Identifying indicators of systemic risk

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Research question

Objective of macroprudential policy: Address systemic risk.

Which indicators measure systemic risk?

- Crucial to inform/evaluate the setting of policies.
- Crucial to understand interaction of macroprudential with monetary policy.
- However, plethora of indicators have been constructed revealing disagreement/uncertainty about (i) indicators and (ii) the concept of systemic risk.

This paper tries to fill this gap as objectively as possible:

1. Start from the official definition of systemic risk
2. Derive principles for an indicator of systemic risk
3. Map it into a two-stage hierarchical testing framework
4. Conduct inference on a set of candidate indicators for the G7

Contributions and Results

Contributions

- Guidance on which indicators qualify for monitoring systemic risk
- Testing framework which is straightforward to implement and easy to interpret
- Enhance our understanding of systemic risk by screening a set of indicators

Results

- Credit-to-GDP gap is not an indicator of systemic risk
- Composite measure of financial cycle performs best in our test
- Individual components of financial cycle do not pass test
- Financial conditions indices do not pass test
- Results are robust to various modifications of our test

Overview

1. Introduction
2. Definition of systemic risk
3. Two-stage hierarchical testing framework
4. Application to G7 data
5. Robustness

Start: Definition of systemic risk

In their report to the G20 finance ministers in 2009, IMF, BIS, and FSB define systemic risk as a

> *"risk of disruption to financial services that is (i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences for the real economy"*

Principle 1

> **"risk of disruption to financial services** *that is (i)* **caused by an impairment** *of all or parts* **of the financial system** *and (ii) has the potential to have serious negative consequences for the real economy"*

- **Risk:** Today's probability of an event in the future (time dimension of systemic risk).
- **Event:** "…disruption to financial services caused by …"

Principle 1:

- **An indicator of systemic risk has to measure as of today the probability of a future event that qualifies as a disruption to financial services caused by an impairment of the financial system.**

Principle 2

> **"risk of disruption** *to financial services that is (i) caused by an impairment of all or parts of the financial system and (ii)* **has the potential to have serious negative consequences for the real economy"**

- **Risk of disruption must affect real economy:** Not all disruptions need to feed into systemic risk.
- **Potential consequences:** Future distribution of real economic variables.
- **Serious negative:** Left tail of the distribution of real economic variables.

Principle 2:

- **The probability of a future disruption must be negatively related to the left tail of real economic variables.**

Hierarchical testing framework: Stage 1

Stage 1:

- Draw on early-warning literature of financial crises (e.g. Demirgüç-Kunt and Detragiache (1998), Kaminsky and Reinhart (1999))
- Logit regression:

$$\text{logit}(\pi_{t,t+h}) = \alpha + \sum_{k=0}^{K} \beta_k x_{t-k} \tag{1}$$

  - $\text{logit}(\pi_{t,t+h}) = \ln(\pi_{t,t+h}/(1-\pi_{t,t+h}))$
  - $\pi_{t,t+h} = P(d_{t+h} = 1|\mathcal{F}_t)$
  - d_t: Disruption dummy
  - x_t: Candidate indicator of systemic risk
  - h: For various horizons

- Candidate passes if $\exists k$ s.t. $\beta_k \neq 0$ (likelihood ratio test)

Hierarchical testing framework: Stage 2

Stage 2:

- Draw on growth-at-risk literature (e.g. Adrian et al. (2019))
- Quantile regression at quantile $\tau = 5\%$:

$$y_{t+h} = \gamma_\tau + \delta_\tau \widehat{\pi}_{t,t+h} + \boldsymbol{\omega}_\tau \mathbf{z}_t + \varepsilon_{t+h} \qquad (2)$$

  - y_{t+h}: GDP growth in $t+h$.
  - $\widehat{\pi}_{t,t+h}$ (from Stage 1): Risk of disruption in $t+h$.
  - $\mathbf{z}_t$: Lagged GDP growth

- Candidate passes if $\delta_\tau < 0$ (one-sided t-test with adjusted standard errors).
- Complement by mean regression ($\delta_\tau < \delta$), investigate notion of "severe negative consequences"

Adjusted standard errors

Predicted probability from Stage 1 is a generated regressor:
$\Longrightarrow$ Adjust standard errors of Stage 2.

