Estimation of a Nonparametric model for Bond Prices from Cross-section and Time series Information

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Overview

- Develop estimation methodology for an **additive nonparametric panel** model suitable for capturing **the pricing of coupon-paying government bonds** over many periods.
- The novelty lies in the combination of (1) **cross-sectional nonparametric** methods and (2) kernel estimation for **time varying dynamics**.
- Estimate the yield curve and its dynamics and Predict **individual bond prices** given the full payment schedule.
- **Asymptotic results** are provided and **simulations** and **US bonds application** show strong performance of the proposed method.

Bond pricing: Panel data framework

Discrete time semiparametric model:

$$p_{it} = \sum_{j=1}^{m_{it}} c_{it}(\tau_{ij}) d(\tau_{ij}, X_t) + \varepsilon_{it}, \qquad (1)$$

where $c_{it}(\tau_{ij})$ are the cash flows at future payment dates, $\{\tau_{ij}\}_{j=1}^{m_{it}}$ for each bond i.

- $X_t \in \mathbb{R}^L$, with $L \ll n(= \min_t n_t)$: (possibly stochastic) **observable covariates** or **factors**
- ε_{it}: a conditional (on X_t) mean zero **pricing error**
- $d(\cdot)$: **discount function** unspecified but smoothly varying over its arguments.

Modelling issues

- Discount (d), yield (y) and forward curves (f)

$$d(\tau, x) = e^{-\tau y(\tau, x)}, \quad f(\tau, x) = -\frac{d'(\tau, x)}{d(\tau, x)}$$

- Diebold and Li (2006):

$$d_{\theta_t}(\tau) = \exp\left(-\tau y_{\theta_t}(\tau)\right) \qquad (2)$$

$$y_{\theta_t}(\tau) = \sum_{j=0}^{2} \beta_{jt}\varphi_j(\tau; \tau_j),$$

where $\varphi_0(\tau; \tau_0) = 1$, for some τ_1 and τ_2, $\varphi_1(\tau; \tau_1) = (1 - \exp(-\tau/\tau_1))(\tau_1/\tau)$, and $\varphi_2(\tau; \tau_2) = \varphi_1(\tau; \tau_2) - \exp(-\tau/\tau_2)$.

- Relevance of our model to Diebold and Li:
  - If X_t are the unobserved dynamic parameters β_t, Diebold and Li is a special case of our model
  - For the case of observable X_t,

$$y(\tau, X_t) = \sum_{j=1}^{L} \beta_j(X_{jt})\varphi_j(\tau), \qquad (3)$$

where $\beta_j(.)$, $\varphi_j(.)$ are smooth but unknown functions

- Static and Dynamic Arbitrage restrictions
  - Under the modelling setting, no static arbitrage opportunities exist but without further restriction, there may be dynamic arbitrage opportunities.
  - Gouriéroux et al. (2002) under dynamic no arbitrage restriction: $d_t(\tau) = \exp(a_\tau^\top X_t + b_\tau)$, where a_τ and b_τ satisfy nonlinear first order difference equations, which is a special case of our model

Forecasting Future Bond Prices

Forecasting bond prices within the model

- Case where $X_t = t/T$, i.e. the deterministic evolution of the yield curve

$$d\left(\tau, \frac{T+k}{T}\right) = d(\tau, 1) + d_2(\tau, 1)\frac{k}{T} + o(kT^{-1}),$$

- Special case in relation to Diebold and Li

$$y(\tau, t/T) = \sum_{j=1}^{L} \beta_{jt}\varphi_j(\tau),$$

where the quantities $\beta_{jt} = \beta_j(t/T)$ are deterministically slowly time varying or have an AR structure.

- Predictive regression whereby the stochastic X_t that enter d, and hence y, are taken to be lagged values
- Let

$$\widehat{g}(\tau, x) = \frac{\sum_{t=1}^{T-1}\sum_{i=1}^{n_t} K_h(\tau - \tau_i)K_h(x - X_t)\widehat{d}(\tau_i, X_{t+1})}{\sum_{t=1}^{T-1}\sum_{i=1}^{n_t} K_h(\tau - \tau_i)K_h(x - X_t)}.$$

Then define the forecast of $p_{i,T+1}$

$$\widehat{p}_{i,T+1|T} = \sum_{j=1}^{m_{i,T+1}} c_{ij,T+1}\widehat{g}(\tau_{ij}, X_T),$$

