the equilibrium conditions in a similar fashion. Thus, we omit the derivations for Example 1 here, and provide only the result regarding parameter t_{fa}. The equilibrium level of $Y_f = C$ in Example 1 are depicted in Table 4.

Phases	$Y_f = C$
$A_a^P A_a^Q \leq \frac{2}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}}$
$\frac{2}{\sqrt{3}} < A_a^P A_a^Q < 2$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(\left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}} \right)$
$A_a^P A_a^Q \geq 2$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}}$

Table 4. Total consumption (or net-output) in Example 1

$$t_{fa} \begin{cases} 0 & \text{if } A_a^P A_a^Q \leq \frac{2}{\sqrt{3}} \left(\frac{L}{H}\right)^{0.5} \\ \dfrac{\frac{H}{L}\left(\frac{A_a^Q A_a^P}{2}\right)^2 \left(\frac{3}{4}\right) - \left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)\left(1 + \frac{H}{L}\left(\frac{A_a^Q A_a^P}{2}\right)^2\right)} & \frac{2}{\sqrt{3}} \left(\frac{L}{H}\right)^{0.5} < A_a^P A_a^Q < 2 \left(\frac{L}{H}\right)^{0.5} \\ 1 & A_a^P A_a^Q \geq 2 \left(\frac{L}{H}\right)^{0.5} \end{cases}$$

Cases	Phases	$Y_f = C$
Case 1. $L = 4$ and $H = 1$	$A_a^P A_a^Q \leq \frac{4}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} L^{\frac{3}{4}} H^{\frac{1}{4}}$
	$\frac{4}{\sqrt{3}} < A_a^P A_a^Q < 4$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(L \left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + H \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}} \right)$
	$A_a^P A_a^Q \geq 4$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}} (LH)^{\frac{1}{2}}$
Case 1. $L = \frac{1}{4}$ and $H = 1$	$A_a^P A_a^Q \leq \frac{1}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} L^{\frac{3}{4}} H^{\frac{1}{4}}$
	$\frac{1}{\sqrt{3}} < A_a^P A_a^Q < 1$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(L \left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + H \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}} \right)$
	$A_a^P A_a^Q \geq 1$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}} (LH)^{\frac{1}{2}}$

Table 5. Total consumption (or net-output) in Example 1 under two additional cases for the labor supply.

7.5 Equilibrium in the n-Sector Economy

Proof of Theorem 1 and Corollary 1:

The cost minimization problem for each firm $i \in N$ is:

$$\min_{\{L_{ij}\}_{j \in 0 \cup K}, H_i, \{X_{ij}\}_{j \in N}} \sum_{j \in 0 \cup K} w_L L_{ij} + w_H H_i + \sum_{j \in N} p_j X_{ij}$$

$$\text{subject to } 1 = A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} \left(L_{ij} + X_{ij}\right)^{\alpha_{ij}^L} \right] \left[\prod_{j \in n_i} (X_{ij})^{\alpha_{ij}^n} \right]$$

The Lagrangian function is:

$$\mathcal{L} = \sum_{j\in 0\cup K} w_L L_{ij} + w_H H_i + \sum_{j\in N} p_j X_{ij} - \lambda_i \left(A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j\in a_i} (L_{ij}+X_{ij})^{\alpha_{ij}^L} \right] \left[\prod_{j\in n_i} (X_{ij})^{\alpha_{ij}^n} \right] - 1 \right)$$

The first order conditions are:

- $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 1 - A_i^P (L_{i0}^*)^{\alpha_{i0}^L} (H_i^*)^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L} \right] \left[\prod_{j\in n_i} (X_{ij}^*)^{\alpha_{ij}^n} \right] = 0$

  $1 = A_i^P (L_{i0}^*)^{\alpha_{i0}^L} (H_i^*)^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L} \right] \left[\prod_{j\in n_i} (X_{ij}^*)^{\alpha_{ij}^n} \right]$

  $\frac{\partial \mathcal{L}}{\partial H_i} = w_H - \frac{\lambda_i^* \alpha_i^H}{H_i^*} = 0$

  $H_i^* = \frac{\lambda_i^* \alpha_i^H}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{i0}} = w_L - \frac{\lambda_i^* \alpha_{i0}^L}{L_{i0}^*} = 0$

  $L_{i0}^* = \frac{\lambda_i^* \alpha_{i0}^L}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial \{X_{ij}\}_{j\in n_i}} = p_j - \frac{\lambda_i^* \alpha_{ij}^n}{\{X_{ij}^*\}_{j\in n_i}} = 0$

  $\{X_{ij}^*\}_{j\in n_i} = \frac{\lambda_i^* \alpha_{ij}^n}{p_j}$

- $\{L_{ij}^*, X_{ij}^*\}_{j\in a_i} =: \begin{cases} \{\lambda_i^* \frac{\alpha_{ij}^L}{w_L}, 0\} & \text{if } \frac{w_L}{p_j} < 1 \\ \{\lambda_i^* \frac{(1-t_{ij}^*)\alpha_{ij}^L}{w_L}, \lambda_i^* \frac{t_{ij}^* \alpha_{ij}^L}{p_j} & \text{if } \frac{w_L}{p_j} = 1 \\ \{0, \lambda_i^* \frac{\alpha_{ij}^L}{p_j}\} & \text{if } \frac{w_L}{p_j} > 1 \end{cases}$

- $\{\mathbf{L_{ij}^*}, \mathbf{X_{ij}^*}\}_{\mathbf{j\notin a_i \cup n_i}} = \mathbf{0}$

The FOCs above together imply:

$1 = A_i^P (\frac{\lambda_i^* \alpha_{i0}^L}{w_L})^{\alpha_{i0}^L} (\frac{\lambda_i^* \alpha_i^H}{w_H})^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L} \right] \left[\prod_{j\in n_i} (\frac{\lambda_i^* \alpha_{ij}^n}{p_j})^{\alpha_{ij}^n} \right]$