1. Starting point: Maximum likelihood framework of Murphy and Topel (JBES 1985 & 2002).
2. Potentially error terms on Stage 2 not identically distributed: Extend general formulas to quasi-MLE Technical details
3. Case: "logit + quantile regression" based on quasi-MLE framework in Komunjer (J Econometrics 2005) Technical details
4. Case: "logit + linear regression" straightforward Technical details

Data: Candidate indicators of systemic risk (G7 countries)

1. Credit growth (non-financial private sector)
   (e.g. Schularick and Taylor (AER 2012))
2. House price growth (residential property prices)
   (e.g. Jorda, Schularick, and Taylor (JME 2015))
3. Stock returns (country indices)
   (e.g. Claessens, Kose, and Terrones (J Intern Econ 2012))
4. Corporate bond price growth
   (e.g. Gilchrist and Zakrajsek (AER 2012))
5. Basel III credit-to-GDP gap
   (Basel Committee on Banking Supervision (2010))
6. Composite financial cycle
   (Schüler, Hiebert, and Peltonen (2015, 2017))

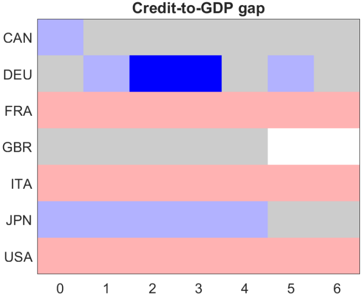
* Candidates transformed to quarterly/semi-annual by averaging (if necessary)

** Candidates deflated by GDP deflator (if necessary)

Data: Dependent variables and model selection

- Stage 1: Dummy variables for disruption to financial services
  - Romer and Romer (AER 2017)
    - Disruption to credit supply; on a 0 to 15 scale.
    - Map 0-15 scale into a 0-1 dummy variable.
    - Semi-annual, 1973H1–2017H2.
- Stage 2: Measure of real economic activity
  - real GDP growth (semi-annual)
- Number of lags
  - Stage 1: Lag length selected by BIC
  - Stage 2: Two lags of GDP growth
- Significance level: 10%

Results: Basel III credit-to-GDP gap

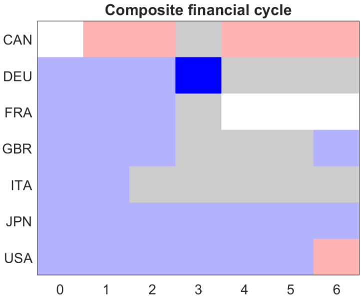


Basel III credit-to-GDP gap

- strong predictive performance (almost always passes Stage 1)
- but incoherent signals across countries
- current Basel regulation is targeting positive (red) coefficients

White color indicates that the variable fails in Stage 1 of the test.
Grey color indicates that the variable fails in Stage 2 of the test.
Blue (red) color means that the sum of the slope coefficients in Stage 1 is negative (positive).
The different shades of blue and red indicate whether Stage 2 is passed only for OLS or quantile regressions (light color) or for both OLS and quantile regressions (dark color)

Results: Composite financial cycle

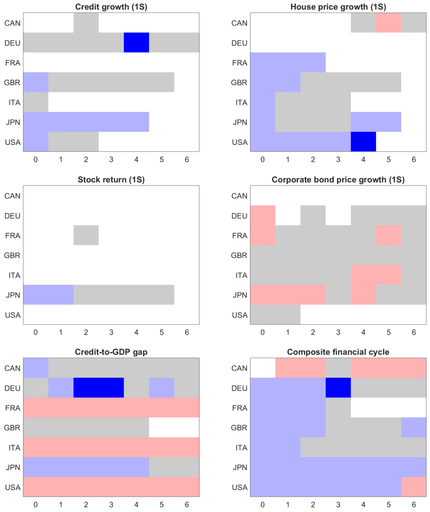


Composite financial cycle

- also strong predictive performance
- signs across countries fairly robust (Canada is special)
- high systemic risk goes hand in hand with **lower** level of financial cycle (after boom period)
- results are in line with **impossibility to predict turning points**

White color indicates that the variable fails in Stage 1 of the test.
Grey color indicates that the variable fails in Stage 2 of the test.
Blue (red) color means that the sum of the slope coefficients in Stage 1 is negative (positive).
The different shades of blue and red indicate whether Stage 2 is passed only for OLS or quantile regressions (light color) or for both OLS and quantile regressions (dark color)

Results: All candidate indicators



- Other indicators largely fail hierarchical test
- Hardly evidence of nonlinear relations
- However, potential losses to real GDP may still be large (see next slide)

Regression results for composite financial cycle

			CAN		DEU		FRA		GBR		I
contmp	St 1		-0.47	[0]	-7.56 ***	[0]	-6.88 ***	[0]	-19.98 ***	[0]	-9.52 *
	St 2	Mreg	-308.21		-13.42 **		-4.1 ***		-13.29 ***		-4.65 *
		Qreg	-464.19		-38.14 **		-7.57		-17.88 ***		-15.46
0.5y	St 1		6.95 **	[2]	-9.74 ***	[0]	-6.49 ***	[0]	-19.41 ***	[0]	-8.37 *
	St 2	Mreg	-16.57 **		-11.27 ***		-3.57 **		-9.3 **		-3.74 *
		Qreg	-23.1 *		-15.42 *		-11.63		-18.78 ***		-7.79 *
1y	St 1		7.44 **	[1]	-9.84 ***	[0]	-5.33 ***	[0]	-17.02 ***	[0]	-7.35 *
	St 2	Mreg	-13.79 **		-15.09 ***		-5.35 **		-5.69 **		-2.49
		Qreg	-38.95 *		-34.14 ***		-12.68		-4.31		-5.09
1.5y	St 1		7.82 ***	[0]	-11.08 ***	[0]	-3.61 *	[0]	-15.02 ***	[0]	-6.84 *
	St 2	Mreg	-4.9		-10.48 **		-3.99		0.78		0.52
		Qreg	7.59		-39.34 ***		8.63		-2.43		2.13
2y	St 1		9.89 ***	[0]	-11.34 ***	[0]	-2.19	[0]	-13.01 ***	[0]	-6.23 *
	St 2	Mreg	-12.61 *		-3.88		1.62		3.72		2.56
		Qreg	-43.72		-12.66		17.21		18.75		-1.38
2.5y	St 1		9.56 ***	[0]	-11.57 ***	[0]	-0.57	[0]	-11.22 ***	[0]	-5.34 *
	St 2	Mreg	-19.45 **		-0.94		15.11		1.09		1.87
		Qreg	-28.91		-9.64		149.93		26.28		1.95

