Unspanned Risks, Negative Local Time Risk Premiums, and Empirical Consistency of Models of Interest-Rate Claims

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What do we do in this paper?

- We show the relevance of unspanned risks in the pricing kernel process.
- We show the relevance of unspanned risks in the volatility dynamics of futures/bond returns.
- We formalize a form of market incompleteness.
- We introduce the notion of local time risk premiums and argue that it should be negative to match data features.

What is our rationale?

- We build a theory that is relevant to modeling the $\mathbb{P}/\mathbb{Q}$ dynamics of claims to the downside and upside; that is, the risk premiums on the returns of puts and calls.
- Models devoid of unspanned components in the pricing kernel and return volatility are inconsistent with the data from Treasury markets.
- The cornerstone of interest-rate theory is that prices of Treasury bonds, futures on bonds, and options on Treasury bond futures can be characterized by postulating the pricing kernel process and the evolution of the spot interest-rate.

Definition of unspanned risks

- Unspanned risks capture risks not spanned by bond or bond futures returns but are, potentially, spanned by options.

Theoretical framework

Our theoretical treatment, which incorporates a salient role for market incompleteness and volatility via the formalism of Tanaka's formula unmasks the conditions for a negative risk premium on local time, a concept decoupled from unspanned stochastic volatility.

$$
\begin{aligned}
d \log M_t &= -r_t dt - \frac{1}{2}\boldsymbol{\lambda}[t,\mathbf{X}]'\boldsymbol{\lambda}[t,\mathbf{X}]dt + \boldsymbol{\lambda}[t,\mathbf{X}]' \underbrace{d\boldsymbol{\omega}_t^{\mathbb{P}}}_{\text{spanned}} \\
&\quad -\frac{1}{2}\boldsymbol{\alpha}[t,\mathbf{X}]'\boldsymbol{\alpha}[t,\mathbf{X}]dt + \boldsymbol{\alpha}[t,\mathbf{X}]' \underbrace{d\mathbf{u}_t^{\mathbb{P}}}_{\text{unspanned}} , \\
dr_t &= \mu_r[t,\mathbf{X}]dt + \boldsymbol{\sigma}_r[t,\mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}, \\
\frac{dB_t^T}{B_t^T} &= \mu_B[t,T,\mathbf{X}]dt - \boldsymbol{\sigma}_B[t,T,\mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}, \quad \text{and} \\
\frac{dF_t^{T_F}}{F_t^{T_F}} &= \mu_F[t,T_F,\mathbf{X}]dt - \boldsymbol{\sigma}_F[t,T_F,\mathbf{X}]' d\boldsymbol{\omega}_t^{\mathbb{P}}.
\end{aligned}
$$

(term $\boldsymbol{\alpha}[t,\mathbf{X}]' d\mathbf{u}_t^{\mathbb{P}}$ in SDF captures unspanned risks)

Empirical puzzle and properties of average returns of options on Treasury bond futures

We construct the option returns as

$$z^{\mathrm{put}}_{t,T_O,\aleph\%} = \frac{\left(K - F^{T_F}_{T_O}\right)^+}{P_t[K]} \; - \; 1, \text{ where } K \text{ corresponds to } K = F_t^{T_F} e^{-\aleph\%}, \; \aleph = 1\%, 3\%, \text{ and } 5\%,$$

$$z^{\mathrm{call}}_{t,T_O,\aleph\%} = \frac{\left(F^{T_F}_{T_O} - K\right)^+}{C_t[K]} \; - \; 1, \text{ where } K \text{ corresponds to } K = F_t^{T_F} e^{+\aleph\%}, \; \aleph = 1\%, 3\%, \text{ and } 5\%,$$

where $P_t[K]$ ($C_t[K]$) is the **settlement** price of a put (call) on the Treasury bond futures with strike K, as reported by the CME.