$1 = A_i^P (\frac{\lambda_i^* \alpha_{i0}^L}{w_L})^{\alpha_{i0}^L} (\frac{\lambda_i^* \alpha_i^H}{w_H})^{\alpha_i^H} \left[\prod_{j\in a_i} \left(\frac{\lambda_i^* \alpha_{ij}^L}{w_L}\right)^{(1-t_{ij}^*)\alpha_{ij}^L} \right] \left[\prod_{j\in a_i} \left(\frac{\lambda_i^* \alpha_{ij}^L}{p_j}\right)^{t_{ij}^* \alpha_{ij}^L} \right] \left[\prod_{j\in n_i} (\frac{\lambda_i^* \alpha_{ij}^n}{p_j})^{\alpha_{ij}^n} \right]$

Lastly, profit maximization implies that the unit cost for each i is equal to p_i. Thus $\lambda_i^* = p_i$ can be rewritten as:

$$p_i = B_i \left[\prod_{j\in N} (p_j)^{t_{ij}^* \alpha_{ij}} \right] (w_L)^{\alpha_{i0}^L + \sum_{j\in N} (1-t_{ij}^*)\alpha_{ij}^L} (w_H)^{\alpha_i^H} \tag{42}$$

where

$$B_i = \frac{1}{(A_i^P)(\alpha_i^H)^{\alpha_i^H}(\alpha_{i0}^L)^{\alpha_{i0}^L}\left[\prod_{j\in N}(\alpha_{ij}^L)^{\alpha_{ij}^L}\right]\left[\prod_{j\in N}(\alpha_{ij}^n)^{\alpha_{ij}^n}\right]} \tag{43}$$

and T^* is the equilibrium expenditure-weight matrix satisfying the following conditions:
$t_{ij}^* = 1$ for all $j \in n_i$
$t_{ij}^* \in [0, 1]$ for all $j \in a_i$.
Then, the equilibrium input-output matrix, Ω, is the Hadamard product of two matrices:

$$\Omega = T^* \circ \Omega^S$$

where Ω^S is the structural input-output matrix satisfying the conditions below:
$[\Omega^S]_{ij} = \alpha_{ij}^n > 0$ is the task-weight of non-automatable task $j \in n_i$ in sector i.
$[\Omega^S]_{ij} = \alpha_{ij}^L > 0$ is the task-weight of automatable task $j \in a_i$ in sector i.
$[\Omega^S]_{ij} = 0$ for any $j \in N$ such that $j \notin a_i \cup n_i$.
Then, we can write the equation for prices as follows:

$$p_i = B_i\left[\prod_{j\in N}(p_j)^{\omega_{ij}}\right](w_L)^{\alpha_i^L}(w_H)^{\alpha_i^H} \tag{44}$$

where

- $\alpha_i^L = \alpha_{i0}^L + \sum_{j\in N}(1 - t_{ij}^*)\alpha_{ij}^L$
- $\omega_{ij} = t_{ij}\alpha_{ij}^L$ for all pairs (i, j) such that $j \in a_i$
- $\omega_{ij} = \alpha_{ij}^n$ for all pairs (i, j) such that $j \in n_i$.

The existence of equilibrium follows from Theorem 1 of Arrow and Debreu [10]. Our equilibrium definition satisfies the assumptions I-IV defined by Arrow and Debreu [10], which guarantees the existence of equilibrium in a given competitive n-sector economy.

Proof of uniqueness:

We can rewrite Equation 42 as follows:

$$p_i = B_i\left[\prod_{j\in N}(p_j)^{\overline{\omega_{ij}}}\right](w_L)^{\overline{\alpha_i^L}}(w_H)^{\alpha_i^H}$$

where $\overline{\omega_{ij}}$s are the input output matrix entries with zero automation in each automatable task, and $\overline{\alpha_i^L}$ is the sum of low-skilled labor shares in each sector again under zero-automation. The equation can be rewritten as follows:

$$\frac{p_i}{w_L} = B_i \left[\prod_{j \in N} (\frac{p_j}{w_L})^{\overline{\omega_{ij}}} \right] \left(\frac{w_H}{w_L} \right)^{\alpha_i^H}$$

$$log \left(\frac{p_i}{w_L} \right) = log B_i + \sum_{j \in N} \overline{\omega_{ij}} log \left(\frac{p_j}{w_L} \right) + \alpha_i^H log \left(\frac{w_H}{w_L} \right)$$

In matrix notation,

$$log \left(\frac{\vec{p_i}}{w_L} \right) = (I - \overline{\Omega})^{-1} \left[log B_i + \alpha_i^H log \left(\frac{w_H}{w_L} \right) \right] \tag{45}$$

For given B_i , at any equilibrium, $\overline{\Omega}$ is fixed. Consider two different equilibrium E' and E'' . Then, Equation 45 implies three cases as follows:

i) For $\left(\frac{w_H}{w_L}\right)' = \left(\frac{w_H}{w_L}\right)''$, $log \left(\frac{\vec{p_i}}{w_L}\right)' = log \left(\frac{\vec{p_i}}{w_L}\right)''$ holds.

ii) For $\left(\frac{w_H}{w_L}\right)' > \left(\frac{w_H}{w_L}\right)''$, $log \left(\frac{p_i}{w_L}\right)' > log \left(\frac{p_i}{w_L}\right)''$ holds for all $i \in N$

ii) For $\left(\frac{w_H}{w_L}\right)' < \left(\frac{w_H}{w_L}\right)''$, $log \left(\frac{p_i}{w_L}\right)' < log \left(\frac{p_i}{w_L}\right)''$ holds for all $i \in N$

So, all entries in vector of $log \left(\frac{\vec{p_i}}{w_L}\right)$ move in the same direction with $log \left(\frac{w_H}{w_L}\right)$.