- coefficients can be sizeable, e.g. Germany 1.5Y ahead: 1% increase in $\widehat{\pi}$ lowers 5% quantile of ann. GDP growth by 0.39%
- many coefficients in fact significant at 1% level

Robustness

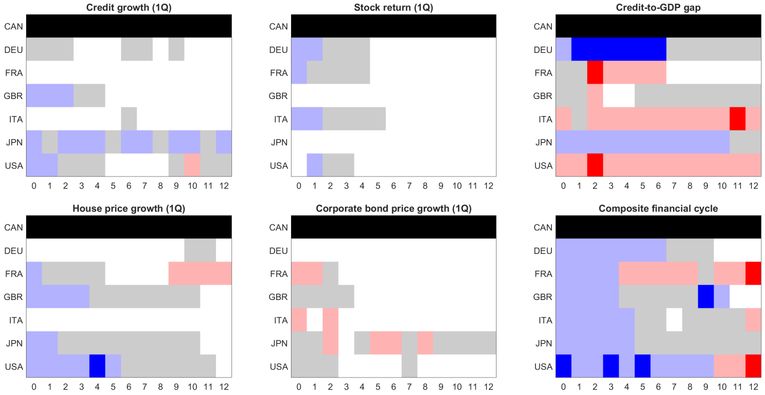
1. Alternative dummy variables for periods of financial disruption
   - Laeven and Valencia (ECB, 2018) ▸ LV
   - Reinhart and Rogoff (2009) ▸ RR
   - Financial crises dates from ESRB ▸ ESRB
   - (Placebo) peak-to-trough dates from ECRI ▸ PT
2. Alternative measures of real economic activity
   - growth rate of industrial production ▸ IP
   - (negative of the) unemployment rate ▸ UR

Robustness

3. Alternative candidate variables
   - financial conditions indices (Adrian et al., AER 2018) ▸ FCI
   - term spread (business cycle indicator) ▸ TS
4. Long-term growth rates of credit or asset prices ▸ LT
5. Econometric procedure
   - Impact of standard error correction ▸ Detecting Nonlinearities
   - Impact of hierarchical test
     - with unadjusted standard errors ▸ Non-corr
     - with adjusted standard errors ▸ Corr
6. Finite sample problems of the standard error correction

Robustness 1: Laeven and Valencia

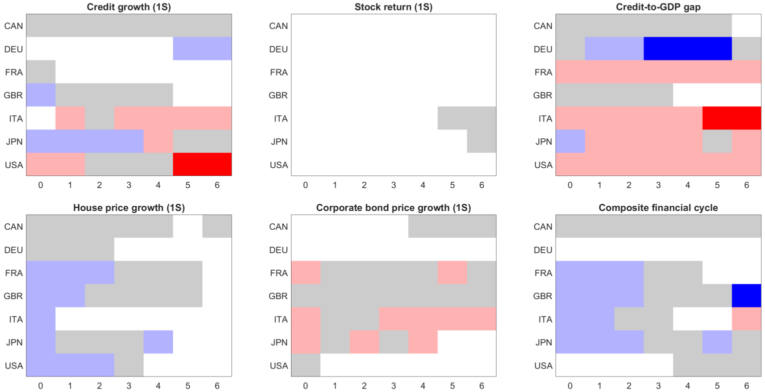
Disruption dummies from Laeven and Valencia (2018) in Stage 1



Almost identical results

Robustness 1: Reinhart and Rogoff

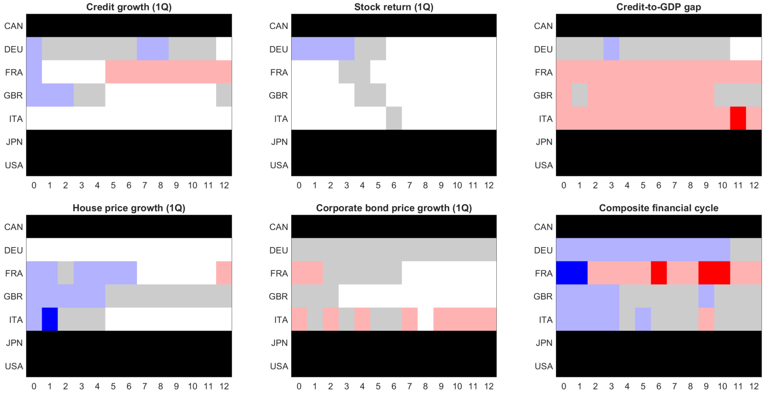
Disruption dummies from Reinhart and Rogoff (2009) in Stage 1



A bit more favorable for credit-to-GDP gap (?)