Average return of options on Treasury bond futures

	OTM (%)	AVG.	90% Bootstrap CI Lower	90% Bootstrap CI Upper	SD
Panel A: Options on futures of the 10-year Treasury bond (330 observations)					
Puts	5	-93	-99	-81	110
Puts	3	-71	-91	-45	260
Puts	1	-41	-54	-26	160
Straddle		-11	-18	-3	80
Calls	1	-11	-27	6	180
Calls	3	-76	-88	-63	140
Calls	5	-91	-99	-82	90
Panel B: Options on futures of the 30-year Treasury bond (337 observations)					
Puts	5	-80	-96	-61	200
Puts	3	-58	-73	-40	180
Puts	1	-28	-42	-13	160
Straddle		-9	-16	-2	80
Calls	1	-3	-22	18	220
Calls	3	-17	-59	37	540
Calls	5	-58	-92	-10	480

Pairwise bootstrap p-values for average return differences across moneyness

Null hypothesis	Underlier of option is futures on 10-year bond	Underlier of option is futures on 30-year bond
$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$	0.000	0.000
$\overline{z}^{\text{put}}_{t,T_O,5\%} - \overline{z}^{\text{put}}_{t,T_O,3\%} \geq 0$	0.000	0.002
$\overline{z}^{\text{put}}_{t,T_O,3\%} - \overline{z}^{\text{put}}_{t,T_O,1\%} \geq 0$	0.009	0.000
$\overline{z}^{\text{call}}_{t,T_O,3\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0$	0.000	0.235
$\overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,3\%} \geq 0$	0.000	0.038
$\overline{z}^{\text{call}}_{t,T_O,5\%} - \overline{z}^{\text{call}}_{t,T_O,1\%} \geq 0$	0.000	0.018

Dollar open interest and dollar trading volume

Options on	Begin date	End date	NOBS	OTM puts		OTM calls	
				Dollar open interest	Dollar trading volume	Dollar open interest	Dollar trading volume
10-year bond futures	7/31/1991	12/31/2018	330	66.9	8.9	55.9	7.2
30-year bond futures	12/31/1990	12/31/2018	337	27.6	3.6	27.1	3.4
S&P 500 equity index	1/31/1990	12/31/2018	348	63.5	8.4	53.1	6.9

Roadmap and Summary

- We present an empirical puzzle in the Treasury bond futures market: Average returns to claims on downside and upside are both negative.

- We show that workhorse models cannot reproduce our empirical findings for claims on the upside of futures return.

- We consider a pricing kernel process that has both spanned and unspanned risks. The presence of unspanned risks and volatility uncertainty is crucial.

- We use Tanaka's formula (under $\mathbb{P}$ and $\mathbb{Q}$) to build model-free general characterizations.

What do models say?

Nearly all extant models of term structure imply that

Expected excess return of puts is negative, while that of calls is positive (the latter coincides with the sign of futures risk premium)!

THIS IS NOT CONSISTENT WITH OUR EMPIRICAL EVIDENCE

Workhorse model classes

- Vasicek (single and multiple-factor)

$$
\begin{aligned}
dr_t &= (\theta^{\mathbb{P}} - \kappa^{\mathbb{P}}\, r_t)\, dt + \sigma\, d\omega_t^{\mathbb{P}} \qquad \text{under } \mathbb{P}, \text{ and} \\
dr_t &= (\theta_t^{\mathbb{Q}} - \kappa^{\mathbb{Q}}\, r_t)\, dt + \sigma\, d\omega_t^{\mathbb{Q}} \qquad \text{under } \mathbb{Q}.
\end{aligned}
$$

- Cox-Ingersoll-Ross (single and multiple-factor formulations)
- Quadratic term structure models
- Long-run risks
- Rare disasters

We show that in all of these models, if the futures risk premium is positive, then the expected excess return of call on bond futures is positive.

We also show that the risk premium on local time is not negative in these models as required by our developed theory.

Tanaka Formula and Expectations under $\mathbb{P}$ and $\mathbb{Q}$

- Let $G_s \equiv \frac{F_s^{T_F}}{F_t^{T_F}}$ be the gross return on the bond futures price over the time period t to s.

Tanaka's formula for continuous semimartingales implies that

$$[\hbar(G_u - k)]^+ = \underbrace{[\hbar(G_t - k)]^+}_{\text{Intrinsic value} \equiv \mathbb{I}_t[\hbar,k]} + \underbrace{\hbar \int_t^u \mathbb{1}_{\{\hbar G_\ell > \hbar k\}} dG_\ell}_{\text{Gain/loss process}} + \underbrace{\mathbb{L}[k; t, u, \langle G \rangle]}_{\text{Local time}}.$$

Consider the case of a put option with $\hbar = -1$.