Then, consider that $\left(\frac{w_H}{w_L}\right)' > \left(\frac{w_H}{w_L}\right)''$, which implies there should be at least one automatable task which is more automated in equilibrium E' than in E'' , otherwise $\frac{w_H}{w_L}$ can not be higher in E' than in E''. However, condition in cse ii) implies that $log \left(\frac{p_i}{w_L}\right)' > log \left(\frac{p_i}{w_L}\right)''$ holds for all $i \in N$ that implies that automation for each task is lower in E' than in E'', which is a contradiction. Similarly, reversed arguments holds for the reversed case: $\left(\frac{w_H}{w_L}\right)' < \left(\frac{w_H}{w_L}\right)''$.

Therefore, at any equilibrium $\frac{w_H}{w_L}$ is unique. Next, we show that the uniqueness of $\frac{w_H}{w_L}$ implies that all prices are unique.

For unique $\frac{w_H}{w_L}$, First Welfare Theorem implies that w_H and w_L are unique as well. Otherwise, there is always an equilibrium that is Pareto dominated. Then, $C = w_H H + w_L L$ is unique. Moreover, uniqueness of $\frac{w_H}{w_L}$ and w_L together imply that $\{p_i\}_{i \in N}$ is unique, which is given by Equation 45. Lastly, for constant $\{\beta_i\}_{i \in N}$ implies that $C_i = \frac{\beta_i C}{p_i}$ is also unique for all $i \in N$.

This is the end of the proof of the uniqueness. The generically uniqueness part follows from Equation 47. For unique values of $\{p_i\}_{i \in N}$, w_L, and w_H, there exists a generically unique input-output matrix Ω satisfying Equation 47. Then, this implies that

$\{m_i\}_{i\in N}$ is generically unique and so there exists a generically unique equilibrium set of quantities $\{Y_i, \{L_{ij}\}_{j\in K\cup 0}, H_i, \{X_{ij}\}_{j\in N}\}_{i\in N}$.

The proof of Corollary 1 follows from the fact that

i) Low- and High-skilled labor wages:
$C_i = \frac{\beta_i C}{p_i}$
$Y_i = (\sum_{j\in N} \frac{\omega_{ji} p_j Y_j}{p_i}) + \frac{\beta_i(w_L L + w_H H)}{p_i}$
$p_i Y_i = (\sum_{j\in N} \omega_{ji} p_j Y_j) + \beta_i(w_L L + w_H H)$
In matrix notation:
$\overrightarrow{p_i Y_i} = (I - \Omega')^{-1}\left[\overrightarrow{\beta_i(w_L L + w_H H)}\right]$
Then, we can write $p_i Y_i$ as follows:
$p_i Y_i = m_i(w_L L + w_H H)$
where $m_i = \sum_{j\in N}[(I-\Omega')^{-1}]_{ij}\beta_j$.
$\alpha_i^H p_i Y_i = \alpha_i^H[m_i(w_L L + w_H H)]$
$\sum_{i\in N}\alpha_i^H p_i Y_i = \sum_{i\in N}\left[\alpha_i^H[m_i(w_L L + w_H H)]\right]$
$w_H H = \sum_{i\in N}\left[\alpha_i^H[m_i(w_L L + w_H H)]\right]$
$w_H H = \sum_{i\in N}\left[\alpha_i^H m_i w_L L + \alpha_i^H m_i w_H H\right]$
$w_H H(1 - \sum_{i\in N}\alpha_i^H m_i) = \sum_{i\in N}\left[\alpha_i^H m_i w_L L\right]$
Thus:

$$\frac{w_L}{w_H} = \left(\frac{H}{L}\right)\frac{1 - \sum_{i\in N}\alpha_i^H m_i}{\sum_{i\in N}\alpha_i^H m_i} \tag{46}$$

where m_i is the Domar weight of firm i.

7.6 Proofs of Propositions 6 and 7

Proof of Proposition 6.

Proof of part i :

The growth of consumption in response to sectoral productivity growth can be rewritten as follows:
$\frac{\text{dlog}C}{\text{dlog}A_i^P} = \frac{A_i^P}{C}\frac{dC}{dA_i^P}$
If we show $\frac{dC}{dA_i^P} = \frac{p_i Y_i}{A_i^P}$, then we are done. We derive $\frac{dC}{dA_i^P}$. The social planner's problem is as follows:

$$\max_{\{C_i\},\{X_{ij}\},\{L_{ij}\},\{L_{i0}\},\{H_i\}} \phi C^L + (1-\phi)\,C^H + \sum_i \lambda_i\left(A_i^P F_i - \sum_j X_{ji} - C_i\right) + \eta\left(L - \sum_i\sum_{j\in a_i\cup 0} L_{ij}\right) + \mu\left(H - \sum_i H_i\right)$$

For $\phi = \frac{1}{2}$, social planner's problem is equivalent to

$$\max_{\{C_i\},\{X_{ij}\},\{L_{ij}\},\{L_{i0}\},\{H_i\}} \left(C^L + C^H\right) + \sum_i \lambda_i \left(A_i^P F_i - \sum_j X_{ji} - C_i\right) + \eta \left(L - \sum_i \sum_{j \in a_i \cup 0} L_{ij}\right) + \mu \left(H - \sum_i H_i\right)$$

Then, the envelope theorem implies:

$\frac{dC}{dA_i^P} = -\lambda_i \frac{Y_i}{A_i^P}$.

If $-\lambda_i = p_i$, then we are done.

