Momentum Turning Points

Ashish Garg
Research Affiliates, LLC, Newport Beach, CA 92660 USA

Christian L. Goulding
Research Affiliates, LLC, Newport Beach, CA 92660 USA

Campbell R. Harvey*
Duke University, Durham, NC 27708 USA
National Bureau of Economic Research, Cambridge, MA 02138 USA

Michele G. Mazzoleni
Research Affiliates, LLC, Newport Beach, CA 92660 USA

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Turning points are the Achilles' heel of time-series momentum portfolios. Slow signals fail to react quickly to changes in trend while fast signals are often false alarms. We examine theoretically and empirically how momentum portfolios of various intermediate speeds, formed by blending slow and fast strategies, cope with turning points. Our model predicts an optimal dynamic speed selection strategy. We apply this strategy across domestic and international equity markets and document efficient out-of-sample performance. We also propose a novel decomposition of momentum strategy alpha, highlighting the role of volatility timing.

Keywords: time-series momentum, volatility timing, market timing, asset pricing, trend following, turning points, momentum speed, mean reversion, behavioral finance

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1. Introduction

Time-series (TS) momentum strategies are based on two main premises. First, expected returns vary over time. Second, the *signs* of expected returns have some degree of persistence. If the expected return is positive (negative) this month, then it is more likely to remain positive (negative) next month than to flip sign. TS momentum strategies attempt to exploit such variation and persistence in expected returns, where it exists, by taking long positions in "uptrend" phases and short positions in "downtrend" phases. In uptrend or downtrend phases, returns will tend to have similar signs—positive or negative, respectively—and so TS momentum strategies that take positions having the same sign as some lookback horizon of trailing returns will tend to place good bets.

The premise of time-varying expected returns has extensive support in the literature (Fama and French, 1988, and Cochrane, 2011, among many). The premise of persistent trends also has support in the literature, which documents that asset returns measured over the recent past are positively correlated with future returns (Jegadeesh and Titman, 1993; 2001, Asness, 1994, Conrad and Kaul, 1998, Lee and Swaminathan, 2000, and Gutierrez and Kelley, 2008) and that these momentum effects are stable across assets and countries (Rouwenhorst, 1998, Griffin, Ji, and Martin, 2003, Israel and Moskowitz, 2012, and Asness, Moskowitz, and Pedersen, 2013). Recent studies provide evidence that TS momentum strategies can successfully exploit these trends (Moskowitz, Ooi, and Pedersen, 2012; Georgopoulou and Wang, 2017; Ehsani and Linnainmaa, 2019).

Unless trend is perennial in one direction, however, occasionally trend will break down. These breaks, or "momentum turning points", mark a reversal in trend from uptrend to downtrend or vice versa. At and after turning points, TS momentum strategies are prone to place bad bets because they rely on observations of realized returns, which reflect a mixture of different trend regimes and noise.

The speed of the momentum signal balances the tension between reducing the impact of noise and reacting quickly. This tension plays out differently for different speeds. Either the momentum signal attempts to reduce the influence of noise by having a relatively long lookback window (e.g., 12 months) but thus is *slow* to react to a turning point, or the momentum signal attempts to be *fast* to react to a turning point by having a relatively short lookback window (e.g., 1 month) and therefore is more influenced by noise. Accordingly, TS momentum strategies face a key trade-off, not emphasized in the literature, with respect to momentum signal speed: go slow and risk incurring a Type II error—failing to react to a turning point when it occurs—or go fast and risk incurring a Type I error—reacting to noise when a turning point has not occurred.

Can agreement and disagreement between slow and fast strategies help us differentiate trends and turning points? When bets indicated by slow and fast strategies disagree, the intuition is that the market is more likely to be at a turning point. The agreement of slow and fast strategies to go long (short) is more likely to indicate the market is in the midst of an uptrend (downtrend). We find this intuition to be consistent with the returns behavior of the U.S. stock market after each of the four different phases, or market cycles, defined by the up or down directions of each of the slow and fast momentum strategies.

Figure 1 summarizes, over the last 50 years of the U.S. stock market, the conditional behavior of the average, volatility, and skewness of returns in months following four market cycles and the monthly relative frequency of such states.[1] When both slow and fast momentum agree on the direction of trend, we call it a "Bull" or "Bear" state, depending on whether the agreement is to take a long or short position, respectively. These labels loosely map to phases of uptrend and downtrend: Bull states are followed by relatively high average returns with low volatility, and Bear states are followed by negative average returns with the highest relative volatility.

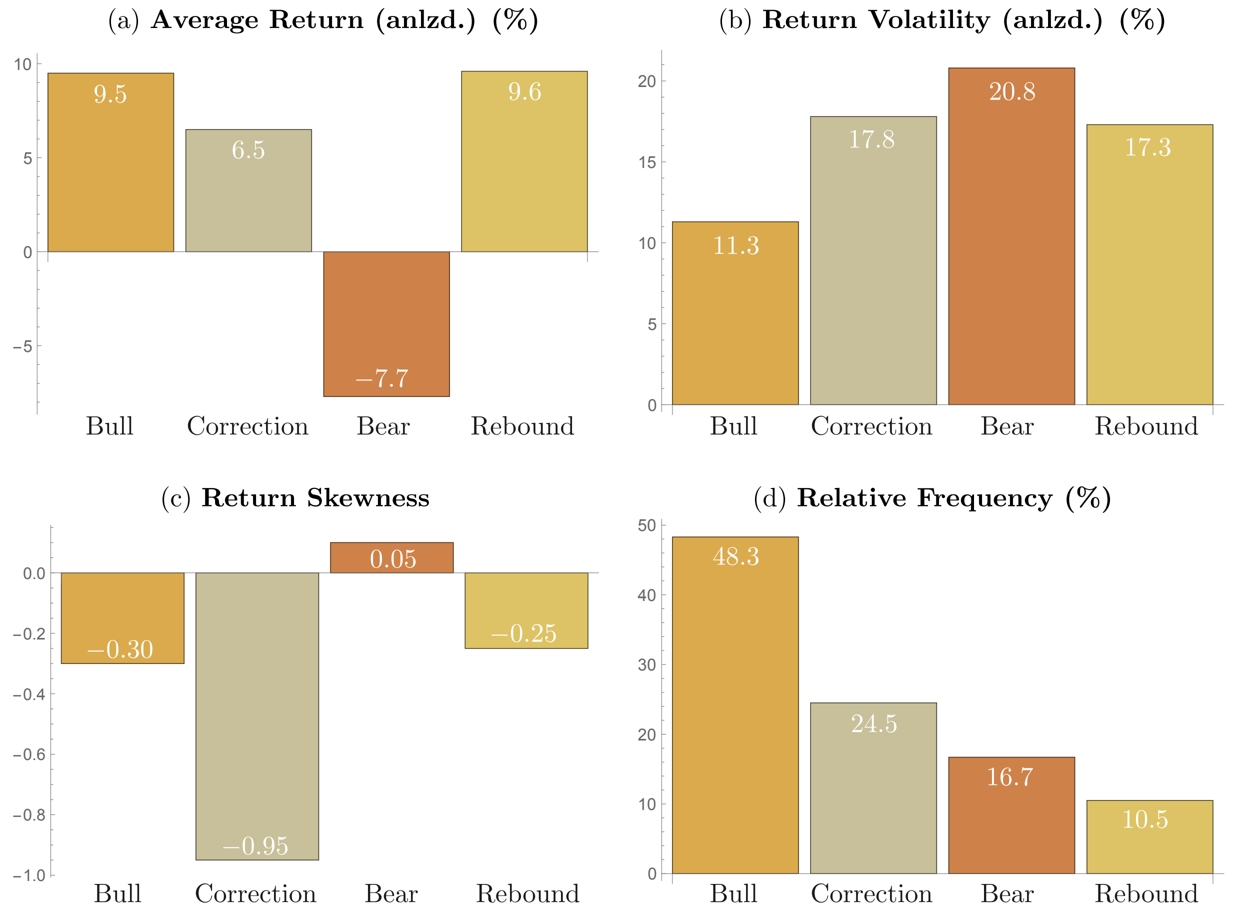
When slow and fast momentum disagree, we call it a "Correction" state if slow momentum indicates a long position and a "Rebound" state if slow momentum indicates a short position. Similarly, these labels loosely map to the potential occurrences of turning points from uptrend to downtrend and vice-versa. Correction states are followed by deteriorating average returns, increased volatility, and severe downside outcomes—possibly a lead up to a Bear state. Rebound states are followed by average returns and skewness similar to Bull phases, but with higher volatility—possibly a lead up to a Bull state. Lastly, Corrections and Rebounds are significant in frequency and combined occur more than one-third of the time covered in our analysis.

These four market cycles implied by slow/fast momentum positions appear to be useful proxies of potential trends and turning points in the macroeconomy as well as in the stock market. We study the behavior of 15 major macroeconomic indicators from the three broad categories of economic activity, risk, and survey-based variables. We find that surprises in these variables exhibit distinct differences across these market cycles.[2] For instance, employment, consumer sentiment, and stock market liquidity all deteriorate more than expected during Bear cycles and improve more than expected during Bull cycles. Corrections and

[1]For most of the paper, we apply TS momentum strategies to the aggregate U.S. stock market to develop and test our ideas. In later sections, we extend our empirical analysis to international equity markets. Also, for most of the paper, we evaluate performance over the last 50 years. We do this for consistency with later analyses in which data prior to the last 50 years is used to warm up our dynamic strategies for out-of-sample evaluation. In Appendix C, we report results based on a longer evaluation period beginning in 1927 from which we draw similar inferences as in our main analyses.

[2]See Section 2.2 for details.

Figure 1: **U.S. Stock Market Cycles**



Notes: This figure reports (a) the conditional average, (b) the conditional volatility, and (c) the conditional skewness of monthly aggregate U.S. stock market returns, based on the market cycle state in the prior month, over the 50-year evaluation period from 1969-01 to 2018-12. A month ending at date t is classified as Bull if both the trailing 12-month return (arithmetic average monthly return), $r_{t-12,t}$, is nonnegative and the trailing 1-month return, $r_{t-1,t}$, is nonnegative. A month is classified as Correction if $r_{t-12,t} \geq 0$ but $r_{t-1,t} < 0$; as Bear if $r_{t-12,t} < 0$ and $r_{t-1,t} < 0$; and as Rebound if $r_{t-12,t} < 0$ but $r_{t-1,t} \geq 0$. The figure also reports (d) the relative frequency of these cycles over the same evaluation period. Market returns are U.S. excess value-weighted factor returns (Mkt-RF) from the Kenneth French Data Library.

Rebounds show neither significant positive nor negative surprises, consistent with the notion that these two cycles may be signaling turning points.

Motivated by this evidence, we form and analyze TS momentum strategies of various intermediate "speeds" by blending slow and fast momentum strategies with various weights on each. Rather than going decisively long or short following Correction and Rebound phases according to the slow (SLOW) or fast (FAST) strategies, we find empirically that scaling back positions after periods of disagreement may be beneficial. When SLOW and FAST disagree, and therefore specify opposite positions (one long and the other short), this could indicate a turning point, which SLOW has failed to reflect: a Type II error. Or, the market might still be in a trend phase and FAST is a false alarm: a Type I error. If both errors are likely, then intermediate-speed strategies, which blend opposing positions to (partially) offset one another, can reduce exposure to the downside associated with turning points.

We first investigate *static* intermediate speeds, which are fixed-proportion blends of slow and fast strategies. Such strategies are not new. Rather, they are common in the literature and in practice, which is a testament to their empirical effectiveness relative to single horizon alternatives. However, explanations for their effectiveness have not been thoroughly developed. We provide new insights and highlight three results.

First, we examine return and volatility risk characteristics. We show analytically that intermediate-speed momentum strategies will have higher Sharpe ratios than the average Sharpe ratios of slow and fast strategies. Our results explicitly link this behavior to the volatility of returns following turning point states (Corrections and Rebounds). Consistent with these results, intermediate-speed strategies empirically exhibit the highest Sharpe ratios across all speeds when applied to the aggregate U.S. stock market. In particular, intermediate-speed portfolios scale down positions after Correction and Rebound months, reducing exposure to volatile months without surrendering average returns in comparison to SLOW and FAST.

Second, we shed new light on the drivers of market beta and alpha of TS momentum strategies of different speeds. Recent studies have argued that the profitability of TS momentum strategies is predominantly attributed to their static (average) tilts.[3] Empirically, we document that TS momentum strategies of all speeds on the U.S. stock market have positive average exposures (i.e., going long more often than short), yet their market betas are lower than suggested by these exposures—beta estimates nearer to zero for slow-to-intermediate speeds and negative for faster speeds. Our analysis reveals that this seeming disparity arises from the ability of these TS momentum portfolios to time volatility states. We derive a novel model-free decomposition of beta into the sum of three components: (i) a static component, which reflects the average market position of the strategy; (ii) a market-timing component, which reflects covariance between strategy weights and subsequent market returns; and (iii) a volatility-timing component, which reflects covariance between strategy weights and subsequent market return volatility.[4] In addition, we derive a related decomposition of alpha into the sum of two components: market timing and volatility timing. Empirically, we estimate that market timing drives about two-thirds of the alpha of TS momentum strategies on the

[3] See Huang et al. (2019) and Goyal and Jegadeesh (2017).

[4] We examine the relationships to the volatility-managed strategies of Moreira and Muir (2017) in Section 3.2.2 and Appendix G.

U.S. stock market over the last 50 years and volatility timing drives the remaining one-third.

Third, we provide new insights on the tail behavior of TS momentum strategies of different speeds through the lens of momentum turning points. By reducing market exposure following Corrections, intermediate-speed strategies exhibit more desirable tail-risk characteristics. We show analytically that the skewness of an intermediate-speed momentum strategy is scaled and shifted relative to the average skewness of SLOW and FAST, with the shift in the same direction at its Sharpe ratio: toward higher (more positive, less negative) skewness whenever the intermediate-speed Sharpe ratio is positive.

We also introduce and show the merits of *dynamic* strategies whose speeds may vary month-by-month depending on observed market cycles.[5] Turning points from up to down may have different properties than turning points from down to up. For example, after Corrections (when SLOW is long and FAST is short), the likelihood of a turning point false alarm (Type I error) might dominate, while after Rebounds (when SLOW is short and FAST is long), the likelihood of missed detection (Type II error) might dominate. In this case, a dynamic-speed strategy that is slower after Corrections (long, but possibly scaled down) and faster after Rebounds (long, but possibly scaled down) could be effective. We derive analytical expressions for the state-dependent speed rule which yields the maximum Sharpe ratio. We estimate the optimal speed rule from historical returns and then evaluate its performance out-of-sample. Empirically, we find that this implementation can efficiently track the best possible state-dependent performance—over 90% efficiency across different evaluation windows. For momentum strategies applied to the aggregate U.S. stock market, relative to static-speed momentum strategies, a dynamic strategy with a slower speed following Corrections and a faster speed following Rebounds improves not only Sharpe ratios but also average returns per unit of drawdown risk.

Lastly, to test the external validity of our findings for the U.S. stock market, we examine the empirical performance of TS momentum strategies of various static and dynamic speeds across international equity markets. We find that our results largely carry over, with the exception of dynamic-speed results applied to Japan, where static intermediate-speed strategies outperform dynamic strategies. Nevertheless, even for Japan, a dynamic speed rule outperforms both slow and fast momentum strategies.

TS momentum has been the subject of several recent articles. Moskowitz, Ooi, and Pedersen (2012) find substantial profitability for 12-month TS momentum strategies and document return predictability evidence across lookback horizons from 1 to 12 months. We

[5]Our approach is not to be confused with moving average crossovers. Levine and Pedersen (2016) show that moving average crossovers are essentially equivalent to static blends of time-series momentum strategies. Moreover, Hurst et al. (2013) show that the returns of trend-following strategies such as Managed Futures funds and CTAs can be explained by static blends of time-series momentum strategies.

focus on the trade-off between TS momentum at the slow and fast ends of this range and document the merits of static and time-varying slow/fast blends.

Huang et al. (2019) argue that predictability of 12-month TS momentum as captured by forecasting regressions does not appear to be statistically significant and, accordingly, the profitability of a diversified 12-month TS momentum portfolio can be largely attributed to its static tilt (net long positions). We offer a different interpretation. Despite their positive static (average) tilts, TS momentum portfolios need not have positive betas to the underlying asset. Applying our novel decomposition of TS momentum beta and alpha, we document the role of innate volatility timing in TS momentum strategies, which can explain this disparity between static tilts and betas and contribute to meaningful alphas.[6] Huang et al. (2019) further point out that it would be worthwhile to investigate horizons other than 12 months, a task we embark on in this paper.

Goyal and Jegadeesh (2017) also argue that the net dollar exposure of TS momentum strategies is a key determinant of their profitability. Specifically, they propose that adding the net dollar exposure of a TS momentum portfolio to a cross-sectional (CS) momentum portfolio can reproduce TS momentum profits. This argument does not apply, however, to the case of a single asset for which a CS momentum strategy is either not defined or trivial. In this paper, we focus on single-asset TS momentum.

Although our focus is on TS momentum, our approach shares some themes with the CS momentum literature. In a TS setting, we employ market cycles as Cooper, Gutierrez, and Hameed (2004), Daniel and Moskowitz (2016), and Daniel, Jagannathan, and Kim (2019) do in CS settings. Cooper, Gutierrez, and Hameed use a slow trailing three-year return to define market states, whereas we use slow and fast trailing returns signals.[7] Daniel and Moskowitz study CS momentum crashes and recoveries and propose a dynamic CS weighting strategy. Daniel, Jagannathan, and Kim use a two-state hidden Markov model of unobserved "turbulent" and "calm" states to predict crashes in CS momentum portfolios. We employ a finer cycle partition of observable states dictated by slow and fast momentum directions and develop a dynamic TS momentum strategy which blends slow and fast strategies using cycle-conditional information.

In a TS setting, we evaluate different horizons as Novy-Marx (2012) does in a CS setting. Novy-Marx points out that CS momentum works best at 7-to-12 month horizons.

[6]Note that we do not volatility-scale our TS momentum positions, so the volatility timing we document is that which is inherent in simple long/short TS momentum strategies. In Appendix G, we analyze the added impacts of volatility scaling on our strategies.

[7]Specifically, these authors rely on trailing three-year market returns to define two states—"up" and "down"—and forecast stock momentum consistent with the predictions of behavioral models (Daniel, Hirshleifer, and Subrahmanyam, 1998, and Hong and Stein, 1999).

We demonstrate that shorter horizons are significant predictors of aggregate stock market returns and can augment slower signals. Liu and Zhang (2008) show in a CS momentum setting that recent winners load temporarily on the growth rate of industrial production and that macroeconomic risk can explain more than half of CS momentum profits. This work relates to our findings of distinct macroeconomic linkages to the market cycles determined by slow and fast TS momentum directions.[8]

Our paper also bridges to the literature on volatility-managed (VOM) portfolios sparked by Moreira and Muir (2017) and Harvey et al. (2018). A well-known empirical regularity, the negative correlation between stock market returns and volatility, underscores the association between TS momentum and VOM strategies. Although volatility timing is a contributor to the alphas of both types of strategies, we find that market timing is a major distinct contributor to the alpha of a TS momentum portfolio. Moreover, we show that VOM strategies applied to the U.S. stock market appear to deliver marginal tactical information beyond what is already captured by TS momentum portfolios.

Our paper is organized as follows. In Section 2, we introduce four market cycles and link these cycles to macroeconomic activity. Section 3 explores static momentum strategies and presents a number of analytical results that help characterize both returns, volatility and skewness. In Section 4, we analyse dynamic blends of slow and fast momentum strategies and present a cycle-dependent strategy that maximizes the Sharpe ratio. In Section 5, we present empirical results across international equity markets. Some concluding remarks are offered in Section 6.

2. Momentum Speed and Market Cycles

In this section, we define a collection of TS momentum strategies of different speeds by blending simple slow and fast TS momentum strategies in various proportions. We also define market cycles stemming from the intersection of slow and fast strategies and discuss their economic linkages to market trends and turning points.

[8]Recent studies focus on applications of momentum to *aggregate* factors. Ehsani and Linnainmaa (2019) show evidence of TS momentum across equity factors that appears to subsume the CS momentum risk factor. A companion paper by Arnott et al. (2019) shows that CS factor momentum is a pervasive phenomenon that, moreover, drives industry momentum. Gupta and Kelly (2019) show that portfolios of TS factor momentum can add value to a wide array of investment strategies.

2.1. Characterizing Speed and Market Cycles

We construct a simple framework to highlight the properties of slow and fast TS momentum. At date t, if the trailing 12-month return (arithmetic average monthly return), $r_{t-12,t}$, is nonnegative, then the slow strategy goes long one unit in the subsequent month, otherwise, it goes short one unit:

$$w_{\text{SLOW},t} := \begin{cases} +1 & \text{if } r_{t-12,t} \geq 0, \\ -1 & \text{if } r_{t-12,t} < 0. \end{cases} \tag{1}$$

If the prior 1-month return, $r_{t-1,t}$, is nonnegative then the fast strategy goes long one unit in the subsequent month, otherwise, it goes short one unit:

$$w_{\text{FAST},t} := \begin{cases} +1 & \text{if } r_{t-1,t} \geq 0, \\ -1 & \text{if } r_{t-1,t} < 0. \end{cases} \tag{2}$$

The realized returns of the slow and fast strategies for month $t+1$ are $r_{\text{SLOW},t+1} = w_{\text{SLOW},t} r_{t+1}$ and $r_{\text{FAST},t+1} = w_{\text{FAST},t} r_{t+1}$, respectively, where r_{t+1} is the realized underlying market return from date t to date $t+1$. This strategy design intentionally omits more complex features and is similar to that used by Goyal and Jegadeesh (2017) and Huang, Li, Wang, and Zhou (2019). In particular, the signal is not scaled by trailing volatility as in the work of Moskowitz, Ooi, and Pedersen (2012) and does not exponentially weight past prices.[9]

The 12-month trailing return is a relatively slow-moving signal from month to month since 11 of 12 months of returns (92%) overlap in subsequent signals.[10] In contrast, the 1-month trailing return is a relatively fast-moving signal, having no overlapping returns between subsequent signals. We could have used horizons longer than 12 months to define slow momentum, but the signal overlap for such horizons in subsequent months would not differ materially from the high 92% overlap of the 12-month signal. Moreover, 12 months is the standard horizon analyzed in the TS momentum literature (Moskowitz, Ooi, and Pedersen, 2012, and Huang et al., 2019, among others). We could have used horizons longer

[9]Note that volatility-targeting may constitute a phenomenon independent from TS momentum (e.g. Kim et al., 2016, Moreira and Muir, 2017, and Harvey et al., 2018). In addition, trend following is often identified with moving-average crossover strategies, which exponentially weight past prices, whereas our SLOW and FAST portfolios average past returns. As shown by Levine and Pedersen (2016), moving-average crossovers and time-series momentum are close to equivalent. More generally, our focus is on blending strategies with different speeds rather than elaborating on a particular definition of a single speed.