- First, $\mathbb{I}_t[-1, k] \equiv [-(G_t - k)]^+ = [k - G_t]^+$ is the intrinsic value
- the term $-\int_t^u \mathbb{1}_{\{G_\ell < k\}} dG_\ell = -\frac{1}{F_t^{T_F}} (\int_t^u \mathbb{1}_{\{F_\ell^{T_F} < K\}} dF_\ell^{T_F})$ is a stochastic integral which represents the gains/losses to a dynamic trading strategy which takes a short position of magnitude $\frac{1}{F_t^{T_F}}$ at time ℓ, in the futures, if, and only if, $F_\ell^{T_F} < K$.
- Finally, the quantity $\mathbb{L}[k; t, u, \langle G \rangle]$ is the local time of G at k, which, in turn, is related to the quadratic variation $\langle G \rangle_t$ of (G_t).

Risk Premium on Local Time

The quantity

$\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}[k; t, u, \langle G \rangle]) - \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}[k; t, u, \langle G \rangle])$ defines the risk premium on local time.

Quadratic variation (i.e., $\langle G \rangle$) and local time are sample path properties and do not vary with the probability measures $\mathbb{P}$ or $\mathbb{Q}$. However, their expectations may differ under $\mathbb{P}$ and $\mathbb{Q}$.

Interpretation of Local time Risk Premiums and Relation to Risk Premiums of Dispersion Uncertainty

We can show that

$$\underbrace{\mathbb{E}_t^{\mathbb{P}}\Big(\Big\{\log\frac{F_{T_O}^{T_F}}{F_t^{T_F}}\Big\}^2\Big) - \mathbb{E}_t^{\mathbb{Q}}\Big(\Big\{\log\frac{F_{T_O}^{T_F}}{F_t^{T_F}}\Big\}^2\Big)}_{\text{risk premium on squared log contract}} \approx \int_0^\infty \omega[k]\,\underbrace{\{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}_t^{T_O}[k]) - \mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}_t^{T_O}[k])\}}_{\text{risk premium on local time}}\,dk,$$

$$\text{where } \omega[k] \equiv \frac{2}{k^2}(1-\log k),$$

Result 1

Result 1 (Expected excess returns of OTM options on bond futures)

Assume that

$$\mathbb{E}_t^{\mathbb{P}}(\int_t^{T_O} \mathbb{1}_{\{G_\ell > k\}} dG_\ell) \text{ is positive and } \mathbb{E}_t^{\mathbb{P}}(-\int_t^{T_O} \mathbb{1}_{\{G_\ell < k\}} dG_\ell) \text{ is negative.}$$

(a) **(Absence of unspanned risks; market is complete)**. *If*

$$\boldsymbol{\alpha}[t, \mathbf{X}] = \mathbf{0}, \text{ for all } t, \qquad \text{(no unspanned risks)}$$

then $\mathrm{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]) = 0$, *and the local time risk premium is zero. Furthermore, the expected excess return of an OTM put (call) option on bond futures is negative (positive).*

(b) **(With spanned and unspanned risks; market is incomplete)**. *Suppose, for all* t,

$$\boldsymbol{\alpha}[t, \mathbf{X}] \neq \mathbf{0} \quad \text{and} \quad \mathrm{cov}_t^{\mathbb{Q}}(\frac{M_t}{M_{T_O}} e^{-\int_t^{T_O} r_\ell d\ell}, \mathbb{L}_t^{T_O}[k; \langle G \rangle]) \ < \ 0. \tag{2}$$

The local time risk premium is negative. In this case, the expected excess return of OTM puts and calls can both be negative.

Result 2

Result 2

The percentage risk premium on local time when $k = \frac{K}{F_t^{T_F}} = 1$ *can be inferred from the expected (excess) return of straddles as*

$$\overbrace{\left.\frac{\mathbb{E}_t^{\mathbb{P}}(\mathbb{L}[k;t,T_O,\langle G\rangle])}{\mathbb{E}_t^{\mathbb{Q}}(\mathbb{L}[k;t,T_O,\langle G\rangle])}\right|_{k=1} - 1}^{\text{Percentage risk premium on local time}} \approx \frac{1}{2} B_t^{T_O} \overbrace{\left(\frac{\mathbb{E}_t^{\mathbb{P}}([F_{T_O}^{T_F} - F_t^{T_F}]^+ + [F_t^{T_F} - F_{T_O}^{T_F}]^+)}{C_t[F_t^{T_F}] + P_t[F_t^{T_F}]} - \frac{1}{B_t^{T_O}} \right)}^{\text{Expected excess return of straddles}}.$$

Empirical data says that the expected (excess) return and hence the risk premium on local time is negative and statistically significant.