The social planner's problem also implies that:

$\frac{dC}{dC_i} = -\lambda_i$

Moreover, FOCs of the profit maximization of the representative firm in the final good sector imply that:

$p_i C_i = \beta_i p_f C$, which further implies

$\sum_i p_i C_i = C$.

Thus, $\frac{dC}{dC_i} = p_i$. By combining this result with the FOCs of the social planner's problem, we get $-\lambda_i = p_i$, which completes the proof.

Proof of part ii and iii :

m_i is the Domar weight of sector i, which implies:

$p_i Y_i = m_i(w_L L + w_H H)$

We multiply both sides by α_i^H and sum across sectors:

$\alpha_i^H p_i Y_i = \alpha_i^H m_i(w_L L + w_H H)$

$\sum_{i \in N} \alpha_i^H p_i Y_i = \sum_{i \in N} \alpha_i^H m_i(w_L L + w_H H)$,

The FOCs of firm maximization imply that $\sum_{i \in N} \alpha_i^H p_i Y_i = w_H H$. Thus,

$w_H H = \sum_{i \in N} \left[\alpha_i^H m_i w_L L + \alpha_i^H m_i w_H H\right]$

$w_H H(1 - \sum_{i \in N} \alpha_i^H m_i) = \sum_{i \in N} \left[\alpha_i^H m_i w_L L\right]$

and so

$$w_L L = (w_H H) \frac{1 - \sum_{i \in N} \alpha_i^H m_i}{\sum_{i \in N} \alpha_i^H m_i}.$$

For $C = w_L L + w_H H$, it follows that:

$$C = \frac{w_H H}{\sum_{i \in N} \alpha_i^H m_i} \quad \text{and} \quad C = \frac{w_L L}{1 - \sum_{i \in N} \alpha_i^H m_i}$$

By taking logs of both sides and totally differentiating both sides, we get:

$$\text{dlog} w_H = \text{dlog} C + \text{dlog}(\sum_{i \in N} \alpha_i^H m_i)$$

$$\text{dlog} w_L = \text{dlog} C - \text{dlog}(1 - \sum_{i \in N} \alpha_i^H m_i).$$

Proof of Proposition 7

Proof of part i)

LEMMA 1 *Consider the $n \times n$ matrix $D = (I - \Omega')^{-1}$, where Ω' is the transpose of the given matrix Ω. Then, the following conditions hold:*

$D_{ij} = \sum\limits_k D_{ik}\Omega_{jk}$ *for all pairs $[i, j\}$ s.t. $i \neq j$*

$D_{ii} = 1 + \sum\limits_k D_{ik}\Omega_{ik}$

Proof of Lemma 1.

For $D = (I - \Omega')^{-1}$, we claim that $D = I + D\Omega'$ holds. Suppose it holds. Then, by plugging $D = (I - \Omega')^{-1}$ into $D = I + D\Omega'$, we get $(I - \Omega')^{-1} = I + (I - \Omega')^{-1}\Omega'$, which can be rewritten as $(I - \Omega')^{-1}(I - \Omega') = I$. Thus, our claim holds. Then, $D = I + D\Omega'$ implies that:

$[D_{ij}]_{i \neq j} = \sum\limits_k D_{ik}\Omega'_{kj}$

$D_{ii} = 1 + \sum\limits_k D_{ik}\Omega'_{ki}$

Then, by using $\Omega'_{kj} = \Omega_{jk}$ and $\Omega'_{ki} = \Omega_{ik}$, we can write:

$$[D_{ij}]_{i \neq j} = \sum_k D_{ik}\Omega_{jk}$$

$$D_{ii} = 1 + \sum_k D_{ik}\Omega_{ik}$$

This completes the proof of Lemma 1.

By Lemma 1,

- $D_{ii} = 1 + \sum\limits_{k \in N} D_{ik}\Omega_{ik}$ for all i,
- $D_{ij} = D_{ii}\Omega_{ji} + \sum\limits_{k \neq i} D_{ik}\Omega_{jk}$ for all pairs $\{i, j\}$ such that $i \neq j$ and $\Omega_{ji} > 0$ (i is a direct supplier of j), and
- $D_{ij} = \sum\limits_{k \neq i} D_{ik}\Omega_{jk}$ for all pairs $\{i, j\}$ such that $i \neq j$ and $\Omega_{ji} = 0$ (i is not a direct supplier of j).

Given that $\Omega_{ij} \geq 0$ for all pairs $\{i, j\}$, the set of equations above imply that each element of matrix D is

i) non-negative,

ii) non-decreasing in each element of matrix Ω.

Proof of part ii):

Denote the initial economy given in part ii) by E^0, and denote the equilibrium input-output matrix of economy E^0 by Ω.

Then, Ω has the following properties:

i) $[\Omega]_{ij} = t_{ij}\alpha_{ij}^L \in [0, \alpha_{ij}^L]$ for all $i \in N$, $j \in a_i$.

ii) $[\Omega]_{ij} = \alpha_{ij}^n$ for all $i \in N$, $j \in n_i$.

iii) $[\Omega]_{ij} = 0$ for all $i \in N$, $j \in N \setminus \{a_i \cup n_i\}$.

In economy E^0, consider the set $\left[S_j^0\right]$ such that there exists a directed upstream supply path from each $s \in S_j^0 \setminus j$ to j and $j \in S_j^0$ as well.

Thus, for any given j, S_j^0 is the set of sectors including sector j and its all direct and indirect suppliers in Economy E^0. In addition, denote the set of sectors that has no upstream supply path to sector j by $\left\{S_j^0\right\}^C$. For any given economy, we can find these sets for each $j \in N$.

Step 1) First, we show that if there exists any $j \in N$ such that $\left\{S_j^0\right\}^C \neq \emptyset$, then $D_{kl}^0 = [(I - \Omega')^{-1}]_{kl} = 0$ holds for all pairs $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$.