Robustness 1: European Systemic Risk Board

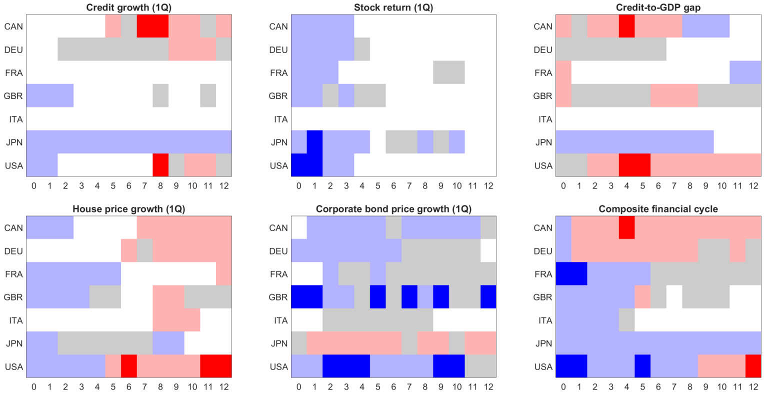
Disruption dummies from ESRB in Stage 1



- UK turns red
- Germany turns grey

Robustness 1: Business cycle peak and trough

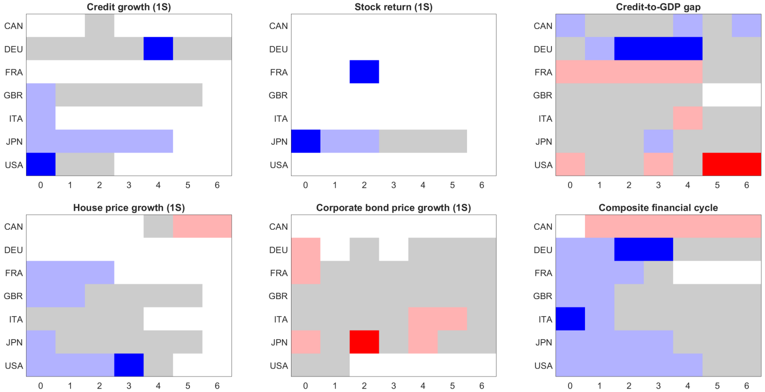
(Placebo) business cycle dummies from ECRI in Stage 1



- Stock and bond prices predict recessions
- Financial cycle inherits this property
- Credit-to-GDP gap has almost no link to business cycles

Robustness 2: Industrial production

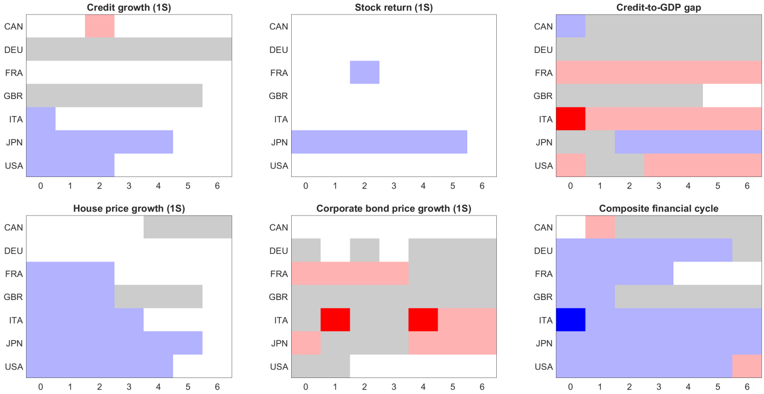
Industrial production growth in Stage 2



- Fewer significant coefficients on Stage 2
- Results with Laeven-Valencia dummies similar to benchmark