Model with a negative risk premium on local time

We now focus on exploring a model design that can be consistent with the data on average returns of options on Treasury bond futures. Encapsulated within this model class is market incompleteness combined with sources of volatility uncertainty and negative risk premiums on local time.

In what follows, we partition the vector of state variables into two sets: $\mathbf{X}_t \equiv [\mathbf{Z}_t' \ \mathbf{U}_t']'$, where $\mathbf{Z}_t$ (respectively, $\mathbf{U}_t$) are spanned (unspanned) by bond returns.

Let t_0 be some arbitrary initial date. Define the quantity

$$M_t^{\mathrm{unspanned}} \ \equiv \ \exp\Big(\int_{t_0}^{t} -\frac{1}{2}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]^{'}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]d\ell \ + \ \boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]^{'}\,d\mathbf{u}_\ell^{\mathbb{P}}\Big).$$

Thus, for $T \geq t$, we have
$\frac{M_T^{\mathrm{unspanned}}}{M_t^{\mathrm{unspanned}}} = \exp(\int_t^T -\frac{1}{2}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]^{'}\boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]d\ell + \boldsymbol{\alpha}[\ell,\mathbf{Z},\mathbf{U}]^{'}\,d\mathbf{u}_\ell^{\mathbb{P}})$.
$M_t^{\mathrm{unspanned}}$ is a martingale.

Continued

Next, we construct the pricing kernel M_t as follows:

$$
\begin{aligned}
M_t &= \overbrace{e^{-\nu t}\,(\phi + \boldsymbol{\psi}'\,\mathbf{Z}_t)}^{\text{as in Filipovic-Larsson-Trolle (2017 JF)}} \times M_t^{\text{unspanned}}, \quad \text{with} \\
d\,Z_{i,t} &= (\boldsymbol{\kappa}_Z(\overline{\mathbf{Z}} - \mathbf{Z}_t))_i\,dt + \sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t]\,\underbrace{d\omega_{i,t}^{\mathbb{P}}}_{\text{spanned}}, \qquad \text{for } i = 1, \ldots, N.
\end{aligned}
$$

We then have

$$
\begin{aligned}
\frac{dM_t}{M_t} &= -r_t\,dt + \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}]'\,d\boldsymbol{\omega}_t^{\mathbb{P}} + \underbrace{\boldsymbol{\alpha}[t, \mathbf{Z}, \mathbf{U}]'\,d\mathbf{u}_t^{\mathbb{P}},}_{\text{term not in Filipovic, Larsson, and Trolle}} \qquad \text{with} \\
r_t &= \nu - \frac{\boldsymbol{\psi}'\,\boldsymbol{\kappa}_Z(\overline{\mathbf{Z}} - \mathbf{Z}_t)}{\phi + \boldsymbol{\psi}'\mathbf{Z}_t} \quad \text{and} \quad \boldsymbol{\lambda}[t, \mathbf{Z}, \mathbf{U}] = \frac{\sum_{i=1}^{N} \sigma_{i,Z}[\mathbf{Z}_t, \mathbf{U}_t]\psi_i}{\phi + \boldsymbol{\psi}'\mathbf{Z}_t}.
\end{aligned}
$$

Our paper shows that this model can be consistent with the data.

Conclusions

- We present an empirical puzzle in the Treasury bond futures market: Average returns to claims on downside and upside are both negative.
- We develop a theoretical framework with unspanned risks and volatility uncertainty to understand the empirical puzzle.
- We use Tanaka's formula (under $\mathbb{P}$ and $\mathbb{Q}$) to build model-free general characterizations.

- Show that many extant models (e.g., Long run risks, Quadratic, one-factor and multi-factor Vasicek and CIR, rare disasters) have implications counterfactual to our empirical findings from Treasury options markets.
- We consider a model class, with unspanned risks and negative local time risk premiums, that can be consistent with the data.