[10]Note that we do not skip the immediately lagged month as is done in some momentum strategies on individual securities in order to avoid short-term idiosyncratic reversals. Our empirical application is stock indices, for which the immediately lagged returns tend not to exhibit such reversals.

than one month (e.g., 2 or 3 months) to define fast momentum. However, longer horizons have significant signal overlap (50% for the 2-month signal and 67% for the 3-month signal), which materially slows changes in their strategy positions. Moreover, the 1-month horizon is the shortest at the monthly level of data. For these reasons, we chose 12-month and 1-month horizons.[11]

Next, we define a continuum of intermediate strategies with signal speeds between SLOW and FAST by

$$w_t(a) := (1-a)w_{\mathrm{SLOW},t} + a\, w_{\mathrm{FAST},t}, \tag{3}$$

$$r_{t+1}(a) := w_t(a)r_{t+1} = (1-a)r_{\mathrm{SLOW},t+1} + a\, r_{\mathrm{FAST},t+1}, \tag{4}$$

where the speed parameter $a \in [0,1]$ is a scalar. At $a = 0$, the speed is slow: $w_t(0) = w_{\mathrm{SLOW},t}$. Here, if the preceding 12-month return is positive, the strategy is long one unit even when the most recent month's return is negative. At $a = 1$, the speed is fast: $w_t(1) = w_{\mathrm{FAST},t}$. Here, any change of sign in the most recent month's return results in a strategy position change. For $a \in (0,1)$, the speed is intermediate and, in particular, at $a = \frac{1}{2}$ we call the speed "medium" (MED):

$$w_{\mathrm{MED},t} := w_t\left(\tfrac{1}{2}\right) = \tfrac{1}{2}w_{\mathrm{SLOW},t} + \tfrac{1}{2}w_{\mathrm{FAST},t}, \tag{5}$$

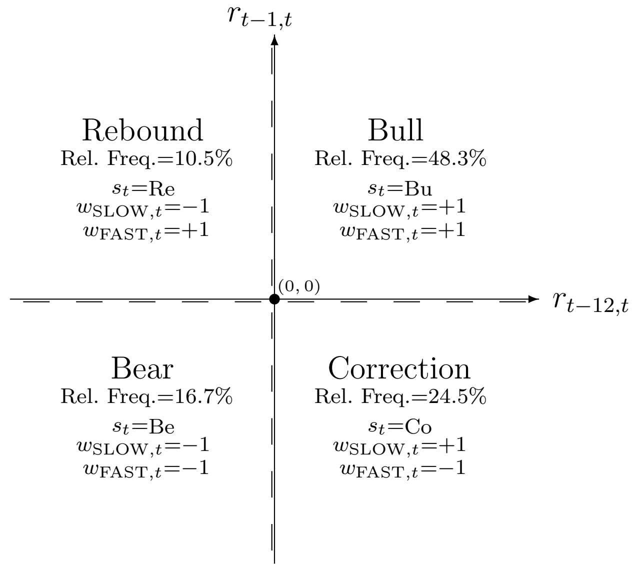
with $r_{\mathrm{MED},t+1} := w_{\mathrm{MED},t}r_{t+1}$.

Blending slow and fast momentum portfolios to form intermediate-speed strategies is a key analytical choice. By doing so, we embed sensitivity to periods of disagreement between the slow and fast strategies, which potentially signal turning points. For example, when SLOW indicates a long position (+1) and FAST indicates a short position (−1), then the intermediate-speed strategy with speed $a = \frac{3}{4}$ takes a lower magnitude short position ($-\frac{1}{2} = \frac{1}{4}(+1) + \frac{3}{4}(-1)$). The MED portfolio is out of the market altogether ($0 = \frac{1}{2}(+1) + \frac{1}{2}(-1)$). In contrast, strategies that go long or short based on the sign of some horizon of trailing returns $r_{t-k,t}$ do not scale down their positions when signals of longer and shorter horizons disagree. For example, the sign of the six-month signal $r_{t-6,t}$ could take several months to reflect a turning point in trend, during which time the strategy is fully long or short.[12]

[11]The empirical evidence pertaining the U.S. stock market supports our rationale. Over the last 50 years, the 3-month TS momentum portfolio showed a higher correlation to 12-month momentum than to 1-month momentum (43% and 34%, respectively), whereas the correlation of the 24-month momentum portfolio to 12-month momentum was above 60%. In addition, the turnover of the 1-month signal was twice as large as the one of 3-month signal, another indication of the differences underlying two apparently similar strategies.

[12]In Appendix B, we compare the performance of the 6-month TS momentum strategy, MOM6, with our intermediate-speed strategies and find that across different metrics MOM6 performs similarly to the SLOW strategy (i.e., MOM12). This result is not surprising because 5 of 6 months (83%) in the 6-month signal

Figure 2: **Stock Market Cycles as a Function of Momentum**



Notes: This diagram defines the four cycles of an asset's price trajectory as a function of SLOW and FAST momentum positions. A month ending at date t is classified as Bull if both the trailing 12-month return (arithmetic average monthly return), $r_{t-12,t}$, is nonnegative, and the trailing 1-month return, $r_{t-1,t}$, is nonnegative. A month is classified as Correction if $r_{t-12,t} \geq 0$ but $r_{t-1,t} < 0$; as Bear if $r_{t-12,t} < 0$ and $r_{t-1,t} < 0$; and as Rebound if $r_{t-12,t} < 0$ but $r_{t-1,t} \geq 0$. Also, the diagram reports the relative frequency of monthly aggregate U.S. stock market cycles over the 50-year period from 1969-01 to 2018-12. Market returns are U.S. excess value-weighted factor returns (Mkt-RF) from the Kenneth French Data Library.

Figure 2 maps the intersection of slow and fast signals to one of four observable *market cycles* or states. We use s_t to denote the state at date t. To capture the notion of a sustained upward trend, we label a month ending at date t as Bull if both the 12-month and 1-month trailing returns are nonnegative ($r_{t-12,t} \geq 0$ and $r_{t-1,t} \geq 0$, on the upper-right quadrant of the diagram). We label the other three quadrants of the diagram (going clockwise) of trailing 12-month and 1-month returns as Correction, Bear, and Rebound, respectively. We will also use the abbreviations Bu, Co, Be, and Re to denote the respective market states (e.g., s_t = Re for a Rebound). Admittedly, the labels assigned to these cycles have a relatively loose meaning. In particular, short-lived market gyrations could lead to brief and unintuitive correction or rebound classifications, which could be addressed by more sophisticated classification rules. Yet, because our simple cycle definitions are one to one with the four possible slow/fast strategy position pairs for the subsequent month, these cycles provide an exact ex-ante specification of the slow and fast momentum strategies, and by extension, all intermediate speed strategies in (3) and (4).

Over the last 50 years, Bull months have been the most common with a relative monthly

$r_{t-6,t}$ overlap in consecutive months. This overlap is close to that of the 12-month signal $r_{t-12,t}$ (92%).

frequency of 48.3%, reflecting the average positive risk premium offered by the U.S. stock market. Bear months are relatively uncommon—approximately one-sixth of the time (16.8%)—whereas Correction and Rebound months amount to the remaining 34.8% of the months. In other words, about once every three months, on average, SLOW and FAST suggest a different position in the stock market. These phases will be the focus of much of our analysis.

Market cycles conveniently map to the properties of SLOW and FAST for analysis. For instance, consider the strategy in (5) that employs the medium-speed signal, which is the equally-weighted average of the slow and fast strategies. Conditioning on the various market cycles as of date t, the equally-weighted strategy return at date $t+1$, $r_{\text{MED},t+1} = w_{\text{MED},t} r_{t+1}$, has long-run expected value:

$$\begin{aligned}\mathbf{E}[r_{\text{MED},t+1}] &= \mathbf{E}[\tfrac{1}{2}\,(w_{\text{SLOW},t} + w_{\text{FAST},t})\, r_{t+1}] \\ &= \mathbf{E}[\tfrac{1}{2}\,(1+1)\, r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] + \mathbf{E}[\tfrac{1}{2}\,(1-1)\, r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}] \\ &\quad + \mathbf{E}[\tfrac{1}{2}\,(-1-1)\, r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}] + \mathbf{E}[\tfrac{1}{2}\,(-1+1)\, r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}] \\ &= \mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]. \end{aligned} \tag{6}$$

This framework can be extended beyond the calculation of expected returns. In particular, the nature of market cycles also has an impact on the Sharpe ratio of a trend strategy (see Section 3.1) and its skewness (see Section 3.3). Moreover, it opens the intriguing possibility of "speed timing" by strategically adjusting the speed parameter a across market states, a challenge so far not tackled by any existing paper (see Section 4). Next, we further motivate the relevance of our cycles by evaluating their economic linkages.

2.2. The Economic Linkages of Market Cycles

Explanations of momentum rely on either behavioral or rational foundations.[13] We conjecture that the four stock market cycles we use in our analysis will relate differently to a broad set of variables, such as macroeconomic activity, liquidity risk, and survey-based

[13] Studies on behavioral foundations include Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), Odean (1998), Hong and Stein (1999, 2007), Gervais and Odean (2001), Grinblatt and Han (2002), Barberis and Shleifer (2003), Chan (2003), Cross et al. (2005), Grinblatt and Han (2005), Cross, Grinfeld, and Lamba (2006), Frazzini (2006), Shefrin (2008), Chui, Titman, and Wei (2010), Haldane (2010), Dasgupta, Prat, and Verardo (2011), Bandarchuk and Hilscher (2012), Antoniou, Doukas, and Subrahmanyam (2013), Avramov, Cheng, and Hameed (2013), and Chen and Lu (2013). Studies on rational foundations include Lo and MacKinlay (1990), Carhart (1997), Berk, Green, and Naik (1999), Ahn et al. (2002), Johnson (2002), Lewellen (2002), Allen, Morris, and Shin (2006), Watanabe (2008), Banerjee, Kaniel, and Kremer (2009), Verardo (2009), McLean (2010), Cespa and Vives (2012), Makarov and Rytchkov (2012), Vayanos and Woolley (2012, 2013), Aretz and Pope (2013), Jordan (2013), Liu and Zhang (2013), and Zhou and Zhu (2013).

expectations that could proxy for one or both types of explanations. In particular, we expect disagreement between SLOW and FAST to be associated with turning points in these variables. For example, we expect Correction to coincide with a worsening economic outlook and deteriorating market-wide liquidity.

We study three groups of macro indicators. The first group includes four indices assembled by the Federal Reserve Bank of Chicago that cover major components of the U.S. economy—production, consumption, employment, and sales—and monetary policy shocks estimated by Gertler and Karadi (2015). The second group includes five risk-related indicators: (i) the National Financial Conditions Index (NFCI) by the Chicago Fed; (ii) the Pastor and Stambaugh (2003) liquidity innovation measure (PS Liquidity); (iii) the Treasury–Eurodollar (TED) spread; (iv) the liquidity metric for the Treasury bond market (Noise) developed by Hu, Pan, and Wang (2013); and (v) the high-volatility/low-volatility valuation spread (Vol Spread) of Pflueger, Siriwardane, and Sunderam (2018). The last group of variables, which are based on media or survey data, includes: (i) the daily news-based Economic Policy Uncertainty Index (News Uncertainty) by Baker, Bloom, and Davis (2016); (ii) the University of Michigan Consumer Sentiment Index; (iii) the Purchasing Manager's Index (PMI) compiled by the Institute for Supply Management; (iv) the current-quarter recession probabilities by the Survey of Professional Forecasters (SPF); and (v) the next-quarter corporate profit expectations also from the SPF. In total, we collected 15 time series of macro indicators.

In Table 1, we map the four market-cycle definitions to innovations in the 15 macro indicators. These innovations are the residuals from individual autoregressive models, whose order is determined by the Bayesian Information Criterion.[14] Our evidence reveals a common theme. During Bull and Bear markets, when SLOW and FAST signals agree, we observe significant positive and negative macroeconomic shocks, respectively. During Corrections and Rebounds, innovations are statistically insignificant. A potential shift or stall in the macro environment, captured by neither significant positive nor negative average innovations, tends to coincide with turning points in the cycles of the stock market. Only monetary policy shocks paint a somewhat different picture, with surprise cuts associated with Rebound phases and hikes with Correction phases.

Risk and survey measures are also correlated with the four market cycles. For instance, mispricing in the bond market (Noise) tends to deteriorate during Bear markets and recover during Rebound phases. Similarly, PMI and News Uncertainty appear to stall in Correction

[14] The only two exceptions to this approach consist of the liquidity innovation metric (PS Liquidity) constructed by Pastor and Stambaugh (2003) and the monetary policy shocks (MP Shock) estimated by Gertler and Karadi (2015).

Table 1: **Macro Innovations t-Statistics by Market Cycle**

Economy (Surprises)					
Market Cycle	Production	Consumption	Employment	Sales	MP Shock
Bull	1.53	2.25	2.30	2.63	0.41
Correction	1.46	−0.29	1.06	−0.13	1.72
Bear	−3.15	−3.24	−4.06	−3.73	0.58
Rebound	−1.41	−0.23	−1.30	−0.65	−3.96
Risk (Surprises)					
Market Cycle	NFCI*	PS Liquidity	TED*	Noise*	Vol Spread
Bull	1.72	4.63	1.67	1.53	3.80
Correction	−1.21	−1.67	−1.67	−0.52	−1.02
Bear	−3.41	−6.22	−3.03	−4.29	−4.19
Rebound	2.32	0.43	2.00	2.15	−0.86
Surveys (Surprises)					
Market Cycle	News Uncertainty*	Consumer Sentiment	PMI	SPF Recession*	SPF Corporate Profits
Bull	2.09	1.64	3.48	2.30	1.80
Correction	0.06	−0.10	−0.77	−1.21	−0.61
Bear	−4.04	−3.30	−3.21	−3.00	−2.83
Rebound	−0.28	0.09	−2.39	0.84	0.76

Notes: This table reports the t-statistics of the innovations in 15 macro series conditional on four states of the U.S. stock market (Bull, Correction, Bear, and Rebound). These innovations are the residuals from individual autoregressive models, whose order is determined by the Bayesian Information Criterion. The only two exceptions to this approach consist of the Pastor and Stambaugh (2003) liquidity innovation metric (PS Liquidity) and the monetary policy shocks (MP Shock) estimated by Gertler and Karadi (2015). The symbol * indicates that for the selected series the sign is reversed, so that a negative shock can be interpreted as a negative outcome. A month ending at date t is classified as Bull if both the trailing 12-month return (arithmetic average monthly return), $r_{t-12,t}$, is nonnegative, and the trailing one-month return, $r_{t-1,t}$, is nonnegative. A month is classified as Correction if $r_{t-12,t} \geq 0$ but $r_{t-1,t} < 0$; as Bear if $r_{t-12,t} < 0$ and $r_{t-1,t} < 0$; and as Rebound if $r_{t-12,t} < 0$ but $r_{t-1,t} \geq 0$. The starting and ending dates of the macro series vary according to availability from the original sources, and they are listed in Appendix A. The stock-market returns are U.S. excess value-weighted factor returns (Mkt-RF) from the Kenneth French Data Library.

and Rebound phases, the phases during which MED exits to cash. Recession expectations by the SPF tend to spike during Bear phases and dissipate during Rebound and Bull phases.

In summary, Correction and Rebound phases are consistent indicators of potential turning points in the economy and in the broad risk appetite. Accordingly, strategies that vary with market cycles, such as the intermediate-speed strategies, have the potential to be more effective than those that do not. In the next section, we analyze which speeds perform better than others and use the lens of market-cycle information to explain why.

3. Static Speed Selection

Our empirical analyses focus on U.S. aggregate stock market returns over the last 50 years.[15] Our model-free analytical results shed light on the fundamental drivers of such empirical performance and apply to all markets. We examine the performance in other stock markets in Section 5 and in other time periods in later sections and in the appendices. Results are gross of transaction costs.

Table 2 summarizes estimates of basic unconditional moments for TS momentum strategies of various speeds alongside the long-only market strategy, whose strategy weights equal to one every month.[16] We highlight these results in three parts: return and risk, market timing, and tail behavior.

Return and risk. First, momentum strategies of every speed exhibit positive average historical returns in our backtest.[17] Although they all report average returns similar in magnitude to the long-only market strategy, intermediate-speed strategies achieve their returns with lower volatilities, and therefore report the highest average return per unit of risk as measured by the Sharpe ratio. We analyze these results in Section 3.1.

Market timing. Second, momentum strategies of every speed have positive average positions in the underlying market (ranging from 0.18 to 0.46 long, on average), yet their market betas are relatively small in magnitude with negative point estimates for intermediate-to-fast speeds. Intermediate-speed strategies are approximately market-beta neutral and deliver alphas having the largest t-statistics. We analyze these results in Section 3.2.

Tail behavior. Third, momentum strategies of every speed exhibit more desirable skewness than the market strategy. Higher positive (or less negative) skewness is a desirable property for most investors, and skewness of intermediate-to-fast strategies performs best in this regard. Moreover, intermediate-speed strategies have less severe maximum drawdowns and record the highest average returns per unit of drawdown risk. We analyze these results in Section 3.3.

These three categories of results emphasize several merits of *intermediate*-speed momentum strategies. In particular, MED delivers the highest Sharpe ratio (0.51) and it does so

[15]We evaluate the performance of our static speed strategies over the last 50 years for comparison with our dynamic strategies in later analyses for which data prior to the last 50 years is used as a warm-up period for out-of-sample evaluation over the last 50 years. In Appendix C, we report results based on a longer evaluation period beginning in 1927 from which we draw similar inferences.

[16]We report the performance since 1927 in Appendix C and over the last 15 years in Appendix D.

[17]Returns are gross of transaction costs, which we estimate to be modest under conservative assumptions. Let the bid–ask spread on S&P 500 Index futures be 2 basis points (bps), roll costs 0.5 bps (with rolls happening every quarter), and commissions 0.1 bps; then the ratio of alpha to transaction costs does not surpass 3% for the strategies reported in Table 2.

Table 2: **Performance Summary by Speed**

	Market	$a=0$ SLOW	$a=\frac{1}{4}$	$a=\frac{1}{2}$ MED	$a=\frac{3}{4}$	$a=1$ FAST
Return and Risk						
Average (%) (anlzd.)	5.91	6.46	6.17	5.88	5.59	5.30
Volatility (%) (anlzd.)	15.64	15.62	12.72	11.60	12.74	15.66
Sharpe Ratio (anlzd.)	0.38	0.41	0.48	0.51	0.44	0.34
Market Timing						
Average Position	1.00	0.46	0.39	0.32	0.25	0.18
Market Beta	1.00	0.15	0.05	−0.04	−0.13	−0.23
Alpha (%) (anlzd.)	0.00	5.58	5.85	6.12	6.39	6.66
Alpha t-statistic	—	2.54	3.24	3.71	3.57	3.07
Tail Behavior						
Skewness	−0.55	−0.43	−0.13	0.02	0.03	0.15
Max. Drawdown (%)	−54.36	−43.43	−37.96	−34.43	−34.07	−44.53
Average (anlzd.)/|Max. DD|	0.11	0.14	0.16	0.17	0.17	0.12

Notes: This table reports the sample average, volatility, Sharpe ratio, average position, market beta and alpha, alpha t-statistic, skewness, maximum drawdown, and ratio of average return to absolute maximum drawdown for monthly returns of the long-only market strategy (Market) and of TS momentum strategies of various speeds. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals −1. The fast strategy weight, $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative, and otherwise equals −1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

with approximate market beta neutrality (beta of −0.04), delivering the most statistically significant alpha (t-statistic of 3.71), without exhibiting the negative skewness behavior of the underlying market nor the severe drawdown behavior of the market and the slow and fast strategies. In the following three subsections, we identify properties of the stock market's return distribution and develop related theory to help explain these findings.

3.1. Return and Risk

3.1.1. Conditional Return Contributions

Table 3 (Panel A) decomposes the unconditional average returns in the first row of Table 2 into their conditional contributions following each market cycle. Specifically, we employ the following decomposition of returns using zero-one indicators of the market cycle at date t:

$$r_{t+1} = r_{t+1}1_{\{\text{Bu}\}} + r_{t+1}1_{\{\text{Be}\}} + r_{t+1}1_{\{\text{Co}\}} + r_{t+1}1_{\{\text{Re}\}}. \tag{7}$$

For example, the conditional contribution to speed-a TS momentum strategy after Bull cycles is the sample estimate of $\mathbf{E}[r_{t+1}(a)1_{\{\text{Bu}\}}]$. The first row of Panel A repeats the unconditional values reported in Table 2 for ease of reference. The middle four values in each column (and the last two values in each column) sum to the value in the first row (subject to rounding).

First, consider the decomposition of average returns to the slow strategy as compared with the long-only market strategy. After Bulls and Corrections, SLOW is long one unit, such as the market strategy, and so gets the same contribution as the market after these phases, namely, 4.59% and 1.59%, respectively. After Bears and Rebounds, SLOW shorts the market and flips the sign of the market contribution from −1.29% to 1.29% and from 1.01% to −1.01% after Bears and Rebounds, respectively.

Second, consider the average return contributions across different speeds. TS momentum strategies of all speeds get the same contribution after Bull and Bear cycles because each is long after Bulls and short after Bears. These states account for the bulk of the average returns to all strategies. Contributions after Corrections and Rebounds explain the differences in average returns by speed. SLOW's loss after Rebounds is similar in magnitude to FAST's loss after Corrections. The net contribution to SLOW after Corrections and Rebounds is 0.58%. In contrast, FAST flips the sign relative to SLOW after Correction and Rebound cycles, yielding a net contribution of −0.58%. MED is simply out of the market after Corrections and Rebounds. These facts explain why the unconditional average returns are close to each other and to that of the long-only market strategy.

Table 3 (Panel B) decomposes the unconditional variance of returns (squares of volatilities reported in second row of Table 2) into their cycle-conditional contributions following each observed market cycle based on the same cycle decomposition of returns as in (7). The cycle-conditional variance contributions are computed as the covariances of each of the terms on the right-hand side of (7) with the overall return. For example, the conditional variance contribution to speed-a TS momentum strategy after Bull cycles is the sample estimate of $\mathbf{Cov}[r_{t+1}(a)1_{\{\text{Bu}\}}, r_{t+1}(a)]$. The middle four values in each column (and the last two values in each column) sum to the value in the first row (subject to rounding).