Estimation: Local constant smoothing

- For s_i. in the neighbourhood of τ_i.,

$$Q_{nT}(d) = \sum_{t=1}^{T}\sum_{i=1}^{n_t}\int\left\{p_{it} - \sum_{j=1}^{m_{it}} c_{it}(\tau_{ij})d(s_{ij}, x)\right\}^2 \prod_{k=1}^{m_{it}}\{K_h(s_{ik} - \tau_{ik})ds_{ik}\}\,\mathscr{K}_h(x - X_t)dx, \qquad (4)$$

where $\mathscr{K}_h(s) = \prod_{l=1}^{L} K_{h_l}(s_l)$ with a bandwidth parameter h_l and $K_{h_l}(\cdot) = K(\cdot/h_l)$.

- The estimator of the discount function is the minimizer of $Q_{nT}(\cdot)$ such that

$$\hat{d}(.) = \arg\min_{d(\cdot)\in\mathscr{D}} Q_{nT}(d) \qquad (5)$$

where $\mathscr{D}$ is the class of all functions for which $Q(\cdot)$ is well defined.

- Obtaining F.O.C
  - Let $\delta_{(\tau,x)}(\cdot,\cdot)$ be the $(L+1)$-dimensional Dirac delta function at (τ, x) such that $\int_{r\in\mathbb{R}^L}\int_{y\in\mathbb{R}^+}\delta_{(\tau,x)}(y, r)g(y, r)dydr = g(\tau, x)$ for any generic function $g(\cdot,\cdot)$ continuous at the point (τ, x)
  - Let $d(\cdot,\cdot) = \hat{d}(\cdot,\cdot) + \eta\delta_{(\tau,x)}(\cdot,\cdot)$, and differentiate $Q_{nT}(d)$ with respect to η at the point of $\eta = 0$.
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$$\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}(\tau_{ij})^2 K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)\hat{d}(\tau, x) - \sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} p_{it}c_{it}(\tau_{ij})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)$$

$$= -\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}}\sum_{\substack{p=1\\p\neq j}} c_{it}(\tau_{ij})c_{it}(\tau_{ip})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)\int\hat{d}(\tau', x)K_h(\tau' - \tau_{ip})d\tau'. \qquad (6)$$

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$$\hat{d}(\tau, x) = \bar{d}(\tau, x) + \int\hat{H}(\tau, \tau', x)\hat{d}(\tau', x)d\tau', \qquad (7)$$

where:

$$\bar{d}(\tau, x) = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} p_{it}c_{it}(\tau_{ij})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)}{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}(\tau_{ij})^2 K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)}, \qquad (8)$$

$$\hat{H}(\tau, \tau', x) = -\frac{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}}\sum_{p=1,p\neq j}^{m_{it}} c_{it}(\tau_{ij})c_{it}(\tau_{ip})K_h(\tau - \tau_{ij})K_h(\tau' - \tau_{ip})\mathscr{K}_h(x - X_t)}{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}(\tau_{ij})^2 K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)}. \qquad (9)$$

Alternatives and other variations

- Linking between bond yields and macro variables

$$\mathscr{Q}_{nT}(y) = \sum_{t=1}^{T}\sum_{i=1}^{n_t}\int\left\{p_{it} - \sum_{j=1}^{m_{it}} c_{it}(\tau_{ij})\exp\{-s_{ij}y(s_{ij}, x)\}\right\}^2 \times \prod_{k=1}^{m_{it}}\{K_h(s_{ik} - \tau_{ik})ds_{ik}\}\,\mathscr{K}_h(x - X_t)dx. \qquad (10)$$

Large sample properties

$$\mathscr{B}_{\hat{d}}^*(\tau, x) = \tilde{\mathscr{B}}_{\hat{d}}^*(\tau, x) + \tilde{\tilde{\mathscr{B}}}_{\hat{d}}^*(\tau, x),$$

$$\tilde{\mathscr{B}}_{\hat{d}}^*(\tau, x) = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}^2(\tau_{ij})[d(\tau_{ij}, X_t) - d(\tau, x)]K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)}{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}^2(\tau_{ij})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)},$$

$$\tilde{\tilde{\mathscr{B}}}_{\hat{d}}^*(\tau, x) = \frac{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}}\sum_{p=1,p\neq j}^{m_{it}} c_{it}(\tau_{ij})c_{it}(\tau_{ip})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)\int K_h(\tau' - \tau_{ip})\left[d(\tau_{ip}, x) - d(\tau', x)\right]d\tau'}{\sum_{t=1}^{T}\sum_{i=1}^{n_t}\sum_{j=1}^{m_{it}} c_{it}^2(\tau_{ij})K_h(\tau - \tau_{ij})\mathscr{K}_h(x - X_t)},$$

$$\mathscr{V}_{\hat{d}}(\tau, x) = (nTh^{L+1})\mathrm{var}\left[\sum_{t=1}^{T}\sum_{i=1}^{n_t}\omega_{it}(\tau, x)\varepsilon_{it}\right], \qquad (11)$$

