# Identification of Auction Models Using Order Statistics


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## Objectives

Identify bidder value distributions in IPV auctions with *nonseparable finite* auction-level unobserved heterogeneity (UH) when only observing three order statistics (OSs) of bids.

Highlights

- Symmetric auction w observed competition
- Symmetric auction w unobserved competition
- Asymmetric auction
- Applications not limited to auctions

## Introduction

- (Auction) data may be imperfect
  1. Missing value relevant covariates ⇒ UH.
     - Unobserved competition
  2. Incomplete bids: only a few OSs of bids
     - Auction format: (second-price/ ascending auctions) winning bid ([1]), the second, third, and fourth highest bids ([7])
     - Data truncation Washington State DOT, publishes only top three low bids

## Literature

1. Tackle UH with all bids ([3], [6], [2])
2. Tackle both UH and incomplete data
   - Auxiliary variable, such as monotone in UH ([4]):
   - Conditional indepen via markov property of OS ([5]).

## Empirical Settings

- Timber auction $m$ is characterized by
  - Auction-level observed charac: $X_m$ (acres, volume etc)
  - Auction-level unobserved charac (UH): $Z_m$ (quality)
  - Number of bidders: $n_m$
  - Bidder's value distribution: $\Phi(v|X,Z,n)$ (IPV)

$$v_{im} = \underbrace{X_m'\beta_x}_{\text{observed}} + \underbrace{Z_m'\beta_z}_{\text{UH}} + \underbrace{\epsilon_{im}}_{\text{private info}}$$

  - Bidder $i$ draws a value $v_{im}$ and bids $b_{im} = s(v_{im})$

## Identification

- Data
  - Ideal data:

$$\{X_m, Z_m, n_m, b_1, ..., b_{n_m}\}_m$$

  - Actual data:

$$\{X_m, \cancel{Z_m}, n_m, b_{r_1:n_m}, ..., b_{r_s:n_m}\}_m$$

    - Order statistics of bids: $b_{1:n} < b_{2:n} < \cdots < b_{n:n}$
- Identification problem
  - Can we recover value distribution $\Phi(v|X,Z,n)$ from actual data?
  - need to recover conditional bid density $f(b|X,Z,n)$
    - First-price: $v = b + \frac{1}{n-1}\frac{F(b|X,Z,n)}{f(b|X,Z,n)}$
    - Second-price: $v = b$

## Existing Method inapplicable

- Existing method of assuming all bids are available
  - Joint distribution of any three bids conditional on $k$

$$f^k(b_1,b_2,b_3) = f^k(b_1)f^k(b_2)f^k(b_3) \quad \text{(cond indep)}$$

  - The correlation of $b_1, b_2, b_3$ reveals information on the UH
    - Intuition: $b_1, b_2, b_3$ are independent without UH.
    - With UH, the joint distribution reflects information on the UH
- The problem of observing only $\{r_1, r_2, r_3\}$ OSs

$$f^k_{r_1,r_2,r_3:n}(b_1,b_2,b_3) \neq f^k_{r_1:n}(b_1)f^k_{r_2:n}(b_2)f^k_{r_3:n}(b_3),$$

the conditional independence fails with OSs.

## Our Contributions

- Our main insight: use <u>consecutive</u> OSs of bids

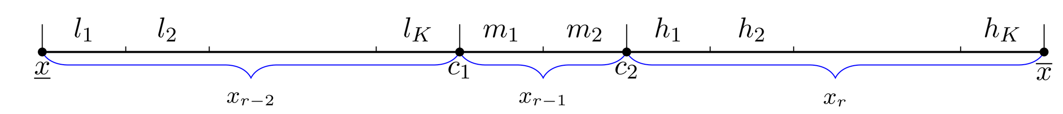
$$f^k_{r-2,r-1,r:n_m}(b_1,b_2,b_3)$$
$$= c \cdot \underbrace{f^k_{r-2:r-2}(b_1)f^k(b_2)f^k_{1:n-r+1}(b_3)}_{\text{separable}} \cdot \underbrace{1(b_1 \le b_2 \le b_3)}_{\text{correlation}}$$

where $b_{r-2:n_m} = b_1, b_{r-1:n_m} = b_2, b_{r:n_m} = b_3$.