The majority of return variance for all speeds of TS momentum strategies and for the long-only market strategy are generated after Bull and Bear markets. However, intermediate-speed strategies scale down their market exposure relative to SLOW and FAST after Corrections and Rebounds and, therefore, experience lower exposure to variance risk emanating

Table 3: **Market-Cycle Decomposition of Returns by Speed**

Panel A: Average Returns

Average (%) (anlzd.)	Market	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Unconditional	5.91	6.46	6.17	5.88	5.59	5.30
Cycle-Conditional Decomposition						
Bull	4.59	4.59	4.59	4.59	4.59	4.59
Correction	1.59	1.59	0.79	0.00	−0.79	−1.59
Bear	−1.29	1.29	1.29	1.29	1.29	1.29
Rebound	1.01	−1.01	−0.51	0.00	0.51	1.01
Bull + Bear	3.31	5.88	5.88	5.88	5.88	5.88
Correction + Rebound	2.60	0.58	0.29	0.00	−0.29	−0.58

Panel B: Variance of Returns

Variance (%) (anlzd.)	Market	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Unconditional	2.44	2.44	1.61	1.34	1.62	2.45
Cycle-Conditional Decomposition						
Bull	0.63	0.63	0.63	0.63	0.63	0.63
Correction	0.77	0.77	0.19	0.00	0.20	0.79
Bear	0.73	0.71	0.71	0.71	0.71	0.71
Rebound	0.31	0.32	0.08	0.00	0.08	0.31
Bull + Bear	1.36	1.34	1.34	1.34	1.34	1.35
Correction + Rebound	1.09	1.10	0.27	0.00	0.28	1.10

Notes: This table reports the unconditional sample average (Panel A) and sample variance (Panel B) of monthly returns of the long-only market strategy and of TS momentum strategies of various speeds and their contribution in months immediately following various market states. Contributions in each column sum to their corresponding unconditional values in the first row of each panel (subject to rounding) either across the four individual states in the middle rows or across the two state pairs in the last two rows. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Market states, s_t, are defined as follows. Bull: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Correction: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; Bear: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; and Rebound: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. The market return is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

from these states. In particular, after these periods of disagreement, MED exits the market altogether and avoids any exposure to the variance risk of these states. These facts explain why the unconditional return variances (and volatilities) are lower for intermediate-speed strategies compared to SLOW and FAST and compared to the long-only market strategy.

3.1.2. Mapping SLOW/FAST Disagreement to Sharpe Ratios

When one blends two strategies that are relatively uncorrelated, such as SLOW and FAST, it is not that surprising to see the variance decrease and the Sharpe ratio increase. In this subsection, however, we provide an alternative characterization of the Sharpe ratio in terms of a disagreement multiplier, which isolates the volatility contribution from states of disagreement between SLOW and FAST after Corrections and Rebounds.[18]

The role of such disagreement in determining Sharpe ratios manifests analytically as a *disagreement multiplier*, $D(a)$, which appears in (11) of Proposition 1. $D(a)$ captures the ratio of the volatility of the unconditional market return to the volatility of the momentum strategy, in terms of volatility contributions from Bull or Bear states and from Correction or Rebound states, respectively.[19]

Proposition 1 (Sharpe ratio decomposition). *The Sharpe ratio of $r_{t+1}(a)$ can be expressed in terms of the Sharpe ratios of $r_{SLOW,t+1}$ and $r_{FAST,t+1}$, respectively, and the market cycles, as follows:*

$$\begin{aligned}\mathbf{Sharpe}[r_{t+1}(a)] = (1-a)\,\mathbf{Sharpe}[r_{SLOW,t+1}]D(a, \mathbf{E}[r_{SLOW,t+1}]) \\ + a\,\mathbf{Sharpe}[r_{FAST,t+1}]D(a, \mathbf{E}[r_{FAST,t+1}]),\end{aligned} \tag{8}$$

where

$$D(a,\mu) := \sqrt{\frac{\mathbf{E}[r_{t+1}^2] - \mu^2}{\mathbf{E}[r_{t+1}^2|{}^{Bu}_{Be}]\mathbf{P}[{}^{Bu}_{Be}] + (2a-1)^2\mathbf{E}[r_{t+1}^2|{}^{Co}_{Re}]\mathbf{P}[{}^{Co}_{Re}] - (\mathbf{E}[r_{t+1}(a)])^2}}. \tag{9}$$

Approximating squared average strategy returns by $(\mathbf{E}[r_{t+1}(a)])^2 \approx 0$ for $a \in [0,1]$, we have

$$\mathbf{Sharpe}[r_{t+1}(a)] \approx ((1-a)\,\mathbf{Sharpe}[r_{SLOW,t+1}] + a\,\mathbf{Sharpe}[r_{FAST,t+1}])\,D(a), \tag{10}$$

[18] This multiplier is also useful for understanding the skewness of a blend, a topic we discuss in a later subsection.

[19] As before, let r_{t+1} denote the return on an underlying security and recall that the speed-a strategy return, $r_{t+1}(a)$, is just the weighted average of SLOW and FAST with weight a applied to FAST. Also, let $\{{}^{\text{Bu}}_{\text{Be}}\}$ denote the union of events Bull or Bear, and $\{{}^{\text{Co}}_{\text{Re}}\}$ the union of events Correction or Rebound. We apply similar notation throughout the paper for the union of any of the four market states. Also, in all analytical results in the paper, we assume all cycle-conditional first and second moments are defined and that second moments are non-zero.

where the term $D(a)$, multiplying the weighted average of Sharpe ratios in (10), is

$$D(a) := D(a,0) = \sqrt{\frac{\mathbf{E}[r_{t+1}^2]}{\mathbf{E}[r_{t+1}^2|{}^{Bu}_{Be}]\mathbf{P}[{}^{Bu}_{Be}] + (2a-1)^2\mathbf{E}[r_{t+1}^2|{}^{Co}_{Re}]\mathbf{P}[{}^{Co}_{Re}]}}. \tag{11}$$

$D(a)$ is greater than or equal to one and is maximized at $a = \frac{1}{2}$ on $a \in [0,1]$:

$$D\left(\tfrac{1}{2}\right) \geq D(a) \geq 1, \quad a \in [0,1]. \tag{12}$$

Moreover, $a = \frac{1}{2}$ is the unique maximizer with $D(\frac{1}{2}) > 1$ if the relative frequency of such states is not zero so that $\mathbf{E}[r_{t+1}^2|{}^{Co}_{Re}]\mathbf{P}[{}^{Co}_{Re}] > 0$.

Proof. Proof of Proposition 1 is in Appendix J. □

Proposition 1 indicates that the risk-adjusted performance of intermediate-speed momentum strategies is greater than the average risk-adjusted performances of the slow and fast momentum strategies, taken separately, as long as squared expected returns are relatively small and can be approximated by zero. For example, if the average return is 5%, then its square is only 0.25%. Note that the approximation is exact at the endpoints of the interval $[0,1]$. Moreover, because Proposition 1 is a model-free result, it extends beyond our running empirical example of the U.S. stock market to momentum strategies of various speeds applied in any market.[20]

Intermediate-speed strategies reduce volatility originating from Correction and Rebound states. Such volatility exposure is largely captured by the conditional average squared returns following these states, $\mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]$, times the relative frequency of these states, $\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]$. Hence, $\mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]$ represents the frequency-weighted contribution of these states to the unconditional volatility of the strategy. Intermediate-speed strategies with parameter a in the vicinity of $\frac{1}{2}$ tend to scale down positions following such periods, boosting the overall risk-adjusted performance by reducing exposure to risk. The factor $(2a-1)^2$ multiplying $\mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]$ captures this scaling-down, and this reduction in the denominator boosts the disagreement multiplier $D(a)$ above one. At $a = \frac{1}{2}$, volatility contributions from Correction and Rebound states are completely eliminated ($(2(\frac{1}{2}) - 1)^2 = 0$), and can explain MED's relatively high Sharpe ratio reported in the third row of Table 2, since average returns are roughly the same magnitude across all speeds.

[20] In Section 5, we report the Sharpe ratios of momentum strategies of various speeds applied to international equity markets.

3.2. Market Timing

3.2.1. The Determinants of Market Beta and Alpha

Given the preponderance of Bull cycles—around half of all months—as reported in Figure 2, it should not come as a surprise that trend following of the U.S. stock market implies a positive static tilt. Indeed, as detailed in Table 2, the average position of the momentum strategies ranged from 46% to 18% from slow to fast speeds. As also detailed in Table 2, however, the betas of the momentum strategy returns are relatively low in magnitude and range from 0.15 to −0.23 from slow to fast speeds, with negative point estimates for intermediate-to-fast speeds—something we might not expect given their positive tilts.

To understand this evidence, we first disentangle the dynamic and static bets in expected returns with a widely-used decomposition:

$$\begin{aligned} r_{t+1}(a) &= w_t(a) r_{t+1} \\ &= \underbrace{\left(w_t(a) - \mathbf{E}[w_t(a)]\right)}_{\text{dynamic}} r_{t+1} + \underbrace{\mathbf{E}[w_t(a)]}_{\text{static}} r_{t+1}, \end{aligned} \tag{13}$$

where the first equality above matches the first equality in (4). Taking expectations, we have

$$\mathbf{E}[r_{t+1}(a)] = \underbrace{\mathbf{Cov}[w_t(a), r_{t+1}]}_{\text{market timing}} + \underbrace{\mathbf{E}[w_t(a)]}_{\substack{\text{static}\\ \text{dollar}\\ \text{exposure}}} \mathbf{E}[r_{t+1}], \tag{14}$$

because $(w_t(a) - \mathbf{E}[w_t(a)])$ is mean zero. The covariance term is also known as the market-timing component, and it represents the share of expected returns generated by the dynamic bets of the signal as reflected by the strategy weights $w_t(a)$. In contrast, the average strategy weight summarizes the static dollar exposure of the strategy.

This decomposition directly relates to evidence that Huang et al. (2019) present. They argue that the profitability of a diversified 12-month TS momentum portfolio can be largely attributed to its static tilt, while the timing component—as captured by forecasting regressions—does not appear to be statistically significant. Because of this evidence they conclude that time-series predictability does not play a preponderant role in explaining the returns of trend portfolios and, moreover, that the 12-month horizon does not appear to offer much alpha.

We offer a different interpretation of the properties of trend portfolios. Our first observation is to caution against dismissing trend signals solely based on forecasting regressions of excess returns. On the one hand, static allocations do constitute a significant share of overall expected returns with the market-timing component adding insignificant or only marginally significant returns. On the other hand, the market-timing component adds negative betas with respect to the underlying market, which offset the beta of the static allocation.[21] These betas imply meaningful alphas, which map to the alphas estimated in Table 2.

To understand this latter point, we next disentangle the dynamic and static components in the market covariance, beta, and alpha:

Proposition 2 (Covariance with underlying market)**.** *The (contemporaneous) covariance between the speed strategy returns and the long-only market strategy returns can be decomposed as follows:*

$$\begin{aligned}&\mathbf{Cov}[r_{t+1}(a), r_{t+1}] \\ &\quad = \mathbf{E}[w_t(a)]\mathbf{Var}[r_{t+1}] + \mathbf{Cov}[w_t(a), r_{t+1}]\mathbf{E}[r_{t+1}] + \mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1})^2], \end{aligned} \tag{15}$$

for $a \in [0, 1]$*. The market beta and alpha, respectively, can be decomposed as follows:*

$$\begin{aligned}&\mathbf{Beta}[r_{t+1}(a)] \\ &\quad = \underbrace{\mathbf{E}[w_t(a)]}_{\substack{\text{static} \\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), r_{t+1}]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]}_{\substack{\text{market timing} \\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}}_{\substack{\text{volatility timing} \\ \text{component}}}, \end{aligned} \tag{16}$$

$$\begin{aligned}&\mathbf{Alpha}[r_{t+1}(a)] \\ &\quad = \mathbf{Cov}[w_t(a), r_{t+1}]\left(1 - \frac{(\mathbf{E}[r_{t+1}])^2}{\mathbf{Var}[r_{t+1}]}\right) - \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}], \end{aligned} \tag{17}$$

for $a \in [0, 1]$*. Approximating squared average market returns by* $(\mathbf{E}[r_{t+1}])^2 \approx 0$*, we have*

$$\mathbf{Alpha}[r_{t+1}(a)] \approx \mathbf{Cov}[w_t(a), r_{t+1}] - \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]. \tag{18}$$

Proof. Proof of Proposition 2 is in Appendix J. □

Equation (16) of Proposition 2 indicates that the market beta of any of our momentum strategies can be decomposed into a static component and a market-timing component, similar to the decomposition of the expected return in (14); however, an important additional volatility-timing component arises, which reflects the predictability of strategy weights for subsequent return volatility. Therefore, if the momentum weights significantly covary with

[21] As reported in Panel A of Table E.1 in Appendix E, the timing share of the portfolios displays insignificant or only marginally significant excess returns, a result that echoes the regression results of Huang et al. (2019). Yet, as reported in Panel B of the same table, these portfolios recorded negative betas with respect to the underlying market.

the subsequent return variance, then the beta of the momentum portfolio is not well approximated by the beta of the average momentum position. Thus, even if the market-timing component is relatively small compared to a larger positive static component, the volatility-timing component could be relatively large, but of opposite sign, and enough to offset the static component of the market beta. As indicated in Table 4, this is indeed the case for the U.S. stock market.

Table 4: **Beta and Alpha Decompositions by Speed**

Market Beta and Alpha	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Beta	0.15	0.05	−0.04	−0.13	−0.23
Alpha (%) (anlzd.)	5.58	5.85	6.12	6.39	6.66
Alpha t-stat.	2.54	3.24	3.71	3.57	3.07
Beta Components					
Static	0.457	0.387	0.317	0.247	0.177
Market Timing	0.008	0.008	0.008	0.008	0.009
Volatility Timing	−0.315	−0.339	−0.364	−0.389	−0.414
Alpha Components (%) (anlzd.)					
Market Timing	3.72	3.84	3.96	4.09	4.21
Volatility Timing	1.86	2.00	2.15	2.30	2.44

Notes: This table reports the sample market beta, alpha, and alpha t-statistic of monthly returns of momentum strategies of various speeds repeated from Table 2. The table also reports the (additive) decomposition of beta into static, market-timing, and volatility-timing components according to estimates of the terms in

$$\mathbf{Beta}[r_{t+1}(a)] = \underbrace{\mathbf{E}[w_t(a)]}_{\substack{\text{static}\\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), r_{t+1}]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]}_{\substack{\text{market timing}\\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}}_{\substack{\text{volatility timing}\\ \text{component}}}; \tag{16}$$

and the (additive) decomposition of alpha into market timing and volatility timing according to estimates of the terms in

$$\mathbf{Alpha}[r_{t+1}(a)] = \mathbf{Cov}[w_t(a), r_{t+1}] \left(1 - \frac{(\mathbf{E}[r_{t+1}])^2}{\mathbf{Var}[r_{t+1}]}\right) - \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]. \tag{17}$$

The slow strategy weight applied to the market return in month $t + 1$, $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals −1. The fast strategy weight, $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative, and otherwise equals −1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1 - a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0, 1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

Table 4 repeats the market beta, alpha, and alpha t-statistic estimates reported in Table 2 for ease of reference and also shows the market beta breakdown into the three (additive) components corresponding to the terms in (16). The static beta component equals the average position as reported in Table 2. The market-timing component of beta is small and near zero for all speeds. The volatility-timing component of beta is roughly as large in magnitude as the static component, but of opposite sign. Together, these terms sum to the market betas in the top row.

Table 4 also reports the market alpha breakdown into its two (additive) components corresponding to the terms in (17). The market-timing component in the alpha approximation of (18) is identical to the market-timing component in the widely-used expected return decomposition of (14). The alpha decomposition reveals, however, that volatility timing, in addition to market timing, can be a driver of alpha. Indeed, Table 4 reports that the volatility-timing component composes a large portion—over 33%—of each strategy's overall alpha estimated over the last 50 years.[22]

To summarize, the average position of a TS momentum portfolio does not need to reflect the beta exposure of the strategy. If weights predict volatility, then the market timing and static dollar exposure terms offer an incomplete picture of the alpha generated by TS momentum.

3.2.2. Relation to Volatility-Managed Portfolios

The relevance of volatility timing for TS momentum portfolios should not entirely be a surprise. The contemporaneous correlation between stock market returns and their monthly volatility has been about −28% over the last 50 years.[23] This well-known empirical regularity is suggestive of market returns' predictive ability of subsequent volatility and it highlights a potential overlap between momentum strategies and so-called volatility-managed portfolios.[24]

[22]Over the last 15 years, the contribution of volatility timing is 65% or more of each strategy's overall alpha. See Appendix F.

[23]This negative association is known as the leverage effect in reference to an economic explanation of the phenomenon offered by Black (1976) and Christie (1982). According to this explanation, negative equity returns lead to higher firm leverage, which in turn should translate into higher equity volatility as firms become riskier. See also Harvey et al. (2018).

[24]In Appendix I we study the predictability of TS momentum portfolio weights for different aspects of returns. We show analytically that intermediate-speed strategy weights have higher correlations with *every* aspect of subsequent market returns (returns, return deviations from their mean, lower-tail return events, upper-tail return events, and so forth) relative to the average correlations of slow and fast strategy weights with such aspects of subsequent returns. We empirically demonstrate this behavior for several key aspects of returns. Weights of intermediate-speed strategies are most negatively correlated with subsequent return volatility, which itself is negatively and contemporaneously correlated with returns. Hence, intermediate-speed strategies tend to go short when next-month return volatility is high and, relatedly, when such returns tend to be lower. Therefore, we expect and report that the alphas, which are partially driven by volatility timing, are more statistically significant for intermediate-speed strategies.

Moreira and Muir (2017) document that managing the leverage of a strategy based on trailing volatility can increase Sharpe ratios and deliver alpha with respect to the underlying strategy. Pertaining to the U.S. stock market, Moreira and Muir highlight two key ingredients: first, trailing volatility tends to be uncorrelated to subsequent returns; and second, volatility tends to be persistent at short horizons. An investor simply needs to increase exposure to the stock market following low-volatility states and decrease it following high-volatility states. Given the negative association between returns and volatility, is TS momentum just a reformulation of volatility management?

As detailed in Appendix G, TS momentum indeed shares similarities with the VOM originally formulated by Moreira and Muir (2017). For instance, different momentum speeds and the VOM portfolio tend to buy (or overweight) the stock market following Bull cycles and sell (or underweight) it following Bear cycles. Yet, over the last 50 years, the VOM strategy failed to deliver a statistically significant alpha (t-statistic of 1.50). Almost the entirety of this alpha—about 2% annualized—is explained by volatility timing, with no meaningful role played by market timing. In contrast, as explained in the previous section, momentum strategies combine both market- and volatility-timing components, with the former constituting the majority of the portfolios' alphas over this time period. To summarize, VOM strategies applied to the U.S. stock market appear to possess little tactical information beyond that offered by momentum portfolios of different speeds.

3.3. Tail behavior

3.3.1. Market-cycle decomposition

Table 5 reports various percentiles of monthly market returns in months following each of the four market cycles. Corrections introduce extreme outcomes and volatility despite most outcomes being positive (median monthly return of +1.07%).[25] Yet extreme outcomes tend to be more extreme on the downside than the upside. The fast strategy tends to flip deeper Correction losses into gains by going short after Corrections, which can help explain its slightly positive point estimate for skewness in Table 2. However, FAST also has full exposure to volatility from both Correction and Rebound states, in which the spread of returns is larger on both the positive and negative sides relative to Bull states. Intermediate-speed strategies reduce exposure to both volatility and extreme events associated with these states. In particular, MED avoided this exposure altogether with a zero position in months

[25] Five of the 10 worst months in the last 50 years were after Correction phases: −23.24% (1987-10); −16.08% (1998-08); −12.90% (1980-03); −11.91% (1978-10); and −10.72% (2000-11). Four of the 10 best months in the last 50 years were after Correction phases: 12.47% (1987-01); 12.16% (1976-01); 11.35% (2011-10); and 10.84% (1991-12).

after these states, which can help explain its approximately zero point estimate for skewness in Table 2.

Table 5: **Cycle-Conditional Market Return Distributions**

Return Percentiles (%)	Bull	Correction	Bear	Rebound
MIN	−9.55	−23.24	−17.23	−10.35
P01	−7.85	−14.62	−12.79	−10.16
P05	−4.64	−7.14	−10.10	−8.41
P10	−3.37	−5.64	−8.06	−5.51
P25	−1.51	−2.08	−4.83	−2.44
P50	1.05	1.07	−0.89	1.15
P75	3.07	3.82	3.98	4.59
P90	4.68	5.84	6.82	7.24
P95	6.13	7.15	7.99	7.98
P99	7.21	11.79	13.68	10.61
MAX	9.59	12.47	16.10	11.30

Notes: This table reports various percentiles of monthly market returns in months immediately following each of four market states, s_t, which are defined in terms of slow and fast momentum strategy positions as follows: The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Bull: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Correction: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; Bear: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; and Rebound: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. MIN and MAX are the lowest and highest observed monthly returns, respectively. PX is the X-th percentile. For example, P10 is the 10th percentile of monthly returns. The market return is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

Downside risk exposure after each market cycle can help explain the pattern of maximum drawdowns across strategies reported in Table 2. After Bear states, the magnitudes of lower percentile (negative) returns are higher in most cases than the magnitudes of symmetrically higher percentile (positive) returns.[26] Momentum strategies of all speeds are short after Bear markets, turning this downside into upside. Consistent with this evidence, Table 2 reports lower maximum drawdowns than the long-only market strategy for all speeds. Intermediate speeds further reduce exposure to extreme downside events by scaling down after Corrections and Rebounds. Accordingly, intermediate speeds report lower maximum drawdowns and higher average returns per unit of absolute maximum drawdown in Table 2.

[26] MIN vs. MAX: $|-17.23\%| > 16.10\%$; P05 vs. P95: $|-10.10\%| > 7.99\%$; P10 vs. P90: $|-8.06\%| > 6.82\%$; P25 vs. P75: $|-4.83\%| > 3.98\%$.

3.3.2. Skewness

We formalize the relationship between skewness of MED relative to the skewness of SLOW and FAST in terms of MED's Sharpe ratio and the disagreement multiplier, $D(a)$, introduced in Section 3.1.2. The connection between the Sharpe ratio and the skewness of a random variable is illuminated in Lemma 3, which we apply to obtain Proposition 4.

Lemma 3. *For any Y such that its first three moments are defined and $\mathbf{SD}[Y] > 0$, then $\mathbf{Skew}[Y] = \frac{\mathbf{E}[Y^3]}{(\mathbf{SD}[Y])^3} - \mathbf{Sharpe}[Y]\,(3 + (\mathbf{Sharpe}[Y])^2)$, where $\mathbf{Sharpe}[Y] = \frac{\mathbf{E}[Y]}{\mathbf{SD}[Y]}$.*

Proof. Proof of Lemma 3 is in Appendix J. □

Proposition 4 (Skewness decomposition)**.** *The skewness of $r_{t+1}(a)$ can be expressed in terms of the skewness of $r_{SLOW,t+1}$ and $r_{FAST,t+1}$, respectively, and a disagreement multiplier. An exact expression as well as an approximation for all $a \in [0,1]$ based on $(\mathbf{E}[r_{t+1}(a)])^2 \approx 0$ for $a \in [0,1]$ are given in (J25) and (J26), respectively, in Appendix J. In the special case of $a = \frac{1}{2}$, we have*

$$\mathbf{Skew}[r_{t+1}(\tfrac{1}{2})] \approx \frac{1}{2}(\mathbf{Skew}[r_{SLOW,t+1}] + \mathbf{Skew}[r_{FAST,t+1}])\left(D(\tfrac{1}{2})\right)^3 + 3\,\mathbf{Sharpe}[r_{t+1}(\tfrac{1}{2})]\left(\left(D(\tfrac{1}{2})\right)^2\left[1 + \left(\frac{\mathbf{Sharpe}[r_{FAST,t+1}] - \mathbf{Sharpe}[r_{SLOW,t+1}]}{2}\right)^2\right] - 1\right), \tag{19}$$

where $D(\frac{1}{2})$ is as defined in (11) for $a = \frac{1}{2}$.