Theorem

Suppose that assumptions (A1)-(A4) and (B1)-(B2) hold. Then,

$$\sqrt{nTh^{L+1}}\mathscr{V}_{\hat{d}}(\tau, x)^{-1/2}\left(\hat{d}(\tau, x) - d(\tau, x) - \mathscr{B}_{\hat{d}}^*(\tau, x)\right) \xrightarrow{d} \mathscr{N}(0, 1). \qquad (12)$$

Corollary

Suppose that all assumptions for Theorem 1 hold. Then,

$$\sqrt{nTh^{L+1}}\mathscr{V}_{\hat{y}}(\tau, x)^{-1/2}\left(\hat{y}(\tau, x) - y(\tau, x) - \mathscr{B}_{\hat{y}}^*(\tau, x)\right) \xrightarrow{d} \mathscr{N}(0, 1),$$

where $\mathscr{B}_{\hat{y}}^*(\tau, x) = (\tau d(\tau, x))^{-1}\mathscr{B}_{\hat{d}}^*(\tau, x)$ *and* $\mathscr{V}_{\hat{y}} = (\tau d(\tau, x))^{-2}\mathscr{V}_{\hat{d}}(\tau, x)$ *with* $\mathscr{V}_{\hat{d}}(\tau, x)$ *specified in (11).*

$$\Sigma_{\hat{d}}(\tau, x) = \lim_{n,T\to\infty}\mathrm{var}\left[\frac{1}{\sqrt{nTh^{L+1}}}\sum_{t=1}^{T}\sum_{i=1}^{n_t}\omega_{it}(\tau, x)\varepsilon_{it}\right]. \qquad (13)$$

$$\hat{V}_{\hat{d}}(\tau, x) = \frac{1}{nTh^{L+1}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{j=1}^{n_t}\sum_{i=1}^{n_t}\omega_{it}(\tau, x)\omega_{js}(\tau, x)\hat{\varepsilon}_{it}\hat{\varepsilon}_{js}.$$

Theorem

Suppose that assumptions (A1)-(A4), (B1)-(B2) and (C1)-(C2) hold. Then, as $n, T \to \infty$

$$\hat{V}_{\hat{d}}(\tau, x) \xrightarrow{p} \Sigma_{\hat{d}}(\tau, x).$$

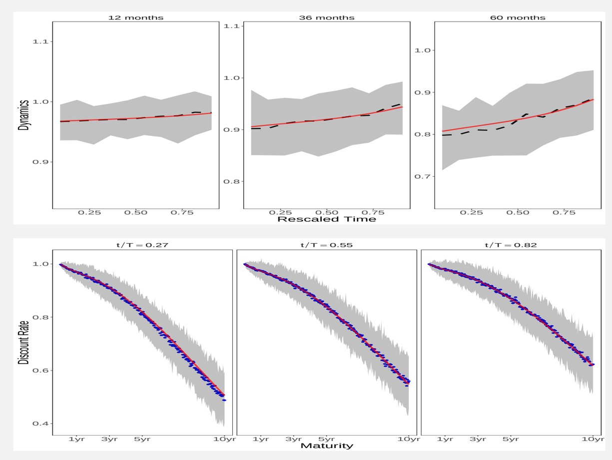
Monte Carlo Study

Simulation design

- Time horizon of 10 years of bi-weekly data
- Number of bonds to $n = 24$ daily on average
- Face value: 100
- Zero coupon bonds (1-12 months), Coupon bonds (1-10 years, biannual payment)
- Replace each expired bond with a new one in the same data structure but with a different identification number
- Bond prices are generated by (1) and for a given i, $\varepsilon_{\cdot t}$ has a ARMA(1,1) structure with AR coefficient (−0.1) and MA coefficient (0.2) to allow for temporal dependence
- Variance is set to increase over duration across bond types
- Discount function is generated from (2) with the parameter vector $\beta_0 = 0$, $\beta_1 = 0.05$, $\beta_2 = 2$, $\tau_1 = 0.75$ and $\tau_2 = 125$.