In order to show this, first, we show the condition below holds:

- $D_{kl}^0 > 0$ if and only if there exists a directed path from k to l.

In order to show this, we use Lemma 1.

Take any $k \in \left\{S_j^0\right\}^C$. For any such k, there exists no directed upstream path from k to any $l \in S_j^0$ holds. Otherwise, if there exists a directed path from k to at least one $l \in S_j^0$, then $k \in S_j^0$ must hold as well.

Then, by Lemma 1:

$D_{kl} = \sum_{i \neq k} D_{ki}\Omega_{li}$ for all pairs $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$.

For any $\Omega_{li} > 0$, $i \in S_l^0$. If $i \in S_l^0$, then $i \in S_j^0$ also holds since there exists a directed path from i to l and from l to j.

Then, for each ordered pair $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$, we have a set of equations:

$[D_{kl}]_{l \in S_j^0} = \sum_{i \in S_j^0} D_{ki}\Omega_{li}$

The same argument applies for each $k \in \left\{S_j^0\right\}^C$. Then, we have a system of equations, which has a unique solution. Otherwise, for given Ω, the matrix D wouldn't be unique as well, because Lemma 1 implies that the set of equations above is the full set of equations that consists any $[D_{kl}]_{k \in \left\{S_j^0\right\}^C, l \in S_j^0}$.

$D_{kl} = 0$ for each ordered pair $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$ is a solution, which gives us the unique values for each such D_{kl}.

Next, we show that $D_{kl}^0 > 0$ if there exists a directed path from k to l.

Suppose that there exists at least one directed upstream path from k to l. Consider any of these directed paths from k to l, and order the firms in a selected directed path as follows: $S_{kl}^0 = \{i_0, i_1, i_2, .., i_n\}$ where $i_0 = k$ and $i_n = l$, and $\Omega_{i_{t+1}, i_t} > 0$ for all $0 \leq t \leq n - 1$.

Then, by using Lemma 1:

$D_{i_0 i_1} = D_{i_0 i_0}\Omega_{i_1 i_0} + \sum_{j \neq i_0} D_{i_0 j}\Omega_{i_1 j}$

Moreover, Lemma 1 implies that $D_{ii} \geq 1$ for all i. Thus, for $\Omega_{i_1 i_0} > 0$, $D_{i_0 i_1} > 0$ holds.

Lastly, the property $D_{ij} = \sum_{k \neq i} D_{ik}\Omega_{jk}$ for all $i, j : \Omega_{ji} = 0$ (i is not a direct supplier of j) implies that $D_{i_j i_{j+t}} > 0$ holds for all $t \leq n - j$, which further implies $D_{kl} > 0$.

Step 2) Next, consider the economy E^* that is more automated than E. Then, the following conditions hold for the equilibrium input-output network at the economy E^*:

i) $[\Omega^*]_{ij} \geq [\Omega]_{ij} = t_{ij}\alpha^L_{ij} \in [0, \alpha^L_{ij}]$ for all $i \in N$, $j \in a_i$.

ii) $[\Omega^*]_{ij} = t^*_{ij}\alpha^L_{ij} > [\Omega]_{ij} = t_{ij}\alpha^L_{ij}$ for some $i \in N$ for some $j \in a_i$.

iii) $[\Omega^*]_{ij} = [\Omega]_{ij} = \alpha^n_{ij}$ for all $i \in N$, $j \in n_i$.

iv) $[\Omega^*]_{ij} = [\Omega]_{ij} = 0$ for all $i \in N$, $j \in N \setminus \{a_i \cup n_i\}$.

Take one increase in automation at a time. In order to do that, take any one of the ordered pairs $\{i, j\}$ such that $[\Omega^*]_{ij} > [\Omega]_{ij}$.

Consider the matrix Ω^1 such that Ω^1 differs from Ω only in its $(ij)^{th}$ element, all else equal, where $[\Omega^1]_{ij} = [\Omega^*]_{ij}$. Call it economy E^1.

Similarly, in economy E^1, consider the set $\left[S^1_j\right]$ such that there exists a directed upstream supply path from each $l \in S^1_j \setminus j$ to the given more automated task j. In addition, denote the set of sectors that has no direct upstream supply path to sector j by $\left\{S^1_j\right\}^C$.

If there is no change in the set of existing paths from Economy E^0 to the economy E^1, but only the weight of an existence link increase, then the result in Step 1 above still holds.

Consider that $\Omega_{ij} = 0$ and an increase in Ω_{ij} results in a new upstream link from j to i in Economy E^1. However, in such a case, the set of upstream suppliers of sector j does not change and the following conditions hold:

$S^0_j = S^1_j$

$\left\{S^0_j\right\}^C = \left\{S^1_j\right\}^C$

Thus for any k such that $k \in \left\{S^1_j\right\}^C$ and $l \in S^1_j$, $[D^0]_{kl} = [D^1]_{kl} = 0$ holds.

Next, we show that for each ordered pair $\{k, l\}$ such that $k \in \left\{S^1_j\right\}^C$ and $l \in \left\{S^1_j\right\}^C$, the following condition holds.

$D^1_{kl} = D^0_{kl}$

In order to show this, first, by Lemma 1:

$D^0_{kl} = D^0_{kk}\Omega_{lk} + \sum_{s \neq k} D^0_{ks}\Omega_{ls}$

$D^1_{kl} = D^1_{kk}\Omega^1_{lk} + \sum_{s \neq k} D^1_{ks}\Omega^1_{ls}$

For any $l \in \left\{S^0_j\right\}^C$ and $k \in \left\{S^0_j\right\}^C$, $\Omega_{lk} = \Omega^1_{lk}$ and $\Omega_{li} = \Omega^1_{li}$ holds if $l \neq i$, where i is the sector that uses more of automation good j. On the other hand, if $l = i$, then since Ω_{ij} rises, $D^0_{kj}\Omega_{ij}$ enters into the equation above. However since $D^0_{kj} = D^1_{kj} = 0$ holds, $D^0_{kj}\Omega_{ij} = D^1_{kj}\Omega^1_{ij} = 0$ holds. Moreover, the set $\left\{S^0_j\right\}^C$ remains same. Thus, the system of equations above remain same in both Economy E^0 and economy E^1. Similar to the previous part, there must exists a unique solution for the system of equations above. Therefore, the unique solution in Economy E^0 is exactly the same as in Economy E_1.