Proof. Proof of Proposition 4 is in Appendix J. □

Proposition 4 indicates that skewness of MED is scaled up relative to the average skewness of SLOW and FAST—since $\left(D(\frac{1}{2})\right)^3 > 1$ per Proposition 1—and shifted to the right when MED has a positive Sharpe ratio—since $\left(D(\frac{1}{2})\right)^2 > 1$ and because the term in the outer parentheses is always positive. The multiplier $D(\frac{1}{2})$ is the same disagreement multiplier which appears in Proposition 1, $D(a)$, evaluated at $a = \frac{1}{2}$. As reported in Table 2, the slow strategy has negative skewness of −0.43, but the fast strategy has positive skewness of 0.15. Their average skewness is negative at −0.14. The disagreement multiplier $D(\frac{1}{2}) = 1.34$ amplifies the first term in (19) by a factor of $\left(D(\frac{1}{2})\right)^3 = 2.42$, bringing its contribution to −0.34. The second term in (19) shifts this value to the right by 0.36, yielding the slight positive skewness of 0.02 for MED as reported in Table 2.

As before, because Proposition 4 is a model-free result, it extends beyond our running empirical example to momentum strategies of various speeds applied in any market. Moreover, as a corollary to this result, if SLOW and FAST both have nonnegative skewness and

Sharpe ratios when applied in some market, then the skewness of MED is going to be positive and higher than the maximum skewness of both SLOW and FAST.

4. Dynamic Speed Selection

The differences in the conditional return distributions following Correction and Rebound cycles (as reported in Table 5 and discussed in Section 3.3.1) raise the question of whether a dynamic momentum strategy could offer improved performance over a static strategy. For example, how might a strategy perform if its weights following Corrections and Rebounds were individually specified instead of implied by the weighted average of slow and fast strategies? Would strategy weights of 0 and $\frac{1}{2}$ following Corrections and Rebounds perform better than the zero weights implied by the MED strategy, or the $-\frac{1}{2}$ and $\frac{1}{2}$ weights of the $a = \frac{3}{4}$ strategy? We address questions of this nature in this section.

Instead of using a static speed parameter a, we let it vary dynamically with the four market cycles. The dynamic speed $a_{s(t)}$ at date t is a function of the observable market state $s(t)$ at that date, which is one of $\{\text{Bu}, \text{Co}, \text{Be}, \text{Re}\}$. For example, if in a Correction cycle at date t ($s(t) = \text{Co}$), then a_{Co} is the parameter which will govern the blending between slow and fast strategy weights in the subsequent month. If the cycle remains in Correction at date $t+1$, then we apply the same a_{Co} for the next month. If the cycle shifts to Bear at date $t+2$, then we apply a_{Be} for the next month, and so on.

The corresponding dynamic strategy return is

$$r_{t+1}(a_{s(t)}) = w_t(a_{s(t)})\, r_{t+1} = [(1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t}]r_{t+1}. \tag{20}$$

Note that since $w_{\text{SLOW},t} = w_{\text{FAST},t}$ with magnitude one after Bull or Bear cycles, the dynamic weight in (20) is invariant to the values of a_{Bu} and a_{Be}. That is, $w_t(a_{\text{Bu}}) = 1$ after Bull for all a_{Bu} and $w_t(a_{\text{Be}}) = -1$ after Bear for all a_{Be}. Nevertheless, the dynamic weight is sensitive to the values of a_{Co} and a_{Re} following Correction and Rebound cycles, respectively. In Proposition 5, we establish the values of these state-conditional speed parameters that maximize the steady-state Sharpe ratio of the dynamic returns.

Proposition 5 (Optimal dynamic speed)**.** *Consider the problem of choosing the state-conditional speeds $a_{s(t)}$ which will be applied following every occurrence of state $s(t)$ in order to achieve the highest steady-state Sharpe ratio:*

$$\max_{a_{s(t)}: s(t) \in \{Co, Re\}} \mathbf{Sharpe}[r_{t+1}(a_{s(t)})]. \tag{21}$$

If $\mathbf{E}[r_{t+1}|_{Bu}]\mathbf{P}[_{Bu}] > \mathbf{E}[r_{t+1}|_{Be}]\mathbf{P}[_{Be}]$*, then*

$$a_{Co} = \frac{1}{2}\left(1 - \frac{\mathbf{E}[r_{t+1}^2|_{Be}^{Bu}]\mathbf{P}\left[_{Be}^{Bu}\right]}{\mathbf{E}[r_{t+1}|_{Bu}]\mathbf{P}[_{Bu}] - \mathbf{E}[r_{t+1}|_{Be}]\mathbf{P}[_{Be}]}\frac{\mathbf{E}[r_{t+1}|_{Co}]}{\mathbf{E}[r_{t+1}^2|_{Co}]}\right), \tag{22}$$

$$a_{Re} = \frac{1}{2}\left(1 + \frac{\mathbf{E}[r_{t+1}^2|_{Be}^{Bu}]\mathbf{P}\left[_{Be}^{Bu}\right]}{\mathbf{E}[r_{t+1}|_{Bu}]\mathbf{P}[_{Bu}] - \mathbf{E}[r_{t+1}|_{Be}]\mathbf{P}[_{Be}]}\frac{\mathbf{E}[r_{t+1}|_{Re}]}{\mathbf{E}[r_{t+1}^2|_{Re}]}\right), \tag{23}$$

is the unique state-conditional speed pair that maximizes (21).

Proof. Proof of Proposition 5 is in Appendix J. □

Proposition 5 specifies the dynamic speed selections that maximize the steady-state Sharpe ratio in terms of cycle-conditional first- and second-population moments of market returns. The condition $\mathbf{E}[r_{t+1}|_{\text{Bu}}]\mathbf{P}[_{\text{Bu}}] > \mathbf{E}[r_{t+1}|_{\text{Be}}]\mathbf{P}[_{\text{Be}}]$ ensures that the weights are maximizers and not minimizers. This condition is typically satisfied because expected returns are typically positive following Bull cycles and negative following Bear cycles.

Because population values of the first and second moments in (22) and (23) are not observable, we use historical estimates of these moments to approximate their values. We use the label "DYN" to denote the investable strategy that employs state-dependent speeds based on estimated versions of (22) and (23), using only data prior to strategy implementation (i.e., no look-ahead bias). Specifically, DYN weights are defined as follows: $w_{\text{DYN},t} := w_t(\widehat{a}_{s(t)}) = (1-\widehat{a}_{s(t)})w_{\text{SLOW},t} + \widehat{a}_{s(t)}w_{\text{FAST},t}$, where $s(t) \in \{\text{Bu}, \text{Co}, \text{Be}, \text{Re}\}$; $\widehat{a}_{\text{Bu}} = \widehat{a}_{\text{Be}} = \frac{1}{2}$; and $\widehat{a}_{\text{Co}}$ and $\widehat{a}_{\text{Re}}$ are estimated using sample averages $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$, which denote estimates of the average and frequency of their given argument, respectively, based on data prior to strategy implementation.[27] For example, $\widehat{\mathbf{E}}[r_{t+1}^2|_{\text{Be}}^{\text{Bu}}]$ is the historical sample average of squared returns in months after Bull or Bear states within the historical estimation window, and $\widehat{\mathbf{P}}\left[_{\text{Be}}^{\text{Bu}}\right]$ is the frequency of Bull or Bear states within that historical sample. If these cycle-conditional return moments are relatively stable over time, then DYN should perform well out of sample.

4.1. Performance

Table 6 reports the out-of-sample performance of DYN over various evaluation windows in the last 50 years—from 50 years ago forward in 5 year increments to the most recent 15 years. We define the efficiency of the DYN strategy over each evaluation window as the ratio of its Sharpe ratio to that of the highest performing ex post state-conditional speed strategy, which we label the "OPT" strategy. OPT's Sharpe ratios vary over different evaluation

[27]If either of the estimates $\widehat{a}_{\text{Co}}$ or $\widehat{a}_{\text{Re}}$ fall outside the unit interval $[0, 1]$, then we set its value to the nearest endpoint, 0 or 1.

Table 6: **DYN Strategy Performance Over the Last 50 Years**

DYN Strategy Estimation Window			Evaluation					
			Evaluation Window			Sharpe Ratio (anlzd.)		
From (yr-mo)	To (yr-mo)	Length (yrs)	From (yr-mo)	To (yr-mo)	Length (yrs)	DYN $(\widehat{a}_{\text{Co}}, \widehat{a}_{\text{Re}})$	"Oracle" OPT	Efficiency DYN/OPT
1926-07	1968-12	42.5	1969-01	2018-12	50.0	0.524 (0.00, 0.58)	0.570	0.920
1926-07	1973-12	47.5	1974-01	2018-12	45.0	0.547 (0.07, 0.59)	0.572	0.956
1926-07	1978-12	52.5	1979-01	2018-12	40.0	0.611 (0.08, 0.65)	0.626	0.977
1926-07	1983-12	57.5	1984-01	2018-12	35.0	0.614 (0.22, 0.67)	0.623	0.985
1926-07	1988-12	62.5	1989-01	2018-12	30.0	0.688 (0.26, 0.69)	0.721	0.954
1926-07	1993-12	67.5	1994-01	2018-12	25.0	0.675 (0.11, 0.71)	0.684	0.988
1926-07	1998-12	72.5	1999-01	2018-12	20.0	0.564 (0.17, 0.69)	0.579	0.975
1926-07	2003-12	77.5	2004-01	2018-12	15.0	0.611 (0.16, 0.69)	0.621	0.984

Notes: This table reports the DYN momentum strategy's Sharpe ratio and its efficiency for various evaluation windows within the last 50 years. DYN is the dynamic (state-dependent) speeds strategy whose weights, $w_t(a_{s(t)}) = (1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t}$, are based on point estimates of the optimal state-dependent speeds, $a_{s(t)}$, using $\widehat{a}_{\text{Bu}} = \widehat{a}_{\text{Be}} = \frac{1}{2}$, $\widehat{a}_{\text{Co}} = \frac{1}{2}\left(1 - \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Co}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Co}]}\right)$, and $\widehat{a}_{\text{Re}} = \frac{1}{2}\left(1 + \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Re}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Re}]}\right)$, where $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$ denote estimates of the average and frequency of their given argument, respectively, based on data in the estimation window; $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative and otherwise equals −1; $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative and otherwise equals −1; and the four observable market states at date t are defined as: Bu: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Be: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; Co: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; and Re: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. If either of the estimates $\widehat{a}_{\text{Co}}$ or $\widehat{a}_{\text{Re}}$ fall outside the unit interval $[0, 1]$, then we set its value to the nearest endpoint, 0 or 1. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. OPT is the dynamic strategy that would have achieved the maximum Sharpe ratio, ex post. State-dependent speeds of both strategies are fixed over the evaluation window. For example, using 57.5 years of data from 1926-07 through 1983-12 to estimate speeds for DYN, its Sharpe ratio over the subsequent 35-year evaluation period from 1981-07 through 2018-12 was 0.614. Relative to the best possible ex-post state-dependent strategy, OPT, which had a Sharpe ratio of 0.623, DYN was 98.5% = 0.614/0.623 efficient.

windows from 0.570 to 0.721, indicating that different periods expose the strategies to different performance opportunities. Nevertheless, DYN consistently exhibits efficiency of 92% or higher across different evaluation windows, reflecting its capacity to exploit performance opportunities out of sample.

The out-of-sample performance of DYN also compares favorably to static-speed strategies. Table 7 reports the Sharpe ratio, its efficiency (relative to OPT), and average returns scaled by the absolute value of the maximum drawdown, for static strategies of various speeds and for DYN. Panel A corresponds to the most recent 50-year evaluation period (same as the

Table 7: **Dynamic vs. Static Performance per Unit Risk**

Panel A: 50-Year Evaluation Window, 1969-01 to 2018-12						
	Static					$\widehat{a}_{\text{Co}} = 0.00$
	$a = 0.00$	0.25	0.50	0.75	1.00	$\widehat{a}_{\text{Re}} = 0.58$
	SLOW		MED		FAST	DYN
Sharpe Ratio (anlzd.)	0.41	0.48	0.51	0.44	0.34	0.52
Efficiency (/OPT)	0.73	0.85	0.89	0.77	0.59	0.92
Avg. (anlzd.)/|Max. DD|	0.15	0.16	0.17	0.16	0.12	0.23

Panel B: 15-Year Evaluation Window, 2004-01 to 2018-12						
	Static					$\widehat{a}_{\text{Co}} = 0.16$
	$a = 0.00$	0.25	0.50	0.75	1.00	$\widehat{a}_{\text{Re}} = 0.69$
	SLOW		MED		FAST	DYN
Sharpe Ratio (anlzd.)	0.55	0.57	0.51	0.36	0.21	0.61
Efficiency (/OPT)	0.88	0.91	0.81	0.58	0.34	0.98
Avg. (anlzd.)/|Max. DD|	0.22	0.26	0.23	0.14	0.08	0.28

Notes: This table reports Sharpe ratios, their efficiencies relative to the OPT strategy, and average returns scaled by absolute maximum drawdowns for static-speed momentum strategies and for the DYN strategy over two different evaluation windows, in Panel A and Panel B, respectively. DYN is the dynamic (state-dependent) speeds strategy whose weights, $w_t(a_{s(t)}) = (1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t}$, are based on point estimates of the optimal state-dependent speeds, $a_{s(t)}$, using $\widehat{a}_{\text{Bu}} = \widehat{a}_{\text{Be}} = \frac{1}{2}$, $\widehat{a}_{\text{Co}} = \frac{1}{2}\left(1 - \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]} \frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Co}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Co}]}\right)$, and $\widehat{a}_{\text{Re}} = \frac{1}{2}\left(1 + \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]} \frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Re}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Re}]}\right)$, where $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$ denote estimates of the average and frequency of their given argument, respectively, based on data before the evaluation window beginning 1926-07; $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative and otherwise equals −1; $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative and otherwise equals −1; and the four observable market states at date t are defined as: Bu: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Be: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; Co: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; and Re: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. If either of the estimates $\widehat{a}_{\text{Co}}$ or $\widehat{a}_{\text{Re}}$ fall outside the unit interval $[0, 1]$, then we set its value to the nearest endpoint, 0 or 1. Static-speed strategy weights are formed according to $w_t(a) = (1 - a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$ for speed parameter $a \in [0, 1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. OPT is the dynamic strategy that would have achieved the maximum Sharpe ratio ex post. State-dependent speeds are fixed over the evaluation window.

first row of Table 6) and Panel B corresponds to the most recent 15-year evaluation period (same as the last row of Table 6). The estimated speed-pair for DYN in Panel A, which is based on data from 1926-07 through 1968-12, is $\widehat{a}_{\text{Co}} = 0.00$ and $\widehat{a}_{\text{Re}} = 0.58$;[28] whereas, in Panel B, estimates based on the 35-year-longer history from 1926-07 through 2003-12 yield $\widehat{a}_{\text{Co}} = 0.16$ and $\widehat{a}_{\text{Re}} = 0.69$. In each case, the estimated speed pair indicates relatively

[28] The point estimate for $\widehat{a}_{\text{Co}}$ is slightly negative, −0.04, so we set it to the nearest value in $[0, 1]$.

slow speed after Corrections and relatively fast speed after Rebounds. Under these state-conditional speeds, DYN exhibits both a higher Sharpe ratio and a higher average return per unit of drawdown risk than all of the static-speed strategies.[29]

4.2. Sensitivity to Estimation Error

Figure 3 graphs Sharpe ratio contour levels for momentum strategies with state-dependent speed pairs $(a_{\text{Co}}, a_{\text{Re}})$ on the unit square over the most recent 15-year evaluation period 2004-01 to 2018-12. Each speed pair is set at the beginning of the evaluation period. The concentric curves represent speed pairs with equal Sharpe ratio levels. Lighter-shaded regions reflect higher Sharpe ratios and darker-shaded regions reflect lower Sharpe ratios. Static strategies—strategies whose speeds do not vary by state—are represented by the points on the dotted diagonal line.[30] As before, OPT is the state-dependent speed pair that would have achieved the highest Sharpe ratio ex post, and DYN indicates the state-dependent pair based on estimates of state-conditional first and second moments using monthly data from 1926-07 to 2003-12 prior to the evaluation period.

The figure illustrates that the Sharpe ratio is less sensitive to deviations in speed pair values near the in-sample optimal speed pair, OPT, than it is farther away. The DYN speed pair based on the historical sample is close to OPT and is approximately as good a performer. The small dots represent the state-dependent speed pairs from 1,000 block bootstrap estimates of state-conditional first and second moments using monthly data from 1926-07 to 2003-12.[31] The bootstrap DYN speed pairs tend to fall into the upper-left quadrant of the graph in Figure 3, which corresponds to strategies that tilt to SLOW after Corrections and to FAST after Rebounds.

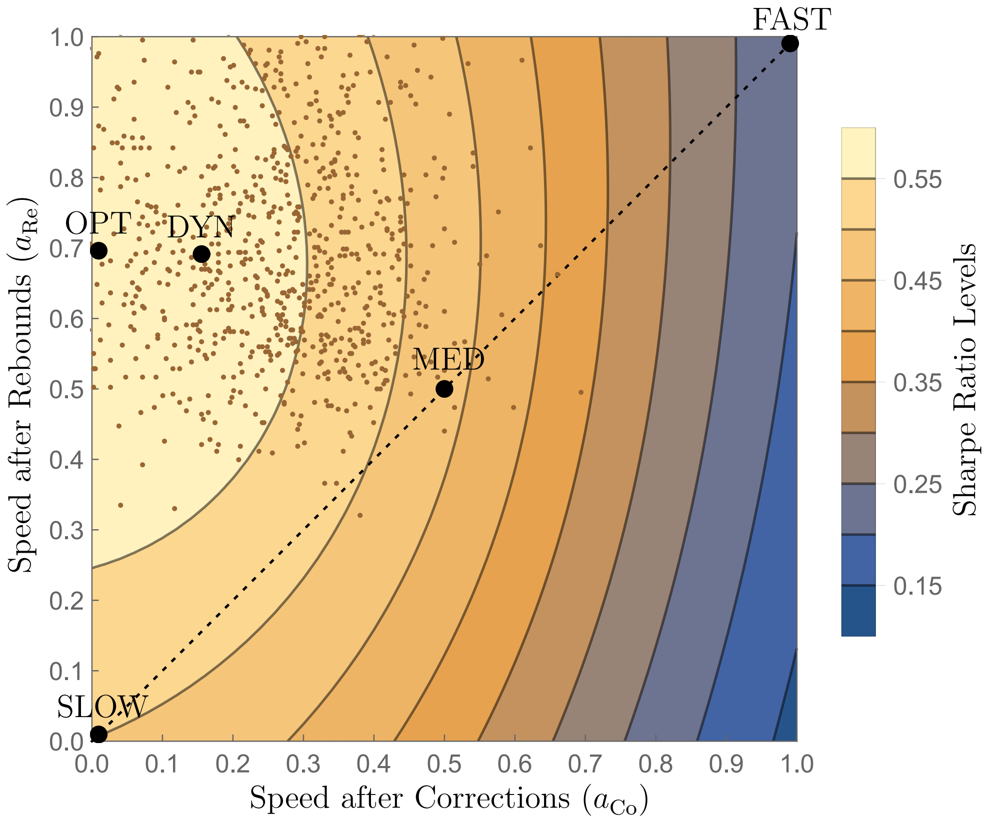
To summarize, state-dependent speed pairs that yield the highest performance tend to be those that are relatively slow after Corrections and relatively fast after Rebounds. Moreover, speed pairs in this region are closer to optimal and, in this region, Sharpe ratio performance is less sensitive to estimation error.

[29] Performance is reported gross of transaction costs, which again are estimated to be marginal. Under the same assumptions reported in footnote 17, we estimate the ratio of alpha to transaction costs for the dynamic strategy to be below 2%.

[30] Note that the intermediate-speed static strategy with $a = \frac{1}{4}$ achieves approximately the highest Sharpe ratio of all static strategies over the measurement period, consistent with values reported in Panel B of Table 7.

[31] The block bootstrap is performed as follows. We partition the 930 historical monthly returns on the U.S. stock market from 1926-07 to 2003-12 into 10 equal-sized nonoverlapping blocks of 93 consecutive months. We randomly sample 10 of these blocks with replacement to form one bootstrap historical sample. For each bootstrap sample, we determine cycle states and estimate state-conditional first and second moments. We plug these estimates into (22) and (23) to generate a speed pair for the sample.

Figure 3: **Sharpe Ratio Performance of Different State-Conditional Speed Pairs Over the Most Recent 15 Years**



Notes: This figure reports Sharpe ratio levels of dynamic-speed momentum strategies over the evaluation period 2004-01 to 2018-12 for speed pairs $(a_{\mathrm{Co}}, a_{\mathrm{Re}})$ where strategy weights take the form: $(1-a_{s(t)})w_{\mathrm{SLOW,t}}+a_{s(t)}w_{\mathrm{FAST,t}}$, where $w_{\mathrm{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative and otherwise equals -1; $w_{\mathrm{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative and otherwise equals -1; and the four observable market states at date t are defined as: Bu: $w_{\mathrm{SLOW},t} = w_{\mathrm{FAST},t} = +1$; Be: $w_{\mathrm{SLOW},t} = w_{\mathrm{FAST},t} = -1$; Co: $w_{\mathrm{SLOW},t} = +1, w_{\mathrm{FAST},t} = -1$; and Re: $w_{\mathrm{SLOW},t} = -1, w_{\mathrm{FAST},t} = +1$. Speeds $a_{\mathrm{Bu}} = a_{\mathrm{Be}} = 0.5$ for all strategies in the figure. The dotted diagonal highlights the continuum of static strategies, which apply the same speeds after all states—SLOW ($a_{s(t)} = 0$), MED ($a_{s(t)} = 0.5$), and FAST ($a_{s(t)} = 1$) are labeled. OPT represents the dynamic strategy pair that would have achieved the best Sharpe ratio ex post. DYN is the dynamic (state-dependent) speeds strategy whose weights are based on point estimates of the optimal state-dependent speeds, $a_{s(t)}$, using $\widehat{a}_{\mathrm{Bu}} = \widehat{a}_{\mathrm{Be}} = \frac{1}{2}$, $\widehat{a}_{\mathrm{Co}} = \frac{1}{2}\left(1 - \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\mathrm{Bu}}_{\mathrm{Be}}]\widehat{\mathbf{P}}[{}^{\mathrm{Bu}}_{\mathrm{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Bu}]\widehat{\mathbf{P}}[\mathrm{Bu}]-\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Be}]\widehat{\mathbf{P}}[\mathrm{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Co}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\mathrm{Co}]}\right)$, and $\widehat{a}_{\mathrm{Re}} = \frac{1}{2}\left(1 + \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\mathrm{Bu}}_{\mathrm{Be}}]\widehat{\mathbf{P}}[{}^{\mathrm{Bu}}_{\mathrm{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Bu}]\widehat{\mathbf{P}}[\mathrm{Bu}]-\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Be}]\widehat{\mathbf{P}}[\mathrm{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\mathrm{Re}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\mathrm{Re}]}\right)$, where $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$ denote estimates of the average and frequency of their given argument, respectively, based on data before the evaluation window beginning 1926-07. If either $\widehat{a}_{\mathrm{Co}}$ or $\widehat{a}_{\mathrm{Re}}$ fall outside the unit interval $[0, 1]$, then we set its value to the nearest endpoint, 0 or 1. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The small dots represent block bootstrap point estimates for DYN using monthly data from 1926-07 to 2003-12.