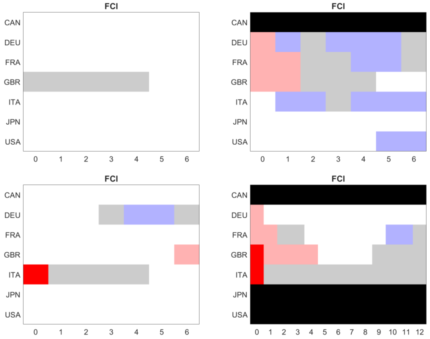
Robustness 2: Unemployment rate

Unemployment rate in Stage 2



Robustness 3: Financial conditions index

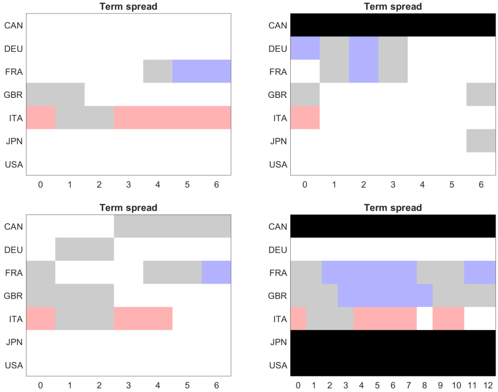
- Alternative candidate variable: **financial conditions index**
  - Capturing financial stress and spillover
  - PC of credit risk, volatility, leverage, credit growth variables



- falls short in Stage 1
- Stage 2 coefficients insignificant due to corrected std errors
- challenges results from Adrian et al. (AER 2018)

Robustness 3: Term spread

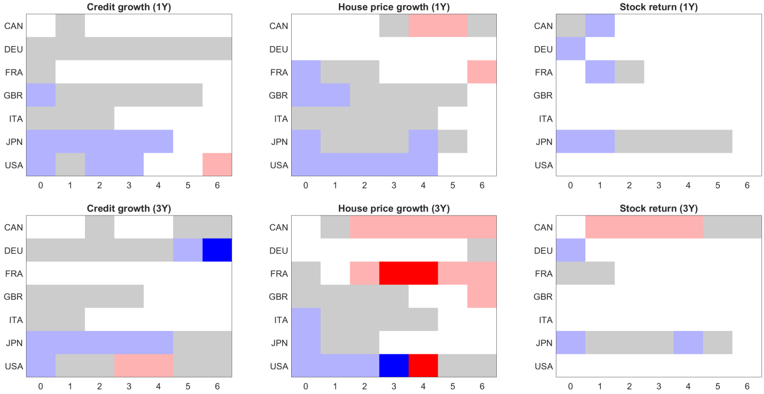
Alternative candidate variable: term spread



Business cycle variables do not predict financial disruptions

Robustness 4: Long-term growth rates

Alternative candidate variables: 1Y or 3Y growth rates of asset prices and credit



Persistent house price growth may signal elevated systemic risk

Robustness 5: Impact of standard error correction

Example: Germany, 1 year horizon, Romer-Romer dummies

			Significance (Stage 1)	Coeff (Stage 2)	Std error (Stage 2)		80% Conf Interval		
Credit	Mreg	non-corr	***	-1.51	2.52		[-4.77	,	1.75]
		corrected			2.37		[-4.57	,	1.55]
	Qreg	non-corr		-16.68	18.05		[-40.01	,	6.65]
		corrected			18.44		[-40.51	,	7.15]
House price	Mreg	non-corr		14.92	45.39		[-43.75	,	73.59]
		corrected			83.19		[-92.6	,	122.44]
	Qreg	non-corr		118.48	81.47		[13.18	,	223.78]
		corrected			492.71		[-518.35	,	755.31]
Credit gap	Mreg	non-corr	***	-4.53	3.07	*	[-8.5	,	-0.56]
		corrected			3.03	*	[-8.45	,	-0.61]
	Qreg	non-corr		-17.85	6.26	***	[-25.94	,	-9.76]
		corrected			6.31	***	[-26.01	,	-9.69]
Fcycle	Mreg	non-corr	***	-15.09	5.02	***	[-21.58	,	-8.6]
		corrected			5.03	***	[-21.59	,	-8.59]
	Qreg	non-corr		-34.14	9.54	***	[-46.47	,	-21.81]
		corrected			10.93	***	[-48.27	,	-20.01]

- Confidence intervals of Mreg and Qreg may overlap with corrected std errors

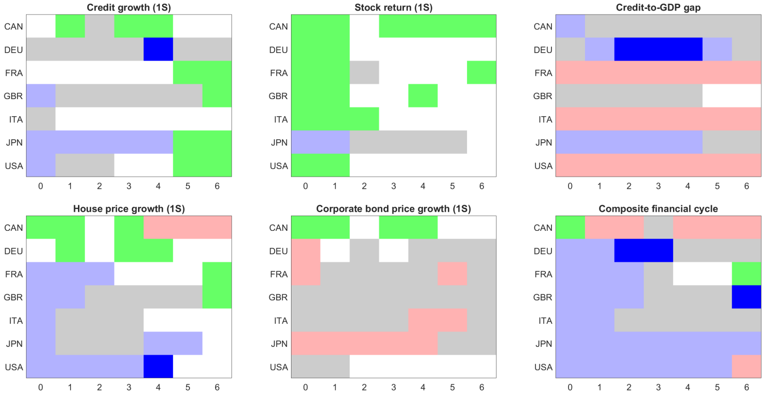
→ less evidence for nonlinearity

- Std error correction can be enormous (if Stage 1 heavily misspecified)