Simulation outputs

Figure: Cubic time-dynamics of the discount curve



US yield curve evolution

Data description

- Daily US treasury data from CRSP
- Seven-year-period from Jan. 2001 to Dec. 2007
- Deterministic time ($u = t/T$) and three-month treasury bill rates (r) as factors

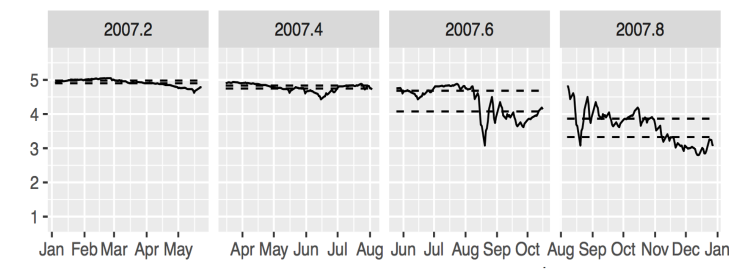


Figure: Three-month Treasury Bill Rates

Estimation results

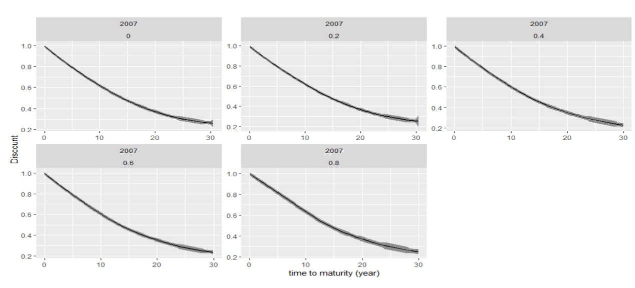


Figure: Estimates for the discount function $\hat{d}(\tau, x)$ with pointwise confidence intervals

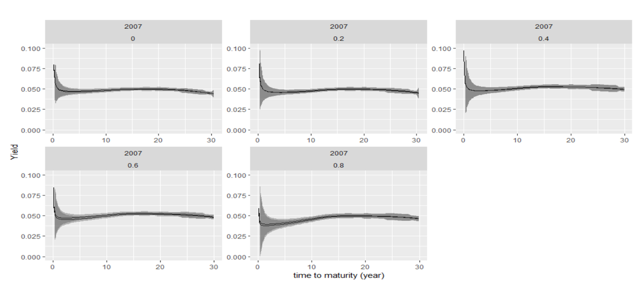


Figure: Estimates for the yield curve $\hat{y}(\tau, x)$ with pointwise confidence intervals

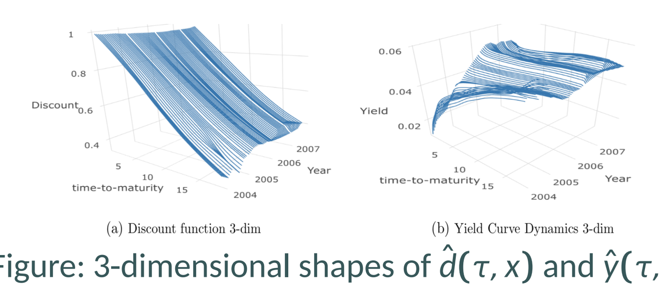


Figure: 3-dimensional shapes of $\hat{d}(\tau, x)$ and $\hat{y}(\tau, x)$

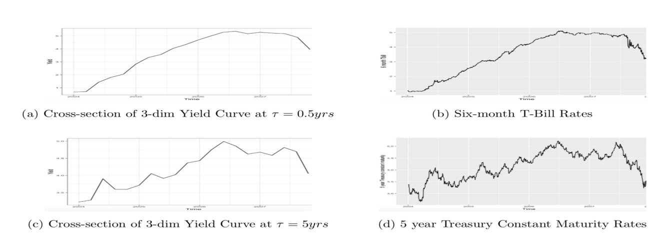


Figure: Cross-sections of $\hat{y}(\tau, x)$ at 6 month and 5 year time-to-maturities

Codes and Package

r-package `ycevo` has been developed in line with this paper and is available at https://github.com/bonsook/ycevo