4.3. Market-Cycle Patterns

The conclusion from our state-dependent speed analysis—elect slower-speed momentum after Correction months and faster-speed momentum after Rebound months—is reinforced by market cycle patterns, as those reported in Table 8.

Table 8: **Market Cycle Transitions**

	Monthly Transition Probability (%)					
	Bull	Correction	Bear	Rebound	Up	Down
Bull	62.8	34.8	2.1	0.3	63.3	36.7
Correction	61.2	29.9	8.8	0.0	61.2	38.8
Bear	9.0	0.0	55.0	36.0	45.0	55.0
Rebound	14.3	1.6	42.9	41.3	55.6	44.4

Notes: This table reports the relative frequency of transitions from one market state (row) in month t to another market state (column) in month $t+1$ (first four columns) and the relative frequency of positive (Up) and negative (Down) returns next month (last two columns) over the 50-year period from 1969-01 to 2018-12. A month ending at date t is classified as Bull if both the trailing 12-month sum of excess returns to date, $r_{t-12,t}$, is nonnegative, and the excess return for the month, $r_{t-1,t}$, is nonnegative. A month is classified as Correction if $r_{t-12,t} \geq 0$ but $r_{t-1,t} < 0$. A month is classified as Bear if $r_{t-12,t} < 0$ and $r_{t-1,t} < 0$. A month is classified as Rebound if $r_{t-12,t} < 0$ but $r_{t-1,t} \geq 0$. Market cycles are evaluated on monthly excess value-weighted factor returns (Mkt-RF) from the Kenneth French Data Library. Evaluation period: 1969-01 to 2018-12.

Table 8 reports for the last 50 years the relative frequency of each market state in the subsequent month (first four columns) given the market state in the preceding month (rows). It also reports the relative frequency of positive and negative returns in the subsequent month (Up and Down states, respectively, in the last two columns) given the market state in the preceding month (rows). First, Corrections tend to revert to Bulls. Given the current month is a Correction state, then most of the time (61.2%) the next month is a Bull month and an Up month. If we view a Correction state as an alarm that uptrend could be turning to downtrend, then this alarm tends to be a false alarm. The fast strategy takes a short position after Corrections, which is a bad bet. Therefore, adopting a slower-speed momentum position after Corrections is a better bet. Second, the market is more likely to go up after Rebounds. Given the current month is a Rebound state, then most of the time (55.6%) the next month is an Up month. If we view a Rebound state as an alarm that downtrend could be turning to uptrend, then regardless of the accuracy of this alarm, FAST takes a long position, which is a good bet. Therefore, adopting a faster-speed momentum position after Rebounds is more effective than adopting slower speeds after Rebounds.

5. Evidence from Other Equity Markets

Table 9 reports the Sharpe ratios for static-speed and dynamic-speed momentum strategies applied in various country equity markets and evaluated over the last 15 years. For all

Table 9: **Sharpe Ratios of Momentum Strategies in International Markets**

	Sharpe Ratio (anlzd.)						
Country	$a = 0.00$ SLOW	0.25	0.50 MED	0.75	1.00 FAST	DYN $(\widehat{a}_{\text{Co}}, \widehat{a}_{\text{Re}})$	DYN Common (0.00, 0.81)
AU	0.367	0.417	0.420	0.350	0.258	0.526 $(0.00, 1.00)$	0.528
CA	0.299	0.417	0.517	0.551	0.527	0.650 $(0.56, 1.00)$	0.601
DE	0.458	0.474	0.429	0.316	0.198	0.542 $(0.00, 0.66)$	0.544
ES	0.165	0.154	0.120	0.067	0.020	0.238 $(0.16, 0.86)$	0.256
FR	0.421	0.487	0.492	0.390	0.269	0.642 $(0.28, 0.81)$	0.649
IT	0.329	0.303	0.236	0.140	0.050	0.329 $(0.00, 0.00)$	0.418
JP	0.529	0.599	0.618	0.559	0.462	0.554 $(0.00, 0.20)$	0.547
NL	0.394	0.343	0.195	0.000	−0.131	0.419 $(0.00, 0.81)$	0.420
SE	0.613	0.700	0.687	0.510	0.322	0.820 $(0.12, 0.59)$	0.803
UK	0.301	0.211	0.052	−0.115	−0.224	0.475 $(0.00, 1.00)$	0.472
US	0.589	0.618	0.565	0.419	0.264	0.669 $(0.07, 0.57)$	0.659

Notes: This table reports the Sharpe ratios (anlzd.) for various strategies applied to different country equity markets evaluated over the 15-year period from 2004-01 to 2018-12. Static-speed strategy weights are formed according to $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$ for speed parameter $a \in [0,1]$, where $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is non-negative and otherwise equals −1; and $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is non-negative and otherwise equals −1. DYN strategy weights take the form $(1 - a_{s(t)})w_{\text{SLOW,t}} + a_{s(t)}w_{\text{FAST,t}}$, where speed $a_{s(t)}$ depends on the four observable market states at date t: Bu: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Be: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; Co: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; and Re: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. DYN speeds are based on point estimates of the optimal state-dependent speeds, $a_{s(t)}$, using $\widehat{a}_{\text{Bu}} = \widehat{a}_{\text{Be}} = \frac{1}{2}$, $\widehat{a}_{\text{Co}} = \frac{1}{2}\left(1 - \frac{\widehat{\mathbf{E}}[r^2_{t+1}|^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Co}]}{\widehat{\mathbf{E}}[r^2_{t+1}|\text{Co}]}\right)$, and $\widehat{a}_{\text{Re}} = \frac{1}{2}\left(1 + \frac{\widehat{\mathbf{E}}[r^2_{t+1}|^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Re}]}{\widehat{\mathbf{E}}[r^2_{t+1}|\text{Re}]}\right)$, where $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$ denote estimates of the average and frequency of their given argument, respectively, based on data before the evaluation window beginning 1980-02. If either $\widehat{a}_{\text{Co}}$ or $\widehat{a}_{\text{Re}}$ fall outside the unit interval $[0,1]$, then we set its value to the nearest endpoint, 0 or 1. The DYN Common strategy uses the median of DYN country speed-pair estimates, $a_{\text{Co}} = 0.00$ and $a_{\text{Re}} = 0.81$, for every country. Monthly strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the excess equity market factor return for the country, obtained via Datastream. Highlighted values exceed all static-speed values for the country.

countries in the table, as predicted by Proposition 1, the Sharpe ratio for MED is higher than the average of the Sharpe ratios of SLOW and FAST. Moreover, for most countries, among static-speed momentum strategies, intermediate speeds exhibit larger Sharpe ratio point estimates than both SLOW and FAST. Therefore, the conclusion to elect intermediate-speed momentum strategies from our static-speed analysis in Section 3 using U.S. equity data largely carries over to international equity markets.

For almost every country, the Sharpe ratio of the DYN strategy is higher than that of the highest static-speed strategy (highlighted in the table). In addition, the optimal combinations are consistent with the notion of slower speeds after Corrections and faster speeds after Rebounds. The two countries for which their DYN Sharpe ratios are not higher than all static speeds are Italy (IT) and Japan (JP), which have estimated slow speeds after both Corrections and Rebounds.

In the column DYN Common, we apply a common speed pair to each country, which is defined as the median estimated DYN speeds after Corrections and Rebounds across all countries. The median speed pair $(0.00, 0.81)$ is slow after Corrections and relatively fast after Rebounds. The Sharpe ratios are similar to DYN, with the largest change for Italy, an increase from 0.329 to 0.418. Thus, with the exception of Japan, the recommendation from our state-dependent speed analysis on U.S. equity in Section 4 largely carries over to international equity markets (slower-speed momentum after Corrections and faster-speed momentum after Rebounds). Nevertheless, even for Japan, DYN exhibits a higher Sharpe ratio than either slow or fast static-speed strategies.

We obtain similar results for average return per unit of drawdown risk, as reported in Table H.1 of Appendix H.

6. Conclusion

The agreement or disagreement between the uptrend or downtrend indications of slow (12-month) and fast (1-month) TS momentum implies four market cycles: Bull, Correction, Bear, and Rebound. These market cycles provide a useful lens to view the challenges that momentum turning points pose for TS momentum strategies. Intermediate-speed strategies, formed by blending slow and fast TS momentum, vary the exposure to the good bets associated with uptrend (Bull) or downtrend (Bear) phases and the bad bets associated with turning points (Correction or Rebound). For the U.S. stock market, we find that intermediate-speed strategies empirically exhibit many advantages over slow and fast strategies, including higher Sharpe ratios, less severe drawdowns, more positive skewness, higher significance of alphas, and stronger predictability of many moments of returns. We provide analytical results that generalize our findings for Sharpe ratios, skewness, and predictability to any market.

The association between market and economic cycles connects potential explanations of momentum—based on either behavioral or rational foundations—to our evidence. We investigate a set of variables related to the economy, risk, and survey-based expectations and find a common theme. During Bull and Bear markets, when momentum signals are in agreement, we observe significant positive and negative economic surprises, respectively. During Corrections and Rebounds, innovations are insignificant, suggesting a potential shift or stall in the macro environment as well as in the stock market. Hence, momentum strategies that vary with market cycles, such as intermediate-speed or dynamic-speed strategies, have the potential to be more effective than those that do not.

We also find that both market timing and volatility timing play important but overlooked roles for TS momentum. In particular, we provide a novel model-free decomposition of momentum's alpha into the sum of market-timing and volatility-timing components. Market timing, the focus of recent TS momentum studies, reflects the covariance between strategy positions and subsequent returns; in contrast, volatility timing reflects the covariance between strategy positions and subsequent return volatility. For the U.S. stock market, we estimate about two-thirds of TS momentum's alpha over the last 50 years is attributable to market timing, and the remaining one-third is attributable to volatility timing.

Finally, we derive a dynamic momentum strategy that varies its speeds based on market cycles so to maximize its Sharpe ratio. We implement an investable version and find consistent improvements, when to compared to static strategies, in out-of-sample risk-adjusted performance. We document that these results, with few exceptions, hold up across international equity markets.

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Appendix A. Data Sources

The U.S. stock market returns of Sections 2, 3, and 4 are the excess-weighted factor returns (Mkt-RF) from the Kenneth French Data Library. The data coverage is from 1926-07 to 2018-12. Section 5 makes use of excess equity market returns from 11 futures contracts: AEX (Netherlands), ASX SPI 200 (Australia), CAC40 10 (France), DAX (Germany), FTSE 100 (United Kingdom), FTSE/MIB (Italy), IBEX 35 (Spain), Nikkei 225 (Japan), OMXS 30 (Sweden), S&P 500 (United States), and S&P/TSX 60 (Canada). The data coverage is from 1980-02 to 2018-02 and the source is Datastream.

Section 2 studies 15 macro time-series. From the FRED database of the Federal Reserve Bank of St. Louis, we obtained five indexes compiled by the Chicago Federal Reserve: CFNAI Personal Consumption and Housing (1967-03 to 2019-03); CFNAI Production and Income (1967-03 to 2019-03); CFNAI Sales, Orders and Inventories (1967-03 to 2019-03); CFNAI Employment, Unemployment and Hours (1967-03 to 2019-03); and the Chicago Fed National Financial Conditions Index (1967-03 to 2019-03). Also, from the FRED database we obtained the TED spread (1986-01 to 2019-03) and the Consumer Sentiment Index of the University of Michigan (1978-01 to 2019-03). The series of monetary policy shocks are estimated as in Gertler and Karadi (2015) and the data coverage is from 1990-01 to 2012-06. The liquidity innovation measure is from Pastor and Stambaugh (2003) (from 1962-08 to 2018-12), the bond illiquidity metric (Noise) is from Hu, Pan, and Wang (2013) (from 1987-01 to 2016-11), and the high-volatility/low volatility spread is from Pflueger, Siriwardane, and Sunderam (2018) (quarterly data from 1970-06 to 2016-06). The U.S. News Uncertainty measure is from Baker, Bloom, and Davis (2016) and covers the period from 1985-01 to 2019-03. Lastly, the PMI ISM series is from Bloomberg (1948-01 to 2019-03) and from the Survey of Professional forecasts we obtained recession expectations (QTR1, median response) and corporate profit expectations (DCPROF2, % change of median response); both are from 1968-12 to 2019-03 at a quarterly frequency.

Appendix B. MOM6 vs. Various Speeds

Table B.1: **Momentum Performance by Speed**

	MOM6	$a = 0$ SLOW MOM12	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST MOM1
Return and Risk						
Average (%) (anlzd.)	4.75	6.46	6.17	5.88	5.59	5.30
Volatility (%) (anlzd.)	15.67	15.62	12.72	11.60	12.74	15.66
Sharpe Ratio (anlzd.)	0.30	0.41	0.48	0.51	0.44	0.34
Market Timing						
Average Position	0.33	0.46	0.39	0.32	0.25	0.18
Market Beta	−0.01	0.15	0.05	−0.04	−0.13	−0.23
Alpha (%) (anlzd.)	4.79	5.58	5.85	6.12	6.39	6.66
Alpha t-statistic	2.15	2.54	3.24	3.71	3.57	3.07
Tail Behavior						
Skewness	−0.61	−0.43	−0.13	0.02	0.03	0.15
Max. Drawdown (%)	−64.56	−43.43	−37.96	−34.43	−34.07	−44.53
Average (anlzd.)/|Max. DD|	0.07	0.14	0.16	0.17	0.17	0.12

Notes: This table reports the sample average, volatility, Sharpe ratio, skewness, maximum drawdown, ratio of average return to absolute maximum drawdown, average position, the market beta and alpha, and the alpha t-statistic for monthly returns of the MOM6 strategy and of TS momentum strategies of various speeds. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The FAST strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. A MOMk strategy is long one unit ($+1$) if the trailing k-month return is nonnegative, and otherwise it is short one unit (-1). The evaluation period is 1969-01 to 2018-12.

Appendix C. Static Speed Performance—Since 1927

Table C.1: **Performance Summary by Speed**

	Market	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Return and Risk						
Average (%) (anlzd.)	7.72	7.60	7.00	6.40	5.81	5.21
Volatility (%) (anlzd.)	18.56	18.56	15.33	14.11	15.38	18.63
Sharpe Ratio (anlzd.)	0.42	0.41	0.46	0.45	0.38	0.28
Market Timing						
Average Position	1.00	0.45	0.39	0.33	0.27	0.21
Market Beta	1.00	0.07	0.05	0.03	0.00	−0.02
Alpha (%) (anlzd.)	0.00	7.03	6.62	6.21	5.80	5.38
Alpha t-statistic	—	3.61	4.11	4.18	3.58	2.74
Tail Behavior						
Skewness	0.19	−0.28	−0.02	0.14	0.12	0.15
Max. Drawdown (%)	−84.69	−68.73	−57.08	−64.00	−74.29	−82.76
Average (anlzd.)/|Max. DD|	0.09	0.11	0.12	0.10	0.08	0.06

Notes: This table reports the sample average, volatility, Sharpe ratio, average position, market beta and alpha, alpha t-statistic, skewness, maximum drawdown, and ratio of average return to absolute maximum drawdown for monthly returns of the long-only market strategy (Market) and of TS momentum strategies of various speeds. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1927-07 to 2018-12.

Appendix D. Static Speed Performance—Last 15 Years

Table D.1: **Performance Summary by Speed**

	Market	$a=0$ SLOW	$a=\frac{1}{4}$	$a=\frac{1}{2}$ MED	$a=\frac{3}{4}$	$a=1$ FAST
Return and Risk						
Average (%) (anlzd.)	7.58	7.60	6.45	5.30	4.15	3.00
Volatility (%) (anlzd.)	13.91	13.91	11.41	10.48	11.50	14.06
Sharpe Ratio (anlzd.)	0.54	0.55	0.57	0.51	0.36	0.21
Market Timing						
Average Position	1.00	0.72	0.62	0.52	0.41	0.31
Market Beta	1.00	0.20	0.14	0.08	0.01	−0.05
Alpha (%) (anlzd.)	0.00	6.08	5.40	4.72	4.04	3.37
Alpha t-statistic	—	1.70	1.83	1.73	1.34	0.91
Tail Behavior						
Skewness	−0.77	0.12	0.27	0.49	0.35	0.00
Max. Drawdown (%)	−51.51	−33.96	−24.67	−23.32	−29.34	−38.36
Avg./|Max. DD|	0.15	0.22	0.26	0.23	0.14	0.08

Notes: This table reports the sample average, volatility, Sharpe ratio, skewness, maximum drawdown, ratio of average return to absolute maximum drawdown, average position, market beta and alpha, and alpha t-statistic for monthly returns of the long-only market strategy (Market) and of TS momentum strategies of various speeds. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 2004-01 to 2018-12.

Appendix E. Dynamic vs. Static Component Performance

Table E.1: **Market Timing and Static Exposure, U.S. Stock Market**

Panel A: Performance of the Market-Timing Component					
$(w_t(a) - \mathbf{E}[w_t(a)])r_{t+1}$	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Average (%) (anlzd.)	3.76	3.88	4.01	4.13	4.26
Volatility (%) (anlzd.)	16.18	13.71	12.85	13.91	16.51
Sharpe Ratio (anlzd.)	0.23	0.28	0.31	0.30	0.26
Skewness	−0.02	0.17	0.23	0.22	0.29
Max. Drawdown (%)	−63.33	−49.85	−46.58	−43.76	−50.23
Avg. (%) (anlzd.)/|Max. DD|	0.06	0.08	0.09	0.09	0.08
Avg. Position	0.00	0.00	0.00	0.00	0.00
Market Beta	−0.31	−0.33	−0.36	−0.38	−0.41
Alpha (%) (anlzd.)	5.58	5.85	6.12	6.39	6.66
Alpha t-stat	2.54	3.24	3.71	3.57	3.07

Panel B: Performance of Static Component					
$\mathbf{E}[w_t(a)]r_{t+1}$	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Average (%) (anlzd.)	2.70	2.28	1.87	1.46	1.04
Volatility (%) (anlzd.)	7.14	6.05	4.95	3.86	2.76
Sharpe Ratio (anlzd.)	0.38	0.38	0.38	0.38	0.38
Skewness	−0.55	−0.55	−0.55	−0.55	−0.55
Max. Drawdown (%)	−28.54	−24.69	−20.66	−16.45	−12.04
Avg. (%) (anlzd.)/|Max. DD|	0.09	0.09	0.09	0.09	0.09
Avg. Position	0.46	0.39	0.32	0.25	0.18
Market Beta	0.46	0.39	0.32	0.25	0.18
Alpha (%) (anlzd.)	0.00	0.00	0.00	0.00	0.00

Notes: This table reports the sample average, volatility, Sharpe ratio, skewness, maximum drawdown, ratio of average return to absolute maximum drawdown, average position, market beta and alpha, and alpha t-statistic for monthly returns of the static (Panel B) and market-timing (Panel A) components of the long-only market strategy (Market) and of TS momentum strategies of various speeds. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals −1. The fast strategy weight, $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative, and otherwise equals −1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted factor return (Mkt-RF) from the Kenneth French Data Library. The static component return is $\mathbf{E}[w_t(a)]r_{t+1}$. The market-timing component return is $(w_t(a) - \mathbf{E}[w_t(a)])r_{t+1}$. The evaluation period is 1969-01 to 2018-12.

Appendix F. Beta and Alpha—Last 15 Years

Table F.1: **Beta and Alpha Decompositions by Speed**

Market Beta and Alpha	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Beta	0.20	0.14	0.08	0.01	−0.05
Alpha (%) (anlzd.)	6.08	5.40	4.72	4.04	3.37
Alpha t-stat.	1.70	1.83	1.73	1.34	0.91
Beta Components					
Static	0.722	0.619	0.517	0.414	0.311
Market Timing	0.007	0.006	0.005	0.003	0.002
Volatility Timing	−0.528	−0.487	−0.445	−0.403	−0.362
Alpha Components (%) (anlzd.)					
Market Timing	2.09	1.72	1.36	0.99	0.63
Volatility Timing	4.00	3.69	3.37	3.06	2.74

Notes: This table reports the sample market beta, alpha, and alpha t-statistic of monthly returns for momentum strategies of various speeds repeated from Table D.1. The table also reports the (additive) decomposition of beta into static, market-timing, and volatility-timing components according to estimates of the terms in

$$\mathbf{Beta}[r_{t+1}(a)] = \underbrace{\mathbf{E}[w_t(a)]}_{\substack{\text{static}\\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), r_{t+1}]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]}_{\substack{\text{market timing}\\ \text{component}}} + \underbrace{\frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}}_{\substack{\text{volatility timing}\\ \text{component}}}; \qquad (16)$$

and the (additive) decomposition of alpha into market timing and volatility timing according to estimates of the terms in

$$\mathbf{Alpha}[r_{t+1}(a)] = \mathbf{Cov}[w_t(a), r_{t+1}]\left(1 - \frac{(\mathbf{E}[r_{t+1}])^2}{\mathbf{Var}[r_{t+1}]}\right) - \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}]. \qquad (17)$$

The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. Intermediate-speed strategy weights, $w_t(a)$ are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0,1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 2004-01 to 2018-12.

Appendix G. Relation to Volatility-Managed Portfolios

We follow the Moreira and Muir (2017) approach and build a volatility-managed (VOM) portfolio for the U.S. stock market as follows:

$$r_{\text{VOM},t+1} := w_{\text{VOM},t} r_{t+1} = \frac{c}{\sigma_t^2} r_{t+1}, \tag{G1}$$

where σ_t^2 is the realized monthly volatility of the stock market, and the term c is such that the in-sample volatility of the VOM portfolio matches the one of the stock market.[32]

Table G.1 reports summary statistics for the positions of four portfolios: VOM, FAST, MED, and SLOW. The first rows of the table indicate the relative positions across cycles, which are defined as $\mathbf{E}[w_t|s(t)] - \mathbf{E}[w_t]$, for cycles $s(t) \in \{\text{Bu}, \text{Co}, \text{Be}, \text{Re}\}$. These relative positions are the conditional differences with respect to the whole-sample average. As a reference, we also report the whole-sample average and the correlations of the TS momentum portfolios to the volatility-managed one.

VOM positions are positively correlated with those of TS momentum strategies of different speeds. VOM shows the highest correlation with MED—at 0.30—and the lowest with FAST. This latter result may appear surprising, because VOM and FAST employ the same lookback period (one month) to construct the signal. The relative positions show that VOM and FAST momentum tilt differently following Rebounds, when *positive* market returns tend to coincide with high volatility states. Instead, VOM and SLOW momentum tilt differently following Correction phases, when volatility tend to spikes. The positions of MED are neutral following Rebound and Correction phases and, as a result, its dynamic bets more closely resemble those of VOM.