Robustness 5: Hierarchical test structure

Green color: variable fails in Stage 1, but would pass Stage 2

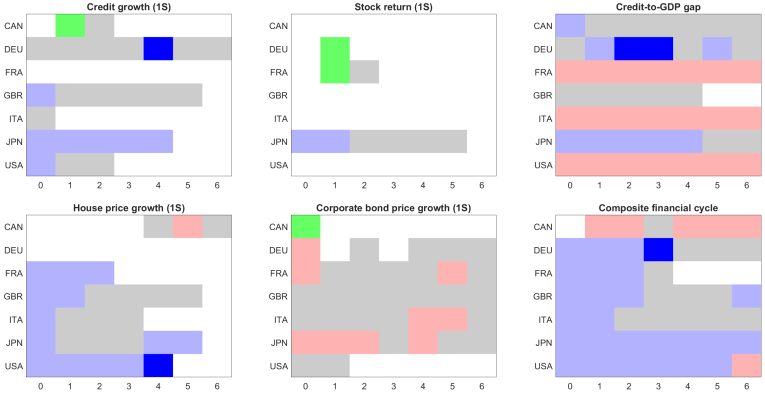
1. Without standard error correction



"Passing Stage 2" alone is not a sufficient criterion (does not automatically imply passing of Stage 1)

Robustness 5: Hierarchical test + standard error correction

2. With standard error correction



Std error correction does not solve this problem entirely

Conclusion

Contributions

- Guidance on which indicators qualify for monitoring systemic risk
- Testing framework which is straightforward to implement and easy to interpret
- Enhance our understanding of systemic risk by screening a set of indicators

- Credit-to-GDP gap is not an indicator of systemic risk
- Composite measure of financial cycle performs best in our test
- Individual components of financial cycle do not pass test
- Financial conditions indices do not pass test
- Results are robust to various modifications of our test
- Macroprudential policy may address systemic risk only indirectly by smoothing the financial cycle in boom phases

Thank you very much!

Technical details: Generated regressors (1/3)

Back

Theorem (Asymptotic distribution of two-step QMLE)

Suppose our model consists of the two marginal distributions $f_1(y_1|x_1, \theta_1)$ *and* $f_2(y_2|x_1, x_2, \theta_1, \theta_2)$. *The estimation proceeds in two steps:*

1. *Estimate* θ_1 *by maximum likelihood in model 1:* $L_1(\theta_1) = \prod_{t=1}^{T} f_1(y_{1t}|x_{1t}, \theta_1)$.
2. *Estimate* θ_2 *by maximum likelihood in model 2, with* $\hat{\theta}_1$ *for* θ_1, *i.e. as if* θ_1 *was known:* $L_2(\theta_1, \theta_2) = \prod_{t=1}^{T} f_2(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2)$.

If the standard regularity conditions for both log-likelihood functions hold and if the quasi maximum likelihood estimate of θ_2 *is consistent, then the MLE of* θ_2 *is asymptotically normally distributed with asymptotic covariance matrix . . .*

Technical details: Generated regressors (2/3)

Theorem (Asymptotic distribution of two-step QMLE)

$$
\begin{aligned}
V_2 = {} & \frac{1}{T}(-H_{22}^{(2)})^{-1}\Sigma_{22}(-H_{22}^{(2)})^{-1} \\
& + \frac{1}{T}(-H_{22}^{(2)})^{-1}\left(H_{21}^{(2)}(-H_{11}^{(1)})^{-1}H_{21}^{(2)\prime} + \Sigma_{21}(-H_{11}^{(1)})^{-1}H_{21}^{(2)\prime} + H_{21}^{(2)}(-H_{11}^{(1)})^{-1}\Sigma_{12}\right)(-H_{22}^{(2)})^{-1}
\end{aligned}
$$

where

$$
\Sigma_{22} = E\left[\frac{1}{T}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2'}\right], \quad \Sigma_{21} = E\left[\frac{1}{T}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2}\frac{\partial \ln L_1(\theta_1)}{\partial\theta_1'}\right],
$$

$$
\Sigma_{12} = E\left[\frac{1}{T}\frac{\partial \ln L_1(\theta_1)}{\partial\theta_1}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2'}\right], \quad H_{11}^{(1)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_1(\theta_1)}{\partial\theta_1\partial\theta_1'}\right],
$$

$$
H_{22}^{(2)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_2(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_2'}\right], \quad H_{21}^{(2)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_2(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_1'}\right].
$$

Technical details: Generated regressors (3/3)

Theorem (Asymptotic distribution of two-step QMLE)