Lastly, the vector of Domar weights (so the centralities) is equal to $\mathbf{m} = (I - \Omega')^{-1}\overrightarrow{\beta}$.

Thus, in Economy E^0, $m_i = \sum_j D^0_{ij}\beta_j$, and in economy E^1, we have $m^1_i = \sum_j D^1_{ij}\beta_j$.

For constant β, for any $k \in \left\{S^0_j\right\}^C \left(= \left\{S^1_j\right\}^C\right)$, $\sum_i D^1_{ki}\beta_i = \sum_i D^0_{ki}\beta_i$ holds, which implies that:

$m^1_k = m_k$ holds for all $k \in \left\{S^1_j\right\}^C$.

Step 3) Next, we show that $D^1_{li} > D^0_{li}$ for all $l \in S^0_j$, where i is the sector that increases its automation in task j.

In order to show that,

$[D^0_{li}]_{l \in S^0_j} = D^0_{ll}\Omega_{il} + \sum_{k \neq l} D^0_{lk}\Omega_{ik}$

$[D^1_{li}]_{l \in S^1_j} = D^1_{ll}\Omega^1_{il} + \sum_{k \neq l} D^1_{lk}\Omega^1_{ik}$, which can be rewritten as:

$[D^1_{li}]_{l \in S^0_j} = D^1_{ll}\Omega^1_{il} + \sum_{k \neq l} D^1_{lk}\Omega^1_{ik}$,

For $l = j$, we have

$D^0_{ji} = D_{jj}\Omega_{ij} + \sum_{k \neq l} D_{jk}\Omega_{ik}$

$D^1_{ji} = D^1_{jj}\Omega^1_{ij} + \sum_{k \neq l} D^1_{jk}\Omega^1_{ik}$

Since, each $[D_{ij}]_{i,j \in N}$ is non-decreasing in any element $[\Omega_{kl}]_{k,l \in N}$ and since $\Omega^1_{ij} > \Omega_{ij}$, we conclude from above that $D^1_{ji} > D^0_{ji}$.

Next, consider any $l \in S^1_j$ and $l \neq j$:

$[D^0_{li}]_{l \in S^0_j \setminus j} = D^0_{ll}\Omega_{il} + D^0_{lj}\Omega_{ij} + \sum_{k \neq j} D^0_{lk}\Omega_{ik}$

$[D^1_{li}]_{l \in S^1_j \setminus j} = D^1_{ll}\Omega^1_{il} + D^1_{lj}\Omega^1_{ij} + \sum_{k \neq j} D^1_{lk}\Omega^1_{ik}$, which can be rewritten as

$[D^1_{li}]_{l \in S^0_j \setminus j} = D^1_{ll}\Omega^1_{il} + D^1_{lj}\Omega^1_{ij} + \sum_{k \neq j} D^1_{lk}\Omega^1_{ik}$

Next, by combining:

i) for each $l \in S^0_j \setminus j$, $D_{lj} > 0$,

ii) each $[D_{ij}]_{i,j \in N}$ is non-decreasing in any element $[\Omega_{kl}]_{k,l \in N}$, and

iii) $\Omega^1_{ij} > \Omega_{ij}$,

we conclude that $\{D^1_{li}\} > \{D^0_{li}\}$ for each $l \in S_j$.

Thus, any sector k that is direct or indirect supplier of sector j ($k \in S^0_j = S^1_j$) has a higher dependency to sector i.

Lastly, for constant β, for any $l \in S^0_j = S^1_j$, $\sum_i D^1_{li}\beta_i > \sum_i D^0_{li}\beta_i$ holds, which implies that:

$m^1_i > m_i$ holds for all $l \in S_j$.

Step 4) We do this iteration one by one for each increase in automation for an ordered pair $\{i, j\}$ such that $[\Omega^{t+1}]_{ij} > [\Omega^t]_{ij} > 0$ until we reach the equilibrium input-output network Ω^*, where the same results above hold at each step, which concludes the proof.

7.6.1 An Example with Non Cobb-Douglas Production Function

Here, we consider production technologies different than the Cobb-Douglas form that we studied so far. We now illustrate how the impact of technological advances depends on how useful low-skilled labor is in other sectors for other forms of production functions. If low-skilled labor is very productive elsewhere, then the technological advances in the automation technology have a higher impact on total consumption.

EXAMPLE **6** *The production functions are:*

$Y_n = A_n L_n$

$Y_a = A_a$

$Y_f = A_f H_f^{\alpha}(L_{fa} + X_{fa})^{1-\alpha} + X_{fn}$

We set $L = H = 1$.

Here, we simplify things by having X_{fn} enter into the production function of the final good in an additively separable way rather than in a Cobb-Douglas form. The expressions have all of the same signs with the Cobb-Douglas form and we report those in the appendix, but this simplifies the expressions substantially.

We also simplify the automation process not to use any labor at all, so that its increase does not impact the production of the other goods other than via the technological advance.

7.6.2 Example 6

In this simple economy, the equilibrium can be understood by maximizing[13] $Y_f = A_f(L_f + A_a)^{1-\alpha} + A_n(1 - L_f)$ where we use that $L = H = 1$.