Table G.2 repeats the beta and alpha decompositions for the MED portfolio and reports those of the VOM portfolio and a *hybrid* version. The positions of the hybrid portfolio are simply the product of the MED and VOM weights, and this design captures the common choice of sizing momentum signals by trailing volatility (e.g., Moskowitz, Ooi, and Pedersen, 2012).[33] Over the last 50 years, the VOM portfolio did not deliver a statistically significant

[32] As noted by Liu, Tang, and Zhou (2019), this approach suffers look-ahead bias. Real-time calibrations show significant drawdowns for the volatility-managed strategy. Yet, for comparison purposes, we opt to adhere to the original formulation proposed by Moreira and Muir (2017).

[33] To maintain the consistency with the work of Moreira and Muir (2017), we scale the MED positions by the trailing 22 business day *variance*. Yet, the time-series momentum literature typically employs exponentially weighted measures of realized standard deviation, which generally deliver more realistic leverage dynamics. In a set of unreported results, we verified the robustness of our conclusions to such alternative approaches and found no major qualitative difference.

Table G.1: **Position Statistics: Vol-Managed Strategy Versus Momentum**

Positions Relative to Average (%)	VOM	$a = 0$ SLOW	$a = \frac{1}{2}$ MED	$a = 1$ FAST
Bull	26	54	68	82
Correction	−10	54	−32	−118
Bear	−36	−146	−132	−118
Rebound	−39	−146	−32	82
Average Position (%)	107	46	32	18
Corr. to VOM Position	1.00	0.27	0.30	0.20

Notes: This table reports the relative monthly position of a volatility-managed (VOM) portfolio and momentum strategies of various speeds. The relative positions, which are conditional on the market cycles, are defined as $\mathbf{E}[w_t|s(t)] - \mathbf{E}[w_t]$, where $s(t) \in \{\text{Bu}, \text{Co}, \text{Be}, \text{Re}\}$. The VOM strategy weight $w_{\text{VOM},t}$ applied to the market return in month $t+1$ equals c/σ_t^2, where σ_t^2 is the previous month volatility of the stock market, and the term c is such that the in-sample volatility of the VOM portfolio matches the volatility of the stock market. The slow strategy weight, $w_{\text{SLOW},t}$, equals $+1$ if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals -1. The fast strategy weight, $w_{\text{FAST},t}$, equals $+1$ if the trailing 1-month return is nonnegative, and otherwise equals -1. The MED strategy weights, $w_t(1/2)$, are formed by mixing slow and fast strategies with equal weights 1/2. Strategy returns are formed as $r_{t+1} = w_t r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

alpha (t-statistic of 1.50). Unsurprisingly, the tactical component generated by the volatility signal is entirely attributable to volatility timing and none to market timing. Lastly, the hybrid portfolio displays significant alpha, although quantitatively smaller than the one of the MED portfolio. Specifically, combining the MED and VOM signals leads to the largest volatility-timing component; yet, this gain is outweighed by the relative loss in market-timing ability.

In conclusion, TS momentum and volatility-management strategies partially overlap in the U.S. stock market. Volatility timing appears to show little tactical information beyond what offered by the medium-speed momentum portfolio. This conclusion is consistent with the evidence by Harvey, Hoyle, Korgaonkar, Rattray, Sargaison, and Hemert (2018), who showed that volatility-managed strategies are most effective when they are in agreement with momentum strategies.

Table G.2: **Beta and Alpha Decompositions: VOM and Momentum**

Market Beta and Alpha	MED	VOM	Hybrid
Beta	−0.04	0.74	0.13
Alpha (%) (anlzd.)	6.12	2.25	5.02
Alpha t-stat.	3.71	1.50	2.83
Beta Components			
Static	0.317	1.071	0.526
Market Timing	0.008	0.001	0.005
Volatility Timing	−0.364	−0.336	−0.401
Alpha Components (%) (anlzd.)			
Market Timing	3.96	0.27	2.66
Volatility Timing	2.15	1.98	2.36

Notes: This table reports the sample market beta, alpha, and alpha t-statistic of monthly returns of MED, the momentum strategy formed as an equal blend of slow and fast momentum strategies, as well as its (additive) decomposition of beta and alpha into static, market-timing, and volatility-timing components repeated from Table 4. Details of the construction of the MED portfolio as well as the estimations of beta and alpha are reported in the same table and the text. In addition, we investigate two portfolios: a volatility-managed (VOM) strategy of the U.S. stock market and a hybrid portfolio (MED and VOM). The VOM strategy weight, $w_{\text{VOM},t}$, applied to the market return in month $t+1$ equals c/σ_t^2, where σ_t^2 is the previous-month volatility of the stock market, and the term c is such that the in-sample volatility of the VOM portfolio matches the volaility of the stock market. The positions of the hybrid portfolio are the product of the MED and VOM weights. The U.S. stock market is the excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

Appendix H. Average Returns per Drawdown Risk, Other Equity Markets

Table H.1: **Average Returns per Drawdown Risk of Momentum Strategies in International Markets**

	Sharpe Ratio (anlzd.)							
Country	$a = 0.00$ SLOW	0.25	0.50 MED	0.75	1.00 FAST	DYN	$(\widehat{a}_{\text{Co}}, \widehat{a}_{\text{Re}})$	DYN Common (0.00, 0.81)
AU	0.099	0.128	0.127	0.092	0.063	0.221	(0.56, 1.00)	0.223
CA	0.107	0.162	0.188	0.214	0.209	0.261	(0.00, 0.66)	0.256
DE	0.160	0.177	0.191	0.169	0.094	0.202	(0.00, 0.66)	0.205
ES	0.047	0.043	0.032	0.019	0.006	0.083	(0.16, 0.86)	0.095
FR	0.126	0.148	0.194	0.137	0.089	0.304	(0.28, 0.81)	0.269
IT	0.094	0.088	0.078	0.042	0.015	0.094	(0.00, 0.00)	0.212
JP	0.224	0.279	0.347	0.305	0.231	0.264	(0.00, 0.20)	0.240
NL	0.138	0.148	0.080	0.000	−0.038	0.167	(0.00, 0.81)	0.168
SE	0.255	0.394	0.360	0.281	0.175	0.491	(0.12, 0.59)	0.442
UK	0.097	0.073	0.018	−0.031	−0.056	0.219	(0.00, 1.00)	0.243
US	0.241	0.279	0.249	0.190	0.118	0.302	(0.07, 0.57)	0.312

Notes: This table reports the average return (anlzd.) per absolute maximum drawdown for various strategies applied to different country equity markets evaluated over the 15-year period 2004-01 to 2018-12. Static-speed strategy weights are formed according to $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$ for speed parameter $a \in [0,1]$, where $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative and otherwise equals −1; and $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative and otherwise equals −1. DYN strategy weights take the form: $(1-a_{s(t)})w_{\text{SLOW,t}} + a_{s(t)}w_{\text{FAST,t}}$, where the speed $a_{s(t)}$ depends on the four observable market states at date t: Bu: $w_{\text{SLOW},t} = w_{\text{FAST},t} = +1$; Be: $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$; Co: $w_{\text{SLOW},t} = +1, w_{\text{FAST},t} = -1$; and Re: $w_{\text{SLOW},t} = -1, w_{\text{FAST},t} = +1$. DYN speeds are based on point estimates of the optimal state-dependent speeds, $a_{s(t)}$, using $\widehat{a}_{\text{Bu}} = \widehat{a}_{\text{Be}} = \frac{1}{2}$, $\widehat{a}_{\text{Co}} = \frac{1}{2}\left(1 - \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Co}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Co}]}\right)$, and $\widehat{a}_{\text{Re}} = \frac{1}{2}\left(1 + \frac{\widehat{\mathbf{E}}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\widehat{\mathbf{P}}[{}^{\text{Bu}}_{\text{Be}}]}{\widehat{\mathbf{E}}[r_{t+1}|\text{Bu}]\widehat{\mathbf{P}}[\text{Bu}] - \widehat{\mathbf{E}}[r_{t+1}|\text{Be}]\widehat{\mathbf{P}}[\text{Be}]}\frac{\widehat{\mathbf{E}}[r_{t+1}|\text{Re}]}{\widehat{\mathbf{E}}[r_{t+1}^2|\text{Re}]}\right)$, where $\widehat{\mathbf{E}}[\cdot]$ and $\widehat{\mathbf{P}}[\cdot]$ denote estimates of the average and frequency of their given argument, respectively, based on data before the evaluation window beginning 1980-02. If either $\widehat{a}_{\text{Co}}$ or $\widehat{a}_{\text{Re}}$ fall outside the unit interval $[0,1]$, then we set its value to the nearest endpoint, 0 or 1. The DYN Common strategy uses the median of DYN country speed-pair estimates, $a_{\text{Co}} = 0.00$ and $a_{\text{Re}} = 0.81$, for every country. Monthly strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is excess equity market factor return for the country, obtained via Datastream. Highlighted values exceed all static-speed values for the country.

For most countries, among static-speed momentum strategies, intermediate speeds exhibit larger average return per drawdown risk point estimates than both SLOW and FAST. For most countries in the table, the average return per unit drawdown risk of the DYN strategy is higher than that of the highest static-speed strategy; in addition, the speeds that achieve such performance are consistent with the notion of slower speeds after Corrections and faster speeds after Rebounds. As before, the only exceptions are Italy and Japan. Nevertheless, for these two countries, their dynamic-speed performance is higher than that of both SLOW and FAST. These results indicate that not only variance risk but also tail risk is moderated by intermediate-speed and state-dependent speed momentum strategies.

Appendix I. Predictability

Why do intermediate-speed momentum strategies have higher alpha t-statistics? To address this question, we look at the capacity of momentum strategy weights to forecast different aspects of subsequent returns. It is intuitive that strategies whose weights tend to align more favorably with beneficial aspects of subsequent returns can generate more reliable outperformance.

Table I.1: **Predictability by Speed**

$\mathbf{Corr}(w_t(a), f(r_{t+1}))$	$a = 0$ SLOW	$a = \frac{1}{4}$	$a = \frac{1}{2}$ MED	$a = \frac{3}{4}$	$a = 1$ FAST
Return: $f(r_{t+1}) = r_{t+1}$	0.078	0.094	0.100	0.093	0.080
Volatility:					
$f(r_{t+1}) = \lvert r_{t+1} - \mathbf{E}[r_{t+1}]\rvert$	−0.235	−0.279	−0.294	−0.270	−0.230
$f(r_{t+1}) = (r_{t+1} - \mathbf{E}[r_{t+1}])^2$	−0.181	−0.226	−0.251	−0.241	−0.214
Lower Tail:					
$f(r_{t+1}) = r_{t+1}1_{\{r_{t+1} \leq 10\% \text{ pctl.}\}}$	0.184	0.224	0.242	0.227	0.197
Upper Tail:					
$f(r_{t+1}) = r_{t+1}1_{\{r_{t+1} \geq 90\% \text{ pctl.}\}}$	−0.100	−0.113	−0.114	−0.100	−0.081

Notes: This table reports the sample correlation of monthly strategy weights of momentum strategies of various speeds with several functions of subsequent monthly market returns. The slow strategy weight applied to the market return in month $t+1$, $w_{\text{SLOW},t}$, equals +1 if the trailing 12-month return (arithmetic average monthly return) is nonnegative, and otherwise equals −1. The fast strategy weight, $w_{\text{FAST},t}$, equals +1 if the trailing 1-month return is nonnegative, and otherwise equals −1. Intermediate-speed strategy weights, $w_t(a)$, are formed by mixing slow and fast strategies with mixing parameter a: $w_t(a) = (1-a)w_{\text{SLOW},t} + aw_{\text{FAST},t}$, for $a \in [0, 1]$. Strategy returns are formed as $r_{t+1}(a) = w_t(a)r_{t+1}$, where r_{t+1} is the U.S. excess value-weighted market factor return (Mkt-RF) from the Kenneth French Data Library. The evaluation period is 1969-01 to 2018-12.

Table I.1 reports the correlation between strategy weights and several functions of subsequent returns for momentum strategies of various speeds. We highlight three results reported in this table. First, although strategy weights of all speeds tend to be relatively weak positive predictors of subsequent returns (first row), they are relatively strong negative predictors of return volatility (second and third rows). Therefore, each strategy has some tendency to take higher (or more positive) positions when returns will be higher. They have an even stronger tendency, however, to take lower (more negative) positions when return volatility will be high, which is also when returns will tend to be lower or more negative (contemporaneous correlation $\mathbf{Corr}[(r_{t+1} - \mathbf{E}[r_{t+1}])^2, r_{t+1}]$ is estimated at −0.277 in our sample). As explained in Section 3.2.1, volatility timing can be an important component of alpha.

Second, there is a trade-off in how these strategies tend to predict extreme returns. On the one hand, strategy weights of all speeds tend to be positive predictors of lower tail returns (fourth row). Therefore, each strategy exhibits skill at forecasting negative tails: it tends to apply lower weights (less positive or more negative) when subsequent returns are in the lowest 10% of all returns (and hence negative)—a desirable strategy property. On the other hand, strategy weights of all speeds tend to be negative predictors of upper tail returns. So, each strategy tends to apply lower weights (less positive or more negative) when subsequent returns are in the highest 10% of all returns (and hence positive). Therefore, momentum strategies of all speeds are missing out on possible high upside outcomes. This trade-off is acceptable under concave utility for which investors are more than willing to forego x% upside for x% avoidance of downside.

Third, the correlation between strategy weights and next month's (i) returns, (ii) absolute return deviations or squared return deviations from the mean, (iii) lower 10th percentile tail returns, and (iv) upper 90th percentile tail returns, are all higher in magnitude (absolute value) for intermediate-speed strategies relative to the average of the corresponding slow and fast strategies. Moreover, MED has the highest magnitude in each row. These results follow intuitively from consideration of the periods of agreement and disagreement between the slow and fast momentum strategies. For example, if the strategy weights of both SLOW and FAST are positively correlated with subsequent returns, then such strategies tend to take long positions when returns are high and short positions when returns are low (and possibly negative). Since SLOW and FAST take precisely the opposite positions following Correction and Rebound phases, whatever forecasting contribution such states add to the slow strategy is negated in the fast strategy and vice versa. Intermediate-speed strategies scale down position size following such states, thereby shifting relative exposure to periods of agreement between SLOW and FAST, amplifying their average predictability capacity following Bull or Bear states of agreement. We generalize this predictability capacity of intermediate-speed momentum strategies in Proposition 6.

Proposition 6 (Correlation between strategy weights and returns)**.** *The correlation of strategy weights $w_t(a)$ with any function of subsequent returns $f(r_{t+1})$ can be expressed in terms of the correlations of $w_{SLOW,t+1}$ and $w_{FAST,t+1}$ with $f(r_{t+1})$, respectively, and the market cycles, as follows:*

$$\begin{aligned}\mathbf{Corr}[w_t(a), f(r_{t+1})] = (1-a)\,\mathbf{Corr}[w_{SLOW,t}, f(r_{t+1})]D_1(a, \mathbf{E}[w_{SLOW,t}]) \\ + a\,\mathbf{Corr}[w_{FAST,t}, f(r_{t+1})]D_1(a, \mathbf{E}[w_{FAST,t}]),\end{aligned} \tag{I1}$$

where

$$D_1(a,\mu) := \sqrt{\frac{1-\mu^2}{\mathbf{P}[^{Bu}_{Be}] + (2a-1)^2\mathbf{P}[^{Co}_{Re}] - (\mathbf{E}[w_t(a)])^2}}. \tag{I2}$$

Approximating squared average strategy positions by $(\mathbf{E}[w_t(a)])^2 \approx 0$ *for* $a \in [0,1]$*, we have*

$$\mathbf{Corr}[w_t(a), f(r_{t+1})] \approx ((1-a)\mathbf{Corr}[w_{SLOW,t}, f(r_{t+1})] + a\mathbf{Corr}[w_{FAST,t}, f(r_{t+1})])\, D_1(a). \tag{I3}$$

The term $D_1(a)$ *multiplying the weighted average of correlations in* (I3) *above,*

$$D_1(a) := D_1(a,0) = \sqrt{\frac{1}{\mathbf{P}[^{Bu}_{Be}] + (2a-1)^2\mathbf{P}[^{Co}_{Re}]}}, \tag{I4}$$

is greater than or equal to one and is maximized at $a = \frac{1}{2}$ *on* $a \in [0,1]$*:*

$$D_1\left(\tfrac{1}{2}\right) \geq D_1(a) \geq 1, \quad a \in [0,1]. \tag{I5}$$

Moreover, $a = \frac{1}{2}$ *is the unique maximizer with* $D_1(\frac{1}{2}) > 1$ *if* $\mathbf{P}[^{Co}_{Re}] > 0$.

Proof. Proof of Proposition 6 is in Appendix J. □

Proposition 6 indicates that the predictive strength of the speed strategy weights for any function of subsequent returns is higher than the average predictive strength of the slow and fast strategy weights, taken separately. Here, the *disagreement multiplier* D_1 is similar to D of Proposition 1 except it only involves market-cycle frequencies and not conditional expected squared returns. As before, the approximation in (I3) is exact at the endpoints of the interval $a \in \{0,1\}$. Also as before, the multiplier $D_1(a)$ in (I4), which scales up the correlation of the two individual slow and fast momentum strategies, is largest at $a = \frac{1}{2}$, the equal split between the two speed extremes.

Appendix J. Proofs

Proof of Proposition 1. We show first that the exact decomposition for the Sharpe ratio in (8) holds. First, note that

$$\begin{aligned}\mathbf{E}[r_{t+1}(a)] &= \mathbf{E}[((1-a)w_{\text{SLOW},t} + a w_{\text{FAST},t})r_{t+1}] \\ &= (1-a)\mathbf{E}[r_{\text{SLOW},t+1}] + a\mathbf{E}[r_{\text{FAST},t+1}], \end{aligned} \tag{J1}$$

where the first equality follows by (4), and the second equality follows by definition of SLOW and FAST in Section 2.1. Second,

$$\begin{aligned}\mathbf{E}[w_{\text{SLOW},t}\, w_{\text{FAST},t}\, r_{t+1}^2] &= \mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] - \mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}] \\ &= (\mathbf{E}[r_{t+1}^2] - \mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]) - \mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}] \\ &= \mathbf{E}[r_{t+1}^2] - 2\mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}], \end{aligned} \tag{J2}$$

where the first equality follows by the law of total expectation, the fact that SLOW and FAST have the same signs after Bull or Bear and the opposite signs after Correction or Rebound, and the fact that the weights of SLOW and FAST each have magnitude one; and the second equality follows by the law of total expectation. Third,

$$\mathbf{Var}[w_{\text{SLOW},t}\, r_{t+1}] = \mathbf{E}[r_{t+1}^2] - (\mathbf{E}[w_{\text{SLOW},t}\, r_{t+1}])^2, \tag{J3}$$

$$\mathbf{Var}[w_{\text{FAST},t}\, r_{t+1}] = \mathbf{E}[r_{t+1}^2] - (\mathbf{E}[w_{\text{FAST},t}\, r_{t+1}])^2, \tag{J4}$$

$$\begin{aligned}\mathbf{Cov}[w_{\text{SLOW},t}\, r_{t+1}, w_{\text{FAST},t}\, r_{t+1}] &= \mathbf{E}[w_{\text{SLOW},t}\, w_{\text{FAST},t}\, r_{t+1}^2] - \mathbf{E}[w_{\text{SLOW},t}\, r_{t+1}]\mathbf{E}[w_{\text{FAST},t}\, r_{t+1}] \\ &= \mathbf{E}[r_{t+1}^2] - 2\mathbf{E}[r_{t+1}^2|{}^{\text{Co}}_{\text{Re}}]\mathbf{P}[{}^{\text{Co}}_{\text{Re}}] \\ &\quad - \mathbf{E}[w_{\text{SLOW},t}\, r_{t+1}]\mathbf{E}[w_{\text{FAST},t}\, r_{t+1}], \end{aligned} \tag{J5}$$

where the first two equalities follow by definition of variance and the fact that the weights of SLOW and FAST have magnitude one; the third equality follows by definition of covariance; and the fourth equality follows by (J2). Using these facts, the variance of a momentum strategy of any speed satisfies

$$
\begin{aligned}
\mathbf{Var}[r_{t+1}(a)] &= \mathbf{Var}[((1-a)w_{\mathrm{SLOW},t} + a\,w_{\mathrm{FAST},t})\,r_{t+1}] \\
&= (1-a)^2\mathbf{Var}[w_{\mathrm{SLOW},t}\,r_{t+1}] + a^2\mathbf{Var}[w_{\mathrm{FAST},t}\,r_{t+1}] \\
&\quad + 2(1-a)a\mathbf{Cov}[w_{\mathrm{SLOW},t}\,r_{t+1}, w_{\mathrm{FAST},t}\,r_{t+1}] \\
&= (1-a)^2[\mathbf{E}[r_{t+1}^2] - (\mathbf{E}[w_{\mathrm{SLOW},t}r_{t+1}])^2] + a^2[\mathbf{E}[r_{t+1}^2] - (\mathbf{E}[w_{\mathrm{FAST},t}r_{t+1}])^2] \\
&\quad + 2(1-a)a\left\{\mathbf{E}[r_{t+1}^2] - 2\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right] - \mathbf{E}[w_{\mathrm{SLOW},t}r_{t+1}]\mathbf{E}[w_{\mathrm{FAST},t}r_{t+1}]\right\} \\
&= \mathbf{E}[r_{t+1}^2] - 4(1-a)a\,\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right] \\
&\quad - \{(1-a)\mathbf{E}[w_{\mathrm{SLOW},t}r_{t+1}] + a\,\mathbf{E}[w_{\mathrm{FAST},t}r_{t+1}]\}^2 \\
&= \left(\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right] + \mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\right) - 4(1-a)a\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right] \\
&\qquad - \{\mathbf{E}[((1-a)w_{\mathrm{SLOW},t} + aw_{\mathrm{FAST},t})r_{t+1}]\}^2 \\
&= \mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right] + (2a-1)^2\mathbf{E}\left[r_t^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right] - (\mathbf{E}[r_{t+1}(a)])^2, \qquad \text{(J6)}
\end{aligned}
$$

where the first equality follows by definition of $r_{t+1}(a)$ in (4); the second equality follows by definition of variance of a sum; the third equality follows by substitution of (J3), (J4), and (J5); the fourth equality follows by collecting terms; the fifth equality follows by the law of total expectation; and the final equality follows by definition of $r_{t+1}(a)$ in (4).