The estimate $\hat{V}_2$ *is given by*

$$\begin{aligned}\hat{V}_2 &= (-\hat{H}_{22}^{(2)})^{-1}[\hat{\Sigma}_{22} + \hat{H}_{21}^{(2)}(-\hat{H}_{11}^{(1)})^{-1}\hat{H}_{21}^{(2)'} + \hat{\Sigma}_{21}(-\hat{H}_{11}^{(1)})^{-1}\hat{H}_{21}^{(2)'} \\ &\quad + \hat{H}_{21}^{(2)}(-\hat{H}_{11}^{(1)})^{-1}\hat{\Sigma}_{12}](-\hat{H}_{22}^{(2)})^{-1}\end{aligned}$$

where $\hat{\Sigma}_{22}, \hat{\Sigma}_{21}$ *and* $\hat{\Sigma}_{12}$ *are the typical BHHH estimators*

$$\hat{\Sigma}_{22} = \sum_{t=1}^{T} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2'}, \quad \hat{\Sigma}_{21} = \sum_{t=1}^{T} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2} \frac{\partial \ln f_{1t}}{\partial \hat{\theta}_1'}, \quad \hat{\Sigma}_{12} = \sum_{t=1}^{T} \frac{\partial \ln f_{1t}}{\partial \hat{\theta}_1} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2'}$$

and the $\hat{H}_{11}$, $\hat{H}_{22}$ *and* $\hat{H}_{21}$ *may be computed as expected Hessians*

$$\hat{H}_{11}^{(1)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{1t}}{\partial \hat{\theta}_1 \partial \hat{\theta}_1'}\right], \quad \hat{H}_{22}^{(2)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{2t}}{\partial \hat{\theta}_2 \partial \hat{\theta}_2'}\right], \quad \hat{H}_{21}^{(2)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{2t}}{\partial \hat{\theta}_2 \partial \hat{\theta}_1'}\right].$$

Technical details: Application to Logit + Linear Regression

Back

Stage 1: Logit model

$$P(y_{1t} = 1) = \Lambda(x_{1t}\theta_1)$$

where $\Lambda(x_t\theta) = \frac{\exp(x_t\theta)}{1+\exp(x_t\theta)}$. The log-likelihood is

$$\ln L_1(\theta_1) = \sum_{t=1}^{T} \ln f_1(y_{1t}|x_{1t}, \theta_1) = \sum_{t=1}^{T} \left[(1 - y_{1t}) \ln[(1 - \Lambda(x_{1t}\theta_1))] + y_{1t} \ln[\Lambda(x_{1t}\theta_1)]\right].$$

Stage 2: Linear regression model

$$E(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2) = x_{2t}\beta + \sum_{k=0}^{p} \Lambda(x_{1t-k}\theta_1)\gamma_k = z_t\theta_2.$$

The log-likelihood is

$$\ln L_2(\theta_1, \theta_2) = \sum_{t=1}^{T} \ln f_2(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma^2) - \sum_{t=1}^{T} \frac{1}{2\sigma^2} u_{2t}^2$$

where $u_{2t} = y_{2t} - z_t\theta_2$.
Derivatives of the log-likelihood w.r.t. θ_1 and θ_2 are straightforward.

Technical details: Application to Logit + Linear Regression

Back

Inputs for the corrected asymptotic covariance matrix:

$$\Sigma_{22} = E\left(\frac{1}{T}\left(\frac{1}{\sigma^2}\right)^2 \sum_{t=1}^{T} u_{2t}^2 z_t' z_t\right), \qquad \Sigma_{21} = E\left(\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} u_{1t} u_{2t} z_t' x_{1t}\right)$$

$$\Sigma_{12} = E\left(\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} u_{1t} u_{2t} x_{1t}' z_t\right), \qquad H_{11}^{(1)} = E\left(-\frac{1}{T}\sum_{t=1}^{T} x_{1t}' x_{1t} \Lambda(x_{1t}\theta_1)(1-\Lambda(x_{1t}\theta_1))\right)$$

$$H_{21}^{(2)} = E\left(-\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} z_t' n_t\right), \qquad H_{22}^{(2)} = E\left(-\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} z_t' z_t\right)$$

with

$$n_t = \frac{\partial \sum_{j=1}^{k_2} z_{tj}\theta_{2j}}{\partial \theta_1'} = \sum_{k=0}^{p} x_{1t-k}\Lambda(x_{1t-k}\theta_1)(1-\Lambda(x_{1t-k}\theta_1))\gamma_k$$

Technical details: Application to Logit + Linear Regression

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Empirical gradients for the BHHH-Type estimators:

$$\frac{\partial \ln f_1}{\partial \hat{\theta}_1} = x'_{1t}\hat{u}_{1t}, \quad \frac{\partial \ln f_2}{\partial \hat{\theta}_2} = \frac{1}{\hat{\sigma}^2}\hat{z}'_t\hat{u}_{2t}$$

Expected Hessians

$$E\left[\frac{\partial^2 \ln f_1}{\partial \hat{\theta}_1 \partial \hat{\theta}'_1}\right] = -x'_{1t}x_{1t}\Lambda(x_{1t}\hat{\theta}_1)(1-\Lambda(x_{1t}\hat{\theta}_1),$$

$$E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}'_1}\right] = -\frac{1}{\hat{\sigma}^2}\hat{z}'_t\hat{n}_t, \qquad E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}'_2}\right] - \frac{1}{\hat{\sigma}^2}\hat{z}'_t\hat{z}_t.$$