The maximizing L_f is

$$\begin{cases} L_f = 1 & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \\ L_f = \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - A_a & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \le A_a \le \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} \\ L_f = 0 & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

and the corresponding Y_f is then

$$\begin{cases} Y_f = A_f(1 + A_a)^{1-\alpha} & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \\ Y_f = A_f\left(\frac{(1-\alpha)A_f}{A_n}\right)^{(1-\alpha)/\alpha} + A_n\left(1 + A_a - \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}\right) & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \le A_a \le \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} \\ Y_f = A_f(A_a)^{1-\alpha} + A_n & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

It then follows that the corresponding $\frac{\partial Y_f}{\partial A_a}$ is:

[13] It is straightforward that the competitive equilibrium in this simple economy is equivalent to a planner maximizing total final good production.

$$\begin{cases} \frac{\partial Y_f}{\partial A_a} = (1-\alpha)A_f(1+A_a)^{-\alpha} & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1, \\ \frac{\partial Y_f}{\partial A_a} = A_n & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 < A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}, \\ \frac{\partial Y_f}{\partial A_a} = (1-\alpha)A_f(A_a)^{-\alpha} & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

Here, we see directly that during automation the rate at which overall production changes in response to technological advances in the automation sector are proportional to the usefulness of labor in the non-automation sector. That is, $\frac{\partial Y_f}{\partial A_a} = A_n$.

Next, we compare the changes in total consumption in different phases. Following the same rate of increase in A_a such that A_a becomes zA_a, the rate of change in Y_f depends on the phase of the economy. The rate of change in $C = Y_f$ during the automation and post-automation phases are given by:

$$(\Delta \text{log} C)^{autom} = log\left(\frac{1+4A_n^2(1+zA_a^{autom})}{1+4A_n^2(1+A_a^{autom})}\right)$$

$$(\Delta \text{log} C)^{post-autom} = log\left(\frac{(zA_a^{post-autom})^{\frac{1}{2}} + A_n}{(A_a^{post-autom})^{\frac{1}{2}} + A_n}\right)$$

Example 6 shows that the rate of change in total consumption rises as the productivity of labor on the alternative uses rises. This result is different than the previous Cobb-Douglas economy example, where only the productivity level of the automation good producer is important, and the level of productivity in resource sector does not play a role in the reallocation effect. However, when we consider the general case for the Cobb-Douglas economy including both direct and indirect substitution effects, the actual levels of productivities in various sectors would play role in determining the reallocation effect. Therefore, as shown in this example, how low-skilled labor is productive in its alternative usage is the main factor that determines how reallocation of labor alters the net-effect of technological changes.

These two examples provide an important lens into why it can be that substitution for labor can have very different effects depending on the alternative uses for labor. These provides new insights into the Solow Paradox and the findings of Brynjolfsson, Rock, and Syverson [27], for instance.

7.7 Discussion of the Reallocation Effects in the Three-Sector Economy

In this part, we consider a discrete technological change in automation sector, $\Delta\left(logA_a^PA_a^Q\right)$, and compare $\Delta logC$ (given by Equation (7)) with $t_{fa}\alpha_{fa}^L\Delta\left(logA_a^PA_a^Q\right)$, where $t_{fa}\alpha_{fa}^L$ is the Domar weight of the automation sector before the technological change.

$$\Delta logC = log\left(\frac{L\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}}{L\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}}\right)$$

$$\left(t_{fa}\alpha_{fa}^L\right)^{before}\Delta logA = log\left(\left(\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}}\right)^{\left(t_{fa}\alpha_{fa}^L\right)^{before}}\right)$$

$$\Delta logC > t_{fa}\alpha_{fa}^L\Delta log\left(A_a^PA_a^Q\right)\ \text{if :}$$

$$log\left(\frac{\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}}{\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}}\right)+log\left(\frac{L+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}\right) > log\left(\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}}\right)^{\frac{-\alpha_f^H-\alpha_n^H\alpha_{fn}^n}{\alpha_a^H}}+$$

$$log\left(\left(\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}}\right)^{\frac{H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H\left(L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}}\right)$$

By further simplification:

$$log\left(\frac{L+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{L+H\left(\left(A_a^PA_a^Q\right)^{before}(\alpha_a^H)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}\right) > log\left(\left(\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}}\right)^{\frac{H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H\left(L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}}\right)$$

Both sides of the equations are greater than 1, so $\Delta logC > t_{fa}\alpha_{fa}^L\Delta log\left(A_a^PA_a^Q\right)$ if :

$$\left(\frac{L+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{L+H\left(\left(A_a^PA_a^Q\right)^{before}(\alpha_a^H)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}\right) > \left(\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}}\right)^{\frac{H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H\left(L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}}$$

or, equivalently,

$$\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}} < \left(\frac{L+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{L+H\left(\left(A_a^PA_a^Q\right)^{before}(\alpha_a^H)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}\right)^{\frac{\alpha_a^H\left(L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}{H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}}$$

$$\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}} < \left(\frac{L+H\left(\left(A_a^PA_a^Q\right)^{after}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{L+H\left(\left(A_a^PA_a^Q\right)^{before}(\alpha_a^H)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}\right)^{\frac{\alpha_a^H\left(L+H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}{H\left(\left(A_a^PA_a^Q\right)^{before}\left(\alpha_a^H\right)^{\alpha_a^H}\left(\alpha_{a0}^L\right)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}}$$

$$\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}} < \left(\frac{s_L^{before}}{s_L^{after}}\right)^{\frac{\alpha_a^H}{1-s_L^{before}}}$$

$$\frac{\left(A_a^PA_a^Q\right)^{after}}{\left(A_a^PA_a^Q\right)^{before}} = \left(\frac{\frac{L}{H}\frac{1-s_L^{after}}{s_L^{after}}}{\frac{L}{H}\frac{1-s_L^{before}}{s_L^{before}}}\right)^{\alpha_a^H} = \left(\frac{1-s_L^{after}}{s_L^{after}}\frac{s_L^{before}}{1-s_L^{before}}\right)^{\alpha_a^H}$$