Next, note that $D(a, \mu)$ in (9), whose definition is repeated here for convenience, satisfies

$$
\begin{aligned}
D(a,\mu) &= \sqrt{\frac{\mathbf{E}[r_{t+1}^2] - \mu^2}{\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Bu}\\ \mathrm{Be}\end{smallmatrix}\right] + (2a-1)^2\mathbf{E}\left[r_{t+1}^2 \middle| \begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right]\mathbf{P}\left[\begin{smallmatrix}\mathrm{Co}\\ \mathrm{Re}\end{smallmatrix}\right] - (\mathbf{E}[r_{t+1}(a)])^2}}, \qquad (9) \\
&= \sqrt{\frac{\mathbf{E}[r_{t+1}^2] - \mu^2}{\mathbf{Var}[r_{t+1}(a)]}}, \qquad \text{(J7)}
\end{aligned}
$$

where the last equality follows from (J6). In particular, applying (J3) and (J4), respectively, gives

$$
D(a, \mathbf{E}[r_{\mathrm{SLOW},t+1}]) = \sqrt{\frac{\mathbf{E}[r_{t+1}^2] - (\mathbf{E}[r_{\mathrm{SLOW},t+1}])^2}{\mathbf{Var}[r_{t+1}(a)]}} = \frac{\mathbf{SD}[r_{\mathrm{SLOW},t+1}]}{\mathbf{SD}[r_{t+1}(a)]}, \qquad \text{(J8)}
$$

$$
D(a, \mathbf{E}[r_{\mathrm{FAST},t+1}]) = \sqrt{\frac{\mathbf{E}[r_{t+1}^2] - (\mathbf{E}[r_{\mathrm{FAST},t+1}])^2}{\mathbf{Var}[r_{t+1}(a)]}} = \frac{\mathbf{SD}[r_{\mathrm{FAST},t+1}]}{\mathbf{SD}[r_{t+1}(a)]}. \qquad \text{(J9)}
$$

Hence, the Sharpe ratio of a momentum strategy of any speed satisfies the following:

$$\begin{aligned}\mathbf{Sharpe}[r_{t+1}(a)] &= \frac{\mathbf{E}[r_{t+1}(a)]}{\mathbf{SD}[r_{t+1}(a)]} \\ &= \frac{(1-a)\mathbf{E}[r_{\mathrm{SLOW},t+1}] + a\mathbf{E}[r_{\mathrm{FAST},t+1}]}{\mathbf{SD}[r_{t+1}(a)]} \\ &= (1-a)\frac{\mathbf{E}[r_{\mathrm{SLOW},t+1}]}{\mathbf{SD}[r_{\mathrm{SLOW},t+1}]}\frac{\mathbf{SD}[r_{\mathrm{SLOW},t+1}]}{\mathbf{SD}[r_{t+1}(a)]} + a\frac{\mathbf{E}[r_{\mathrm{FAST},t+1}]}{\mathbf{SD}[r_{\mathrm{FAST},t+1}]}\frac{\mathbf{SD}[r_{\mathrm{FAST},t+1}]}{\mathbf{SD}[r_{t+1}(a)]} \\ &= (1-a)\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}]D(a, \mathbf{E}[r_{\mathrm{SLOW},t+1}]) \\ &\quad + a\,\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}]D(a, \mathbf{E}[r_{\mathrm{FAST},t+1}]), \end{aligned} \tag{J10}$$

where the first equality follows by definition of the Sharpe ratio; the second equality follows by (J1); the third equality follows by factoring out each ratio of standard deviations in each summand; and the final equality follows by definition of the Sharpe ratio and equalities (J8) and (J9). The approximation in (10) of the proposition follows immediately by the assumption in the proposition.

Finally, we derive tight bounds for the multiplier $D(a, 0)$ in (11) on $a \in [0, 1]$. First, note that

$$(2a-1)^2 \leq 1, \quad a \in [0,1], \tag{J11}$$

because $(2a-1)^2$ is a convex function of a, which therefore achieves its maximum value at the boundary of its domain, and is equal to 1 at both of its boundaries. Subtract 1 from both sides of (J11) and then multiply each side of the resultant inequality by the nonnegative quantity $\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Co}}_{\mathrm{Re}}]\mathbf{P}[{}^{\mathrm{Co}}_{\mathrm{Re}}]$ to obtain:

$$[(2a-1)^2 - 1]\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Co}}_{\mathrm{Re}}]\mathbf{P}[{}^{\mathrm{Co}}_{\mathrm{Re}}] \leq 0, \quad a \in [0,1]. \tag{J12}$$

Add $\mathbf{E}[r_{t+1}^2]$ to both sides of (J12) and use the law of total expectation, $\mathbf{E}[r_{t+1}^2] = \mathbf{E}[r_{t+1}^2|{}^{\mathrm{Bu}}_{\mathrm{Be}}]\mathbf{P}[{}^{\mathrm{Bu}}_{\mathrm{Be}}] + \mathbf{E}[r_{t+1}^2|{}^{\mathrm{Co}}_{\mathrm{Re}}]\mathbf{P}[{}^{\mathrm{Co}}_{\mathrm{Re}}]$, to obtain:

$$\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Bu}}_{\mathrm{Be}}]\mathbf{P}[{}^{\mathrm{Bu}}_{\mathrm{Be}}] + (2a-1)^2\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Co}}_{\mathrm{Re}}]\mathbf{P}[{}^{\mathrm{Co}}_{\mathrm{Re}}] \leq \mathbf{E}[r_{t+1}^2], \quad a \in [0,1]. \tag{J13}$$

Therefore,

$$\begin{aligned} D(a,0) &= \sqrt{\frac{\mathbf{E}[r_{t+1}^2]}{\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Bu}}_{\mathrm{Be}}]\mathbf{P}[{}^{\mathrm{Bu}}_{\mathrm{Be}}] + (2a-1)^2\mathbf{E}[r_{t+1}^2|{}^{\mathrm{Co}}_{\mathrm{Re}}]\mathbf{P}[{}^{\mathrm{Co}}_{\mathrm{Re}}]}} \qquad (11) \\ &\geq 1, \end{aligned}$$

since the numerator inside the square root is always at least as large as the denominator, by (J13). The term $(2a-1)^2\mathbf{E}[r_{t+1}^2|^{\text{Co}}_{\text{Re}}]\mathbf{P}[^{\text{Co}}_{\text{Re}}]$ in the denominator is always nonnegative and equals zero at $a=\frac{1}{2}$, which therefore maximizes $D(a,0)$ on $[0,1]$. If $\mathbf{E}[r_{t+1}^2|^{\text{Co}}_{\text{Re}}]\mathbf{P}[^{\text{Co}}_{\text{Re}}]>0$ then the term is only zero at $a=\frac{1}{2}$, making it the unique maximizer of $D(a,0)$, which completes the proof. □

Proof of Proposition 6. First, note that $w_t(0)=w_{\text{SLOW},t}$ and $w_t(1)=w_{\text{FAST},t}$, and

$$\mathbf{SD}[w_t(0)]=\sqrt{\mathbf{E}[w_t^2(0)]-(\mathbf{E}[w_t(0)])^2}=\sqrt{1-\mathbf{E}[w_t(0)])^2}, \tag{J14}$$

$$\mathbf{SD}[w_t(1)]=\sqrt{\mathbf{E}[w_t^2(1)]-(\mathbf{E}[w_t(1)])^2}=\sqrt{1-\mathbf{E}[w_t(1)])^2}, \tag{J15}$$

where the first equality in each line above follows by definition of standard deviation; and the second equality in each line follows from the fact that the strategy weight at extreme speeds ($a=0$ or $a=1$) is always magnitude one. Second,

$$\begin{aligned}\mathbf{SD}[w_t(a)]&=\sqrt{\mathbf{E}[w_t^2(a)]-(\mathbf{E}[w_t(a)])^2}\\&=\sqrt{\mathbf{E}[((1-a)w_t(0)+aw_t(1))^2]-(\mathbf{E}[w_t(a)])^2}\\&=\sqrt{\mathbf{P}[^{\text{Bu}}_{\text{Be}}]+(2a-1)^2|^{\text{Co}}_{\text{Re}}]\mathbf{P}[^{\text{Co}}_{\text{Re}}]-(\mathbf{E}[w_t(a)])^2},\end{aligned} \tag{J16}$$

where the first equality follows by definition of standard deviation; the second equality follows by definition of $w_t(a)=(1-a)w_t(0)+aw_t(1)$; and the last equality follows from the weights $w_t(0)=w_t(1)$ being equal following Bull or Bear so that $((1-a)w_t(0)+aw_t(1))^2-1$ and $w_t(0)=-w_t(1)$ being opposite sign following Correction or Rebound so that $((1-a)w_t(0)+aw_t(1))^2=(2a-1)^2$. Next,

$$\begin{aligned}\mathbf{Corr}[w_t(a),f(r_{t+1})]&=\frac{\mathbf{Cov}[(1-a)w_t(0)+aw_t(1),f(r_{t+1})]}{\mathbf{SD}[w_t(a)]\mathbf{SD}[f(r_{t+1})]}\\&=(1-a)\frac{\mathbf{Cov}[w_t(0),f(r_{t+1})]}{\mathbf{SD}[w_t(0)]\mathbf{SD}[f(r_{t+1})]}\frac{\mathbf{SD}[w_t(0)]}{\mathbf{SD}[w_t(a)]}\\&\quad+a\frac{\mathbf{Cov}[w_t(1),f(r_{t+1})]}{\mathbf{SD}[w_t(1)]\mathbf{SD}[f(r_{t+1})]}\frac{\mathbf{SD}[w_t(1)]}{\mathbf{SD}[w_t(a)]}\\&=(1-a)\mathbf{Corr}[w_t(0),f(r_{t+1})]D_1(a,\mathbf{E}[w_t(0)])\\&\quad+a\mathbf{Corr}[w_t(1),f(r_{t+1})]D_1(a,\mathbf{E}[w_t(1)]),\end{aligned} \tag{J17}$$

where the first equality follows by definition of correlation and $w_t(a)=(1-a)w_t(0)+aw_t(1)$; the second correlation follows by bilineatity of the covariance operator; and the last equality follows by (J14), (J15), and (J16), together with the definition of $D_1(a,\mu)$ in (I2), which is repeated here for convenience: $D_1(a,\mu) = \sqrt{\frac{1-\mu^2}{\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]+(2a-1)^2\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]-(\mathbf{E}[w_t(a)])^2}}$. The correlation approximation stated in the proposition in (I3) follows immediately from (J17) (equivalently, (I1)) evaluated under the approximation $D_1(a,\mu) = D_1(a,0)$, which is given in (I4).

Finally, we derive tight bounds for the multiplier $D_1(a,0)$ in (I4) on $a \in [0,1]$. First, note that

$$(2a-1)^2 \leq 1, \quad a \in [0,1], \tag{J18}$$

because $(2a-1)^2$ is a convex function of a, which therefore achieves its maximum value at the boundary of its domain, and is equal to 1 at both of its boundaries. Multiply $\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]$, which is nonegative, on both sides of (J18) and then add $\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]$ to each side to obtain:

$$\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + (2a-1)^2\mathbf{P}[{}^{\text{Co}}_{\text{Re}}] \leq \mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + \mathbf{P}[{}^{\text{Co}}_{\text{Re}}] = 1, \quad a \in [0,1]. \tag{J19}$$

Note that the final equality in (J19) holds because $\{{}^{\text{Bu}}_{\text{Be}}\}$ and $\{{}^{\text{Co}}_{\text{Re}}\}$ are complementary events. Therefore, $D_1(a,0) = \sqrt{\frac{1}{\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]+(2a-1)^2\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]}} \geq 1$, since the numerator inside the square root is always at least as large as the denominator, by (J19). The term $(2a-1)^2\mathbf{P}[{}^{\text{Co}}_{\text{Re}}]$ in the denominator is always nonnegative and equals zero at $a = \frac{1}{2}$, which therefore maximizes $D_1(a,0)$ on $[0,1]$. If $\mathbf{P}[{}^{\text{Co}}_{\text{Re}}] > 0$ then the term is only zero at $a = \frac{1}{2}$, making it the unique maximizer of $D_1(a,0)$, which completes the proof. □

Proof of Lemma 3. The proof proceeds according to the following chain of equalities:

$$\begin{aligned}
\mathbf{Skew}[Y] &= \frac{\mathbf{E}[(Y-\mathbf{E}[Y])^3]}{(\mathbf{SD}[Y])^3} \\
&= \frac{\mathbf{E}[Y^3 - 3Y^2\mathbf{E}[Y] + 3Y\mathbf{E}[Y]^2 - \mathbf{E}[Y]^3]}{(\mathbf{SD}[Y])^3} \\
&= \frac{\mathbf{E}[Y^3]}{(\mathbf{SD}[Y])^3} - 3\frac{\mathbf{E}[Y](\mathbf{E}[Y^2]-\mathbf{E}[Y]^2)}{(\mathbf{SD}[Y])^3} - \frac{\mathbf{E}[Y]^3}{(\mathbf{SD}[Y])^3} \\
&= \frac{\mathbf{E}[Y^3]}{(\mathbf{SD}[Y])^3} - 3\frac{\mathbf{E}[Y]}{\mathbf{SD}[Y]} - \left(\frac{\mathbf{E}[Y]}{\mathbf{SD}[Y]}\right)^3 \\
&= \frac{\mathbf{E}[Y^3]}{(\mathbf{SD}[Y])^3} - \mathbf{Sharpe}[Y]\left(3 + (\mathbf{Sharpe}[Y])^2\right),
\end{aligned}$$

where the first equality is by definition of skewness; the second equality follows by expanding the cube inside the expectation of the numerator; the third equality follows by linearity of the expectation operator; the fourth equality follows by the fact that $(\mathbf{SD}[Y])^2 = \mathbf{Var}[Y] = \mathbf{E}[Y^2] - \mathbf{E}[Y]^2$; and the final equality follows by substitution $\mathbf{Sharpe}[Y] = \frac{\mathbf{E}[Y]}{\mathbf{SD}[Y]}$ and gathering terms, which completes the proof. □

Proof of Proposition 4. First, note that

$$\begin{aligned}
\mathbf{E}[r_{t+1}^3(a)] &= \mathbf{E}[(((1-a)\,w_{\text{SLOW},t} + a\,w_{\text{FAST},t})r_{t+1})^3] \\
&= \mathbf{E}\left[((1-a)^3 w_{\text{SLOW},t}^3 + 3(1-a)^2 a\, w_{\text{SLOW},t}^2\, w_{\text{FAST},t}\right. \\
&\qquad \left. + 3(1-a)a^2 w_{\text{SLOW},t}\, w_{\text{FAST},t}^2 + a^3 w_{\text{FAST},t}^3)\, r_{t+1}^3\right] \\
&= \mathbf{E}\left[((1-a)^3 w_{\text{SLOW},t} + 3(1-a)^2 a\, w_{\text{FAST},t}\right. \\
&\qquad \left. + 3(1-a)a^2 w_{\text{SLOW},t} + a^3 w_{\text{FAST},t})\, r_{t+1}^3\right] \\
&= (1-a)((1-a)^2 + 3a^2)\mathbf{E}[w_{\text{SLOW},t}\, r_{t+1}^3] + a(3(1-a)^2 + a^2)\mathbf{E}[w_{\text{FAST},t}\, r_{t+1}^3] \\
&= (1-a)((1-a)^2 + 3a^2)\frac{\mathbf{E}[(r_{\text{SLOW},t+1})^3]}{(\mathbf{SD}[r_{\text{SLOW},t+1}])^3}(\mathbf{SD}[r_{\text{SLOW},t+1}])^3 \\
&\qquad + a(3(1-a)^2 + a^2)\frac{\mathbf{E}[(r_{\text{FAST},t+1})^3]}{(\mathbf{SD}[r_{\text{FAST},t+1}])^3}(\mathbf{SD}[r_{\text{FAST},t+1}])^3,
\end{aligned} \tag{J20}$$

where the first equality follows by definition of $r_{t+1}(a)$ in (4); the second equality follows by expanding the cube inside the expectation; the third equality follows from the fact that $w_{\text{SLOW},t}^3 = w_{\text{SLOW},t}$, $w_{\text{FAST},t}^3 = w_{\text{FAST},t}$, and $w_{\text{SLOW},t}^2 = w_{\text{FAST},t}^2 = 1$; the fourth equality follows by linearity of the expectation operator and collecting terms; and the last equality follows from the fact that $w_{\text{SLOW},t}^3 = w_{\text{SLOW},t}$ and $w_{\text{FAST},t}^3 = w_{\text{FAST},t}$, by definitions $r_{\text{SLOW},t+1} = w_{\text{SLOW},t}\, r_{t+1}$ and $r_{\text{FAST},t+1} = w_{\text{FAST},t}\, r_{t+1}$, and by multiplying and dividing each summand by the corresponding cubed standard deviation term. Second,

$$\begin{aligned}
\mathbf{Skew}[r_{t+1}(a)] &= \frac{\mathbf{E}[r_{t+1}^3(a)]}{(\mathbf{SD}[r_{t+1}(a)])^3} - \mathbf{Sharpe}[r_{t+1}(a)](3 + (\mathbf{Sharpe}[r_{t+1}(a)])^2) \\
&= (1-a)((1-a)^2 + 3a^2)\frac{\mathbf{E}[(r_{\text{SLOW},t+1})^3]}{(\mathbf{SD}[r_{\text{SLOW},t+1}])^3}\frac{(\mathbf{SD}[r_{\text{SLOW},t+1}])^3}{(\mathbf{SD}[r_{t+1}(a)])^3} \\
&\quad + a(3(1-a)^2 + a^2)\frac{\mathbf{E}[(r_{\text{FAST},t+1})^3]}{(\mathbf{SD}[r_{\text{FAST},t+1}])^3}\frac{(\mathbf{SD}[r_{\text{FAST},t+1}])^3}{(\mathbf{SD}[r_{t+1}(a)])^3} \\
&\quad - \mathbf{Sharpe}[r_{t+1}(a)](3 + (\mathbf{Sharpe}[r_{t+1}(a)])^2) \\
&= (1-a)((1-a)^2 + 3a^2)D^3(a, \mathbf{E}[r_{\text{SLOW},t+1}]) \\
&\qquad \times \left(\mathbf{Skew}[r_{\text{SLOW},t+1}] + \mathbf{Sharpe}[r_{\text{SLOW},t+1}](3 + (\mathbf{Sharpe}[r_{\text{SLOW},t+1}])^2)\right) \\
&\quad + a(3(1-a)^2 + a^2)D^3(a, \mathbf{E}[r_{\text{FAST},t+1}]) \\
&\qquad \times \left(\mathbf{Skew}[r_{\text{FAST},t+1}] + \mathbf{Sharpe}[r_{\text{FAST},t+1}](3 + (\mathbf{Sharpe}[r_{\text{FAST},t+1}])^2)\right) \\
&\quad - 3\,\mathbf{Sharpe}[r_{t+1}(a)] - (\mathbf{Sharpe}[r_{t+1}(a)])^3,
\end{aligned} \tag{J21}$$

where the first equality follows by application of Lemma 3 at $Y = r_{t+1}(a)$; the second equality follows by substitution of (J20) for $\mathbf{E}[r_{t+1}^3(a)]$; and the last equality follows by application of Lemma 3 at $Y = r_{\mathrm{SLOW},t+1}$ and $Y = r_{\mathrm{FAST},t+1}$ and by definition of $D(a,\mu)$ in (9), for which $D(a,\mathbf{E}[r_{\mathrm{SLOW},t+1}]) = \frac{\mathbf{SD}[r_{\mathrm{SLOW},t+1}]}{\mathbf{SD}[r_{t+1}(a)]}$ and $D(a,\mathbf{E}[r_{\mathrm{FAST},t+1}]) = \frac{\mathbf{SD}[r_{\mathrm{FAST},t+1}]}{\mathbf{SD}[r_{t+1}(a)]}$. Continuing from (J21) above and substituting for $\mathbf{Sharpe}[r_{t+1}(a)]$ with (8) of Proposition 1, we have

$$
\begin{aligned}
&\mathbf{Skew}[r_{t+1}(a)] \\
&= (1-a)((1-a)^2+3a^2)D^3(a,\mathbf{E}[r_{t+1}(0)]) \\
&\qquad \times \left(\mathbf{Skew}[r_{\mathrm{SLOW},t+1}] + \mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}](3+(\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}])^2)\right) \\
&\quad + a(3(1-a)^2+a^2)D^3(a,\mathbf{E}[r_{t+1}(1)]) \\
&\qquad \times \left(\mathbf{Skew}[r_{\mathrm{FAST},t+1}] + \mathbf{Sharpe}[r_{\mathrm{FAST},t+1}](3+(\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}])^2)\right) \\
&\quad - 3[(1-a)\,\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)]) + a\,\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)])] \\
&\quad - [(1-a)\,\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)]) + a\,\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)])]^3.
\end{aligned}
\tag{J22}
$$

Continuing from (J22) above, expanding the cube and grouping terms, we have

$$
\begin{aligned}
&\mathbf{Skew}[r_{t+1}(a)] \\
&= (1-a)((1-a)^2+3a^2)X^3(a,\mathbf{E}[r_{t+1}(0)])\mathbf{Skew}[r_{\mathrm{SLOW},t+1}] \\
&\quad + a(3(1-a)^2+a^2)X^3(a,\mathbf{E}[r_{t+1}(1)])\mathbf{Skew}[r_{\mathrm{FAST},t+1}] \\
&\quad + 3[(1-a)\,\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)])\left(((1-a)^2+3a^2)D^2(a,\mathbf{E}[r_{t+1}(0)])-1\right) \\
&\qquad + a\,\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)])\left((3(1-a)^2+a^2)D^2(a,\mathbf{E}[r_{t+1}(1)])-1\right)] \\
&\quad + 3(1-a)a[(1-a)\,\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)]) - a\,\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)])] \\
&\qquad \times \left((\mathbf{Sharpe}[r_{\mathrm{FAST},t+1}])^2D^2(a,\mathbf{E}[r_{t+1}(1)]) - (\mathbf{Sharpe}[r_{\mathrm{SLOW},t+1}])^2D^2(a,\mathbf{E}[r_{t+1}(0)])\right).
\end{aligned}
\tag{J23}
$$

Define

$$\gamma(a) := a(a^2+3(1-a)^2), \tag{J24}$$

and note that $\gamma(a) \in [0,1]$ with $\gamma(0)=0$, $\gamma(\frac{1}{2})=\frac{1}{2}$, and $\gamma(1)=1$. Also, note that

$1-\gamma(a) = (1-a)((1-a)^2+3a^2)$. Then,

$$\begin{aligned}
&\textbf{Skew}[r_{t+1}(a)] \\
&= (1-\gamma(a))D^3(a,\mathbf{E}[r_{t+1}(0)])\textbf{Skew}[r_{\text{SLOW},t+1}] + \gamma(a)D^3(a,\mathbf{E}[r_{t+1}(1)])\textbf{Skew}[r_{\text{FAST},t+1}] \\
&\quad + 3[(1-a)\,\textbf{Sharpe}[r_{\text{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)])\left(((1-a)^2+3a^2)D^2(a,\mathbf{E}[r_{t+1}(0)])-1\right) \\
&\qquad + a\,\textbf{Sharpe}[r_{\text{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)])\left((3(1-a)^2+a^2)D^2(a,\mathbf{E}[r_{t+1}(1)])-1\right)] \\
&\quad + 3(1-a)a[(1-a)\,\textbf{Sharpe}[r_{\text{FAST},t+1}]D(a,\mathbf{E}[r_{t+1}(1)]) - a\,\textbf{Sharpe}[r_{\text{SLOW},t+1}]D(a,\mathbf{E}[r_{t+1}(0)])] \\
&\qquad \times \left((\textbf{Sharpe}[r_{\text{FAST},t+1}])^2D^2(a,\mathbf{E}[r_{t+1}(1)]) - (\textbf{Sharpe}[r_{\text{SLOW},t+1}])^2D^2(a,\mathbf{E}[r_{t+1}(0)])\right),
\end{aligned} \tag{J25}$$

where $\gamma(a)$ is as defined in (J24).