Technical details: Application to Logit + Quantile Regression

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Stage 1: Logit model

$$P(y_{1t} = 1) = \Lambda(x_{1t}\theta_1)$$

where $\Lambda(x_t\theta) = \frac{\exp(x_t\theta)}{1+\exp(x_t\theta)}$. The log-likelihood is

$$\ln L_1(\theta_1) \quad = \quad \sum_{t=1}^{T} \ln f_1(y_{1t}|x_{1t}, \theta_1) = \sum_{t=1}^{T} \left[(1 - y_{1t}) \ln[(1 - \Lambda(x_{1t}\theta_1))] + y_{1t} \ln[\Lambda(x_{1t}\theta_1)]\right].$$

Stage 2: Quantile regression model

$$Q_\tau(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2^\tau) = x_{2t}\beta^\tau + \sum_{k=0}^{p} \Lambda(x_{1t-k}\theta_1)\gamma_k^\tau = z_t\theta_2^\tau.$$

Log-likelihood function (Komunjer 2005):

$$\ln L_2(\theta_1, \theta_2^\tau) = \sum_{t=1}^{T} \quad -(1-\tau)\left(\frac{1}{\tau(1-\tau)}(z_t\theta_2^\tau - y_{2t})1_{\{y_{2t} \le z_t\theta_2^\tau\}}\right)$$
$$+\tau\left(\frac{1}{\tau(1-\tau)}(z_t\theta_2^\tau - y_{2t})1_{\{y_{2t} > z_t\theta_2^\tau\}}\right)$$

Derivatives of log-likelihood: only exist in the "distributional" (generalized) sense.

Technical details: Application to Logit + Quantile Regression

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Inputs for the corrected asymptotic covariance matrix:

$$
\begin{aligned}
\Sigma_{22} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{2t}^{(2)} g_{2t}^{(2)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left[\frac{1}{T}\sum_{t=1}^{T} z_t' z_t\right] \\
\Sigma_{21} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{2t}^{(2)} g_{1t}^{(1)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} u_{1t}(\tau - 1_{\{y_{2t}\le z_t\theta_2^\tau\}}) z_t' x_{1t}\right) \\
\Sigma_{12} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{1t}^{(1)} g_{2t}^{(2)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} u_{1t}(\tau - 1_{\{y_{2t}\le z_t\theta_2^\tau\}}) x_{1t}' z_t\right) \\
H_{11}^{(1)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{11t}^{(1)}\right) = -E\left(\frac{1}{T}\sum_{t=1}^{T} x_{1t}' x_{1t} \Lambda(x_{1t}\theta_1)(1-\Lambda(x_{1t}\theta_1))\right) \\
H_{21}^{(2)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{21t}^{(2)}\right) = -\frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} z_t' n_t f_{y_{2t}|z_t\theta_2^\tau}(z_t\theta_2^\tau)\right) \\
H_{22}^{(2)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{22t}^{(2)}\right) = -\frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} z_t' z_t f_{y_{2t}|z_t\theta_2^\tau}(z_t\theta_2^\tau)\right)
\end{aligned}
$$

Technical details: Application to Logit + Quantile Regression

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Empirical gradients for the BHHH-Type estimators

$$\frac{\partial \ln f_1}{\partial \hat{\theta}_1} = x_{1t}' \hat{u}_{1t}, \quad \frac{\partial \ln f_2}{\partial \hat{\theta}_2} = \hat{z}_t'(\tau - 1_{\{y_{2t} \leq \hat{z}_t \hat{\theta}_2^\tau\}})$$

Expected Hessians

$$E\left[\frac{\partial^2 \ln f_1}{\partial \hat{\theta}_1 \partial \hat{\theta}_1'}\right] = -x_{1t}' x_{1t} \Lambda(x_{1t}\hat{\theta}_1)(1 - \Lambda(x_{1t}\hat{\theta}_1),$$

$$E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}_1'}\right] = -\frac{1}{\tau(1-\tau)} \hat{z}_t' \hat{n}_t \hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^\tau}(\hat{z}_t\hat{\theta}_2^\tau), \quad E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}_2'}\right] = -\frac{1}{\tau(1-\tau)} \hat{z}_t' \hat{z}_t \hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^\tau}(\hat{z}_t\hat{\theta}_2^\tau).$$

We estimate the density of the errors using the kernel method of Powell (1991):

$$\hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^\tau}(\hat{z}_t\hat{\theta}_2^\tau) = \frac{1}{2c_T} 1(|\hat{u}_{2t}| < c_T)$$

where

$$c_T = \kappa(\Phi^{-1}(\tau + h_T) - \Phi^{-1}(\tau - h_T))$$

κ is a robust scale estimate and h_T is chosen according to Hall and Sheather (1988).