Then, the condition is:

$$\left(\frac{1-s_L^{after}}{s_L^{after}}\frac{s_L^{before}}{1-s_L^{before}}\right) < \left(\frac{s_L^{before}}{s_L^{after}}\right)^{\frac{1}{1-s_L^{before}}}$$

$$\left(\frac{1-\left(s_L^{before}+\Delta s_L\right)}{s_L^{before}+\Delta s_L}\frac{s_L^{before}}{1-s_L^{before}}\right) < \left(\frac{s_L^{before}}{s_L^{before}+\Delta s_L}\right)^{\frac{1}{1-s_L}}$$

$$\left(1-\left(s_L^{before}+\Delta s_L\right)\right)\left(s_L^{before}+\Delta s_L\right)^{\frac{s_L^{before}}{1-s_L^{before}}} < \left(1-s_L^{before}\right)\left(s_L^{before}\right)^{\frac{s_L^{before}}{1-s_L^{before}}}$$

$$\frac{1-\left(s_L^{before}+\Delta s_L\right)}{1-s_L^{before}} < \left(\frac{s_L^{before}}{s_L^{before}+\Delta s_L}\right)^{\frac{s_L^{before}}{1-s_L^{before}}}$$

which always holds for $\Delta s_L < 0$ and $s_L^{before} + \Delta s_L > 0$.

So, $\Delta logC > t_{fa}\alpha_{fa}^{L}\Delta log\left(A_a^P A_a^Q\right)$ always holds.

7.8 Discussion of the Reallocation Effects in an n-Sector Economy

We can rewrite equation 42 as

$$log\left(\frac{p_i}{w_L}\right) = (I-\Omega)^{-1}\left[logB_i + \vec{\alpha_i^H} log\left(\frac{w_H}{w_L}\right)\right] \tag{47}$$

where Ω is the equilibrium input-output network. Consider an automation good sector j that is in transition, which implies that $p_j = w_L$. Then, following the changes in productivities, $p_j = w_L$ still holds for any such sector j and, thus, $dlog\left(\frac{p_j}{w_L}\right) = 0$ holds. Then, for $K = (I-\Omega)^{-1}$, we have $dlog\left(\frac{w_H}{w_L}\right) = \frac{\sum_{i\in N} K_{ji}\frac{dlogA_i^P}{\alpha_i^H}}{\sum_{i\in N} K_{ji}}$.

Next, we show the change in total consumption during a transition phase.

$$\left[\vec{logp_i}\right] = (I-\Omega)^{-1}\left[logB_i + \alpha_i^L logw_L + \alpha_i^H logw_H\right]$$

We multiply both sides by $\left[\vec{\beta_i}\right]'$ from left, and get

$$\sum_{i\in N}\beta_i logp_i = \sum_{i\in N} m_i logB_i + \sum_{i\in N}\left(\alpha_i^L m_i\right) logw_L + \sum_{i\in N}\left(\alpha_i^H m_i\right) logw_H \tag{48}$$

By taking logs of both sides of $\beta_i C = p_i C_i$, we get:

$$logC + log\beta_i = logC_i + logp_i$$

By multiplying both sides with β_i and summing up, we get:

$$\sum_{i\in N}\beta_i logC + \sum_{i\in N}\beta_i log\beta_i = \sum_{i\in N}\beta_i logC_i + \sum_{i\in N}\beta_i logp_i$$

$C = C_i^{\beta_i}$ implies $\sum_{i\in N}\beta_i logC_i = logC$, which then implies

$$\sum_{i\in N}\beta_i log\beta_i = \sum_{i\in N}\beta_i logp_i \tag{49}$$

By plugging this Equation 49 into the Equation 48, we get:

$$\sum_{i\in N}\beta_i log\beta_i = \sum_{i\in N} m_i logB_i + s_L logw_L + s_H logw_H \tag{50}$$

Then we get

$$\Delta\left(s_L logw_L + s_H logw_H\right) = -\Delta\left(\sum_{i\in N} m_i logB_i\right)$$

$$\Delta\left(s_L logw_L + s_H logw_H\right) = \sum_{i\in N} m_i\left(\triangle logA_i^P\right) - \sum_{i\in N} log\left(B_i^*\right)^{after}\left(\Delta m_i\right)$$

$$(s_L \Delta log w_L + log w_L^* \Delta s_L + s_H \Delta log w_H + log w_H^* \Delta s_H) = \sum_{i \in N} m_i \left(\triangle log A_i^P \right) - \sum_{i \in N} log B_i^* \left(\Delta m_i \right)$$

By using $\Delta log w_L = \Delta log C + \Delta log \left(s_L \right)$, and $\Delta log w_H = \Delta log C + \Delta log \left(s_H \right)$, we can rewrite the equation above as follows:

$$\Delta log C = \sum_{i \in N} m_i \left(\triangle log A_i^P \right) - \sum_{i \in N} log B_i^* \left(\Delta m_i \right) - \left[s_L \left(\Delta log s_L \right) + s_H \left(\Delta log s_H \right) + log w_L^* \left(\Delta s_L \right) + log w_H^* \left(\Delta s_H \right) \right]$$

where

$$B_i^* = \frac{1}{(A_i^P)^* (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L} \left[\prod_{j \in N} (\alpha_{ij}^L)^{\alpha_{ij}^L} \right] \left[\prod_{j \in N} (\alpha_{ij}^n)^{\alpha_{ij}^n} \right]}.$$