Approximate $D(a,\mathbf{E}[r_{t+1}(0)]) \approx D(a,\mathbf{E}[r_{t+1}(1)]) \approx D(a,0)$. Then,

$$\begin{aligned}
\textbf{Skew}[r_{t+1}(a)] \approx\ & [(1-\gamma(a))\textbf{Skew}[r_{\text{SLOW},t+1}] + \gamma(a)\textbf{Skew}[r_{\text{FAST},t+1}]]D^3(a,0) \\
& + 3[(1-a)\,\textbf{Sharpe}[r_{\text{SLOW},t+1}]\left(((1-a)^2+3a^2)D^2(a,0)-1\right) \\
& \quad + a\,\textbf{Sharpe}[r_{\text{FAST},t+1}]\left((3(1-a)^2+a^2)D^2(a,0)-1\right)]D(a,0) \\
& + 3(1-a)a[(1-a)\,\textbf{Sharpe}[r_{\text{FAST},t+1}] - a\,\textbf{Sharpe}[r_{\text{SLOW},t+1}]] \\
& \quad \times \left((\textbf{Sharpe}[r_{\text{FAST},t+1}])^2 - (\textbf{Sharpe}[r_{\text{SLOW},t+1}])^2\right)D^3(a,0).
\end{aligned} \tag{J26}$$

For the special case of $a=\frac{1}{2}$, this approximation becomes:

$$\begin{aligned}
&\textbf{Skew}[r_{t+1}(\tfrac{1}{2})] \qquad\qquad (J27)\\
&\approx \tfrac{1}{2}(\textbf{Skew}[r_{\text{SLOW},t+1}] + \textbf{Skew}[r_{\text{FAST},t+1}])D^3(\tfrac{1}{2};0) \\
&\quad + 3[\tfrac{1}{2}\,\textbf{Sharpe}[r_{\text{SLOW},t+1}] + \tfrac{1}{2}\,\textbf{Sharpe}[r_{\text{FAST},t+1}]]D(\tfrac{1}{2};0)\left(D^2(\tfrac{1}{2};0)-1\right) \\
&\quad + \tfrac{3}{4}\left(\textbf{Sharpe}[r_{\text{FAST},t+1}] - \textbf{Sharpe}[r_{\text{SLOW},t+1}]\right)^2 \\
&\qquad \times \left(\tfrac{1}{2}(\textbf{Sharpe}[r_{\text{FAST},t+1}] + \textbf{Sharpe}[r_{\text{SLOW},t+1}])D(\tfrac{1}{2};0)\right)D^2(\tfrac{1}{2};0) \\
&= \tfrac{1}{2}(\textbf{Skew}[r_{\text{SLOW},t+1}] + \textbf{Skew}[r_{\text{FAST},t+1}])D^3(\tfrac{1}{2};0) \\
&\quad + 3\,\textbf{Sharpe}[r_{t+1}(\tfrac{1}{2})]\left(D^2(\tfrac{1}{2};0)-1\right) \\
&\quad + \tfrac{3}{4}\left(\textbf{Sharpe}[r_{\text{FAST},t+1}] - \textbf{Sharpe}[r_{\text{SLOW},t+1}]\right)^2\textbf{Sharpe}[r_{t+1}(\tfrac{1}{2})]D^2(\tfrac{1}{2};0) \\
&= \tfrac{1}{2}(\textbf{Skew}[r_{\text{SLOW},t+1}] + \textbf{Skew}[r_{\text{FAST},t+1}])D^3(\tfrac{1}{2};0) \\
&\quad + 3\,\textbf{Sharpe}[r_{t+1}(\tfrac{1}{2})]\left(D^2(\tfrac{1}{2};0)\left[1+\left(\frac{\textbf{Sharpe}[r_{\text{FAST},t+1}] - \textbf{Sharpe}[r_{\text{SLOW},t+1}]}{2}\right)^2\right]-1\right),
\end{aligned} \tag{J28}$$

where the middle equality follows from substitution of the approximation of $\mathbf{Sharpe}[r_{t+1}(\frac{1}{2})]$ as in (10) of Proposition 1, which holds under the same assumptions as made above. This completes the proof. □

Proof of Proposition 2. First, note that

$$\begin{aligned}\mathbf{Cov}[w_t(a),(r_{t+1}-\mathbf{E}[r_{t+1}])^2] &= \mathbf{Cov}[w_t(a), r_{t+1}^2 - 2r_{t+1}\mathbf{E}[r_{t+1}] + (\mathbf{E}[r_{t+1}])^2] \\ &= \mathbf{Cov}[w_t(a), r_{t+1}^2] - 2\mathbf{Cov}[w_t(a), r_{t+1}]\mathbf{E}[r_{t+1}], \end{aligned} \tag{J29}$$

where the first equality follows by squaring the term inside the second covariance argument; and the second equality follows by bilinearity of the covariance operator and the fact that $(\mathbf{E}[r_{t+1}])^2$ is a constant and hence has zero covariance with $w_t(a)$.

Second,

$$\begin{aligned}\mathbf{Cov}[r_{t+1}(a), r_{t+1}] &= \mathbf{Cov}[w_t(a)r_{t+1}, r_{t+1}] \\ &= \mathbf{E}[w_t(a)r_{t+1}^2] - \mathbf{E}[w_t(a)r_{t+1}]\mathbf{E}[r_{t+1}] \\ &= (\mathbf{Cov}[w_t(a), r_{t+1}^2] + \mathbf{E}[w_t(a)]\mathbf{E}[r_{t+1}^2]) \\ &\quad - (\mathbf{Cov}[w_t(a), r_{t+1}] + \mathbf{E}[w_t(a)]\mathbf{E}[r_{t+1}])\mathbf{E}[r_{t+1}] \\ &= \mathbf{E}[w_t(a)](\mathbf{E}[r_{t+1}^2] - (\mathbf{E}[r_{t+1}])^2) \\ &\quad - \mathbf{Cov}[w_t(a), r_{t+1}]\mathbf{E}[r_{t+1}] + \mathbf{Cov}[w_t(a), r_{t+1}^2] \\ &= \mathbf{E}[w_t(a)]\mathbf{Var}[r_{t+1}] - \mathbf{Cov}[w_t(a), r_{t+1}]\mathbf{E}[r_{t+1}] + \mathbf{Cov}[w_t(a), r_{t+1}^2] \\ &= \mathbf{E}[w_t(a)]\mathbf{Var}[r_{t+1}] + \mathbf{Cov}[w_t(a), r_{t+1}]\mathbf{E}[r_{t+1}] \\ &\quad + \mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2], \end{aligned} \tag{J30}$$

where the first equality follows by definition of $r_{t+1}(a) = w_t(a)r_{t+1}$; the second equality follows by definition of covariance; the third equality follows by two applications of the definition of covariance (once with $\mathbf{E}[w_t(a)r_{t+1}^2] = \mathbf{Cov}[w_t(a), r_{t+1}^2] + \mathbf{E}[w_t(a)]\mathbf{E}[r_{t+1}^2]$ and again with $\mathbf{E}[w_t(a)r_{t+1}] = \mathbf{Cov}[w_t(a), r_{t+1}] - \mathbf{E}[w_t(a)]\mathbf{E}[r_{t+1}]$) similar to the decomposition in (14); the fourth equality follows by rearrangement of terms; the fifth equality follows by definition of variance; and the last equality follows by substitution for $\mathbf{Cov}[w_t(a), r_{t+1}^2]$ from its implicit relation in (J29).

Next, the expression for beta in (16) follows immediately from (J30) (equivalently, (15)) and the definition of beta as the contemporaneous covariance scaled by the underlying market return variance. Finally,

$$
\begin{aligned}
&\mathbf{Alpha}[r_{t+1}(a)] \\
&\quad = \mathbf{E}[r_{t+1}(a)] - \mathbf{Beta}[r_{t+1}(a)]\mathbf{E}[r_{t+1}] \\
&\quad = \mathbf{Cov}[w_t(a), r_{t+1}] + \mathbf{E}[w_t(a)]\mathbf{E}[r_{t+1}] \\
&\quad\quad - \left(\mathbf{E}[w_t(a)] + \frac{\mathbf{Cov}[w_t(a), r_{t+1}]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}] + \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]} \right) \mathbf{E}[r_{t+1}] \\
&\quad = \mathbf{Cov}[w_t(a), r_{t+1}] \left(1 - \frac{(\mathbf{E}[r_{t+1}])^2}{\mathbf{Var}[r_{t+1}]} \right) - \frac{\mathbf{Cov}[w_t(a), (r_{t+1} - \mathbf{E}[r_{t+1}])^2]}{\mathbf{Var}[r_{t+1}]}\mathbf{E}[r_{t+1}], \qquad \text{(J31)}
\end{aligned}
$$

where the first equality follows by definition of alpha; the second equality follows by covariance decomposition of $\mathbf{E}[r_{t+1}(a)]$ as in (14) and substitution of the expression for beta in (16); and the last equality follows by rearrangement. The approximation in (18) follows immediately from the assumption in the proposition, and completes the proof. □

Proof of Proposition 5. First, the expected dynamic return satisfies:

$$
\begin{aligned}
\mathbf{E}[r_{t+1}(a_{s(t)})] &= \mathbf{E}[((1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t})r_{t+1}] \\
&= \mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] + (1 - 2a_{\text{Co}})\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}] \\
&\quad + (2a_{\text{Re}} - 1)\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}], \qquad \text{(J32)}
\end{aligned}
$$

where the first equality follows by definition of $r_{t+1}(a_{s(t)}) = w_t(a_{s(t)})r_{t+1} = ((1-a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t})r_{t+1}$; and the second equality follows from the law of total expectation and the facts that $w_{\text{SLOW},t} = w_{\text{FAST},t} = 1$ after Bull, $w_{\text{SLOW},t} = 1$ and $w_{\text{FAST},t} = -1$ after Correction, $w_{\text{SLOW},t} = w_{\text{FAST},t} = -1$ after Bear, and $w_{\text{SLOW},t} = -1$ and $w_{\text{FAST},t} = 1$ after Rebound.

Second,

$$
\begin{aligned}
\mathbf{Var}[r_{t+1}(a_{s(t)})] &= \mathbf{Var}[((1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t})r_{t+1}] \\
&= \mathbf{E}[((1 - a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t})^2 r_{t+1}^2] - (\mathbf{E}[r_{t+1}(a_{s(t)})])^2 \\
&= \mathbf{E}[[(1 - a_{s(t)})^2 + 2(1 - a_{s(t)})a_{s(t)}w_{\text{SLOW},t}w_{\text{FAST},t} + a_{s(t)}^2]r_{t+1}^2] \\
&\quad - (\mathbf{E}[r_{t+1}(a_{s(t)})])^2 \\
&= \mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + (2a_{\text{Co}} - 1)^2\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}] \\
&\quad + (2a_{\text{Re}} - 1)^2\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}] - (\mathbf{E}[r_{t+1}(a_{s(t)})])^2, \qquad \text{(J33)}
\end{aligned}
$$

where the first equality follows by definition of $r_{t+1}(a_{s(t)}) = w_t(a_{s(t)})r_{t+1} = ((1-a_{s(t)})w_{\text{SLOW},t} + a_{s(t)}w_{\text{FAST},t})r_{t+1}$; the second equality follows by definition of variance; the third equality follows by expanding the square inside the expectation of the first summand; and the last equality follows by the law of total expectation and the facts that $w_{\text{SLOW},t}w_{\text{FAST},t} = 1$ after Bull or Bear, and $w_{\text{SLOW},t}w_{\text{FAST},t} = -1$ after Correction or Rebound, plus simplification.

Note that the expected return and variance of the dynamic strategy do not depend on the values of $a_{s(t)}$ in Bull or Bear cycles since both SLOW and FAST agree in these states.

Next, we compute the partial derivatives of the expected return, variance, and standard deviation with respect to each decision variable. Expected return:

$$\frac{\partial}{\partial a_{\text{Co}}}\mathbf{E}[r_{t+1}(a_{s(t)})] = -2\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}], \tag{J34}$$

$$\frac{\partial}{\partial a_{\text{Re}}}\mathbf{E}[r_{t+1}(a_{s(t)})] = 2\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}], \tag{J35}$$

Variance:

$$\begin{aligned}
\frac{\partial}{\partial a_{\text{Co}}}\mathbf{Var}[r_{t+1}(a_{s(t)})] &= 4(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}] - 2\mathbf{E}[r_{t+1}(a_{s(t)})]\frac{\partial}{\partial a_{\text{Co}}}\mathbf{E}[r_{t+1}(a_{s(t)})] \\
&= 4(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}] - 2\mathbf{E}[r_{t+1}(a_{s(t)})](-2\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}]) \\
&= 4\mathbf{P}[\text{Co}]\{(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}] + \mathbf{E}[r_{t+1}(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Co}]\}, \qquad \text{(J36)} \\
\frac{\partial}{\partial a_{\text{Re}}}\mathbf{Var}[r_{t+1}(a_{s(t)})] &= 4(2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}] - 2\mathbf{E}[r_{t+1}(a_{s(t)})]\frac{\partial}{\partial a_{\text{Re}}}\mathbf{E}[r_{t+1}(a_{s(t)})] \\
&= 4(2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}] - 2\mathbf{E}[r_{t+1}(a_{s(t)})](2\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}]) \\
&= 4\mathbf{P}[\text{Re}]\{(2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}] - \mathbf{E}[r_{t+1}(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Re}]\}, \qquad \text{(J37)}
\end{aligned}$$

where the second equality in the equality chain of (J36) and (J37) follow from (J34) and (J35), respectively. Standard deviation:

$$\begin{aligned}
\frac{\partial}{\partial a_{\text{Co}}}\mathbf{SD}[r_{t+1}(a_{s(t)})] &= \frac{1}{2}\frac{1}{\mathbf{SD}[r_{t+1}(a_{s(t)})]}\frac{\partial}{\partial a_{\text{Co}}}\mathbf{Var}[r_{t+1}(a_{s(t)})] \\
&= \frac{2\mathbf{P}[\text{Co}]\{(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}] + \mathbf{E}[r_{t+1}(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Co}]\}}{\mathbf{SD}[r_{t+1}(a_{s(t)})]}, \qquad \text{(J38)} \\
\frac{\partial}{\partial a_{\text{Re}}}\mathbf{SD}[r_{t+1}(a_{s(t)})] &= \frac{1}{2}\frac{1}{\mathbf{SD}[r_{t+1}(a_{s(t)})]}\frac{\partial}{\partial a_{\text{Re}}}\mathbf{Var}[r_{t+1}(a_{s(t)})] \\
&= \frac{2\mathbf{P}[\text{Re}]\{(2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}] - \mathbf{E}[r_{t+1}(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Re}]\}}{\mathbf{SD}[r_{t+1}(a_{s(t)})]}, \qquad \text{(J39)}
\end{aligned}$$

where (J38) and (J39) follow from (J36) and (J37), respectively.

Next, we compute the partial derivatives of the Sharpe ratio with respect to each decision variable. Sharpe ratio:

$$\begin{aligned}
\frac{\partial}{\partial a_{\text{Co}}}\mathbf{Sharpe}[r_{t+1}(a)] &= \frac{\partial}{\partial a_{\text{Co}}}\frac{\mathbf{E}[r_{t+1}(a_{s(t)})]}{\mathbf{SD}[r_{t+1}(a_{s(t)})]} \\
&= \frac{\mathbf{SD}[r_{t+1}(a_{s(t)})]\frac{\partial}{\partial a_{\text{Co}}}\mathbf{E}[r_{t+1}(a_{s(t)})] - \mathbf{E}[r_{t+1}(a_{s(t)})]\frac{\partial}{\partial a_{\text{Co}}}\mathbf{SD}[r_{t+1}(a_{s(t)})]}{\mathbf{Var}[r_{t+1}(a_{s(t)})]} \\
&= \frac{\mathbf{SD}[r_{t+1}(a_{s(t)})](-2\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}])}{\mathbf{Var}[r_{t+1}(a_{s(t)})]} \\
&\quad - \frac{\mathbf{E}[r_{t+1}(a_{s(t)})]\left(\frac{2\mathbf{P}[\text{Co}]\{(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]+\mathbf{E}[r_{t+1}(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Co}]\}}{\mathbf{SD}[r_{t+1}(a_{s(t)})]}\right)}{\mathbf{Var}[r_{t+1}(a_{s(t)})]} \\
&= -2\mathbf{P}[\text{Co}]\left[\frac{(\mathbf{Var}[r_{t+1}(a_{s(t)})] + (\mathbf{E}[r_{t+1}(a_{s(t)})])^2)\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{SD}[r_{t+1}(a_{s(t)})]\mathbf{Var}[r_{t+1}(a_{s(t)})]}\right. \\
&\quad \left. + \frac{(2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{E}[r_{t+1}(a_{s(t)})]}{\mathbf{SD}[r_{t+1}(a_{s(t)})]\mathbf{Var}[r_{t+1}(a_{s(t)})]}\right] \\
&= -2\mathbf{P}[\text{Co}]\frac{\mathbf{E}[r_{t+1}^2(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Co}] + (2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{E}[r_{t+1}(a_{s(t)})]}{\mathbf{SD}[r_{t+1}(a_{s(t)})]\mathbf{Var}[r_{t+1}(a_{s(t)})]},
\end{aligned} \tag{J40}$$

where the second equality follows by the quotient rule for derivatives; the third equality follows from (J34) and (J38); and the last equality follows from the definition of variance. Similar steps for a_{Re} yield the following:

$$\frac{\partial}{\partial a_{\text{Re}}}\mathbf{Sharpe}[r_{t+1}(a)] = 2\mathbf{P}[\text{Re}]\frac{\mathbf{E}[r_{t+1}^2(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Re}] - (2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{E}[r_{t+1}(a_{s(t)})]}{\mathbf{SD}[r_{t+1}(a_{s(t)})]\mathbf{Var}[r_{t+1}(a_{s(t)})]}. \tag{J41}$$

Next, we apply the first order conditions to derive stationary points of the maximization problem. The first order conditions are: $\frac{\partial}{\partial a_{\text{Co}}}\mathbf{Sharpe}[r_{t+1}(a)] = 0$ and $\frac{\partial}{\partial a_{\text{Re}}}\mathbf{Sharpe}[r_{t+1}(a)] = 0$, or equivalently, by (J40) and (J41),

$$\mathbf{E}[r_{t+1}^2(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Co}] + (2a_{\text{Co}}-1)\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{E}[r_{t+1}(a_{s(t)})] = 0, \tag{J42}$$

$$\mathbf{E}[r_{t+1}^2(a_{s(t)})]\mathbf{E}[r_{t+1}|\text{Re}] - (2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{E}[r_{t+1}(a_{s(t)})] = 0. \tag{J43}$$

Dividing each condition (J42) and (J43) through by $\mathbf{E}[r_{t+1}^2|\text{Co}]$ and $\mathbf{E}[r_{t+1}^2|\text{Re}]$, respectively, and rearranging yields the equivalent conditions:

$$(2a_{\text{Co}} - 1)\mathbf{E}[r_{t+1}(a_{s(t)})] = -\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}\mathbf{E}[r_{t+1}^2(a_{s(t)})], \tag{J44}$$

$$(2a_{\text{Re}} - 1)\mathbf{E}[r_{t+1}(a_{s(t)})] = \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}\mathbf{E}[r_{t+1}^2(a_{s(t)})]. \tag{J45}$$

By (J32) and (J33), and the fact that $\mathbf{E}[r_{t+1}^2(a_{s(t)})] = \mathbf{Var}[r_{t+1}^2(a_{s(t)})] + (\mathbf{E}[r_{t+1}(a_{s(t)})])^2$, these conditions are equivalent to $(2a_{\text{Co}} - 1)(\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}] + (2a_{\text{Re}} - 1)\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}]) = -\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}(\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + (2a_{\text{Re}} - 1)^2\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}])$ and $(2a_{\text{Re}} - 1)(\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] + (1 - 2a_{\text{Co}})\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]) = \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}(\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + (2a_{\text{Co}} - 1)^2\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}])$, so we can isolate the decision variables on either side of the condition equations: $(2a_{\text{Co}} - 1) = -\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]+(2a_{\text{Re}}-1)^2\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}]-\mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]+(2a_{\text{Re}}-1)\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}]}$ and $(2a_{\text{Re}}-1) = \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]+(2a_{\text{Co}}-1)^2\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}])}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}]+(1-2a_{\text{Co}})\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}]-\mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}$. Next, rewrite these conditions as

$$x(y) = -\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + y^2\mathbf{E}[r_{t+1}^2|\text{Re}]\mathbf{P}[\text{Re}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}] + y\mathbf{E}[r_{t+1}|\text{Re}]\mathbf{P}[\text{Re}]}, \tag{J46}$$

$$y(x) = \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}] + x^2\mathbf{E}[r_{t+1}^2|\text{Co}]\mathbf{P}[\text{Co}])}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - x\mathbf{E}[r_{t+1}|\text{Co}]\mathbf{P}[\text{Co}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}. \tag{J47}$$

To find a fixed point (x^*, y^*) of the above system such that $x^* = x(y^*)$ and $y^* = y(x^*)$, note that $x(0) = x(y(0))$ and $y(0) = y(x(0))$. Hence, $x^* = x(y(0))$ and $y^* = y(x(0))$ are fixed points. Specifically, we have $x(0) = -\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}]-\mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}$ and $y(0) = \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}]-\mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}$. Therefore,

$$a^*_{\text{Co}} = \frac{1}{2}\left(1 - \frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}\right), \tag{J48}$$

$$a^*_{\text{Re}} = \frac{1}{2}\left(1 + \frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}\frac{\mathbf{E}[r_{t+1}^2|{}^{\text{Bu}}_{\text{Be}}]\mathbf{P}[{}^{\text{Bu}}_{\text{Be}}]}{\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}]}\right), \tag{J49}$$

comprise a stationary point of the Sharpe ratio maximization problem.

Moreover, the necessary stationary conditions (J44) and (J45) imply that $(2a_{\text{Co}} - 1)$ and $(2a_{\text{Re}} - 1)$ must be proportional to $-\frac{\mathbf{E}[r_{t+1}|\text{Co}]}{\mathbf{E}[r_{t+1}^2|\text{Co}]}$ and $\frac{\mathbf{E}[r_{t+1}|\text{Re}]}{\mathbf{E}[r_{t+1}^2|\text{Re}]}$, respectively, with the same proportionality constant, $\frac{\mathbf{E}[r_{t+1}^2(a_{s(t)})]}{\mathbf{E}[r_{t+1}(a_{s(t)})]}$. Since the Sharpe ratio is bounded above and below, and since this proportionality constant can be either positive or negative—depending on the sign of its denominator—the stationary point defined by (J48) and (J49), and the negative version of this point, are the only two stationary points. If $\mathbf{E}[r_{t+1}|\text{Bu}]\mathbf{P}[\text{Bu}] - \mathbf{E}[r_{t+1}|\text{Be}]\mathbf{P}[\text{Be}] >$

0, then expected return at stationary point (J48) and (J49) is positive, and this point is the unique maximizer, while the negative version of this point is the unique minimizer. This completes the proof. □