  - Symmetric auctions
    - observe competition: $\{X_m, \cancel{Z_m}, n_m, b_{r-2:n_m}, b_{r-1:n_m}, b_{r:n_m}\}_m$
    - two dimensional UH: $\{X_m, \cancel{Z_m}, \cancel{n_m}, b_{r-2:n_m}, b_{r-1:n_m}, b_{r:n_m}\}_m$
  - Asymmetric auctions

## Identification Sketch

1. Divide the support into three segments $l$(lower), $m$(middle), and $h$(high).
   - To control the correlation $\mathbb{I}(b_1 \le b_2 \le b_3)$




   - Iden only uses variation of the $r-2^{th}$, $r-1^{th}$, and $r^{th}$ OSs w.r.t segment $l$, $m$, and $h$, respectively.
   - $\mathbb{I}(b_1 \le b_2 \le b_3) = 1$ if $b_1 \in l, b_2 \in m, b_3 \in h$.
2. Matrix representation of the joint dist:

$$\mathbb{J}_{\ell,m_{i'},h} = \mathbb{L}\mathbb{D}_{m_{i'}}\mathbb{D}_p\mathbb{H}^T, \quad i' = 1,2$$

where

$$\mathbb{J}_{l,m_{i'},h} \equiv \{\int_{b_1\in l_i, b_2\in m_{i'}, b_3\in h_j} f_{r-2,r-1,r:n}(b_1,b_2,b_3)db_1db_2db_3\}_{i,j}$$
$$\mathbb{L} \equiv \{\int_{b_1\in l_i} G_1(f^k(b_1))db_1\}_{i,k},$$
$$\mathbb{D}_{m_{i'}} \equiv diag\{\int_{b_2\in m_{i'}} f^k(b_2)db_2\}_k$$
$$\mathbb{D}_p \equiv diag\{\lambda_k\}_k$$
$$\mathbb{H} \equiv \{\int_{b_3\in h_j} G_2(f^k(b_3))db_3\}_{j,k},$$

3. Eigenvalue-eigenvector representation

$$\mathbb{J}_{l,m_1,h}J^{-1}_{l,m_2,h} = \mathbb{L}\mathbb{D}_{m_1/m_2}\mathbb{L}^{-1},$$

   - With full rank assumption ($\mathbb{L}$ and $\mathbb{H}$)
4. Iden $\mathbb{L}$ up to permutation and scales.
   - Iden $\mathbb{L}_b = G_1(f^k(b)), b \in l$, up to permutation and scales $\mathbb{J}_{b,m_{i'},h} = \mathbb{L}_b\mathbb{D}_{m_{i'}}\mathbb{D}_p\mathbb{H}^T$
   - Iden $f^k(b), b \in l$, up to permutation and scales via the one-to-one mapping between $G_1(f^k(b))$ and $f^k(b)$.
5. Identify $\mathbb{H}$ up to permutation and scales
   - Iden $f^k(b), b \in h$, up to permutation and scales via the one-to-one mapping between $G_2(f^k(b))$ and $f^k(b)$.
6. Iden $f^k(b), b \in m$, up to permutation and scales

## Pin Down Scales

- Iden type $k$ bid dist in segments $l, m, h$ to scales:

$$f^k(x) = \begin{cases} s^k_l \cdot f^k_l(x) & \text{if } b \in l = (-\infty, c_1] \\ s^k_m \cdot f^k_m(x) & \text{if } b \in m = [c_1, c_2] \\ s^k_h \cdot f^k_h(x) & \text{if } b \in h = [c_2, +\infty) \end{cases}$$

where $s^k_l, s^k_m, s^k_h$ are the unknown scales
- scale conditions:

$$s^k_l \cdot f^k_l(c_1) = s^k_m \cdot f^k_m(c_2)$$
$$s^k_m \cdot f^k_m(c_2) = s^k_h \cdot f^k_h(c_3)$$
$$s^k_l \int_{b\in l} f^k_l(x)dx + s^k_m \int_{b\in m} f^k_m(x)dx + s^k_h \int_{b\in l} f^k_h(x)dx = 1$$

## Conclusion

- Iden auctions with UH using only OS
- Takeaways:
  - Separability, instead of Conditional indepen is the key
  - Separability can be provided via Consecutiveness of OSs

## Companion Papers

- "Identification of Auction Models Using Order Statistics," with Yao Luo, 2020
  – discrete, nonseparable UH
  – use three consecutive OS of bids
- "Order Statistics Approaches to Unobserved Heterogeneity in Auctions," with Yao Luo and Peijun Sang, 2020
  – continuous, nonseparable UH
  – use three consecutive OS of bids
- "Accounting for Unobserved Heterogeneity in Ascending Auctions," with Yao Luo, 2020
  – continuous, separable UH
  – ratio of characteristic functions of OS identifies the parent distribution

## References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