# Wiener-Hopf Parametrization of Possibly Non-Invertible SVARMA Models

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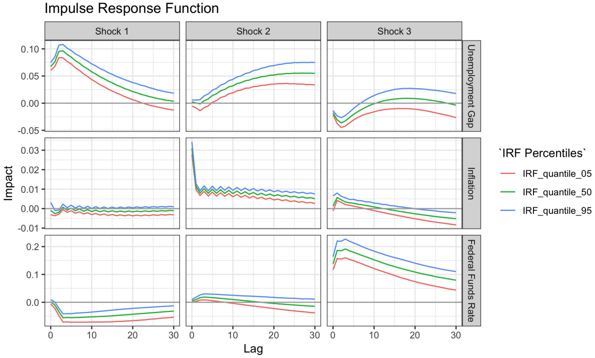
31 Dec 2020

# Outline

# Impulse Response Function (IRF)

- Trace out the **response** of an economic variable of interest with respect **to true underlying economic shocks**
  - Important part of macroeconomic analysis since "Sims (1980): Macroeconomics and Reality"



# Reconstructing Shocks from IRF



## Example: Productivity $y_t = \varepsilon_t + b\varepsilon_{t-1}$

- $\varepsilon_t =$ Shock on productivity (i.i.d.)
  - Underlying true economic shock
- $b > 1$: **Maximal impact occurs with lag**

1. Can we **reconstruct** the shocks from present and past observables? No!
   $\varepsilon_t = y_t - b\varepsilon_{t-1} = y_t - by_{t-1} + b^2\varepsilon_{t-2} = \cdots$
2. **Reconstruct** true underlying shock from future observables? Yes! $\varepsilon_t = \frac{1}{b}\sum_{j=1}^{\infty}\left(-\frac{1}{b}\right)^j y_{t+j}$

# Outline

# Structural Vector Autoregressive Moving Average Models

$$\underbrace{\left(I_n - a_1 z - \cdots - a_p z^p\right)}_{=a(z)} y_t = \underbrace{\left(b_0 + b_1 z + \cdots + b_q z^q\right)}_{=b(z)} B\varepsilon_t, \qquad a_i, b_j \in \mathbb{R}^{n\times n} \tag{1}$$

- Stable AR polynonmial: $\det\left(a(z)\right) \neq 0$ for all $|z| \leq 1$
- Possibly non-invertible MA polynomial: $\det\left(b(z)\right) \neq 0$ for all $|z| = 1$
  - $b_0$ may be singular, identifiability conditions on factorised $b(z)$
- $(\varepsilon_t)$ independent across time, non-Gaussian, independent in cross-section with $\mathbb{E}\left(\varepsilon_t\right) = 0$ and $\mathbb{E}\left(\varepsilon_t \varepsilon_s'\right) = \delta_{ts}\mathrm{diag}\left(\sigma_1, \ldots, \sigma_n\right) = \delta_{ts}\Sigma$
- $(B, \Sigma)$ jointly identified (signed permutations)

# Parametrization for Possibly Non-Invertible SVARMA

"Past" $p(z)$ : All zeros outside the unit circle

$$b(z) = \left[ p(z) \underbrace{\operatorname{diag}\left(z^{\kappa+1}, \ldots, z^{\kappa+1}, z^{\kappa}, \ldots, z^{\kappa}\right)}_{=s(z)} f(z) \right] B\varepsilon_t$$

- stable polynomial in $z$, i.e. $\det(p(z)) \neq 0$ for all $|z| \leq 1$; all zeros outside the unit circle
- Zero and one restrictions to obtain uniqueness

$$p(z) = \begin{pmatrix} I_k & 0_{k\times(n-k)} \\ p_{0,21} & I_{n-k} \end{pmatrix} + \begin{pmatrix} p_{1,11} & 0_{k\times(n-k)} \\ p_{1,21} & p_{1,22} \end{pmatrix} z + \cdots$$

$$\cdots + p_{q-\kappa-1} z^{q-\kappa-1} + \begin{pmatrix} 0_{k\times k} & p_{q-\kappa,12} \\ 0_{(n-k)\times k} & p_{q-\kappa,22} \end{pmatrix} z^{q-\kappa}$$

- $\left(a_p, \left[\begin{array}{cc} p_{[q-\kappa-1,\bullet 1]} & p_{[q-\kappa,\bullet 2]} \end{array}\right]\right)$ of full rank

# Parametrization for Possibly Non-Invertible SVARMA

"Future" $f(z)$: All zeros inside the unit circle

$$f(z) = f_0 + f_1 z^{-1} + \cdots + \begin{pmatrix} f_{\kappa,[1:k,\bullet]} \\ f_{\kappa,[k+1:n,\bullet]} \end{pmatrix} z^{-\kappa} + \begin{pmatrix} f_{\kappa+1,[1:k,\bullet]} \\ 0_{(n-k)\times n} \end{pmatrix} z^{-\kappa-1}$$

- stable polynomial in $\frac{1}{z}$: $\det\left(f\left(\frac{1}{z}\right)\right) \neq 0$ for all $|z| \leq 1$; all zeros inside the unit circle
- $f_0$ of full rank
  - Natural normalization: $f_0 = I_n$
  - Unnatural normalization: $b_0 = I_n$, implying

$$p_0^{-1} = \begin{pmatrix} f_{\kappa+1,[1:k,\bullet]} \\ f_{\kappa,[k+1:n,\bullet]} \end{pmatrix} = \begin{pmatrix} I_k & 0_{k\times(n-k)} \\ -p_{0,21} & I_{n-k} \end{pmatrix}$$

# Parametrization for Possibly Non-Invertible SVARMA

"Shifts" $s(z)$ with partial indices $(\kappa, k)$

$$a(z) = \left[ p(z) \underbrace{\operatorname{diag}\left(z^{\kappa+1}, \ldots, z^{\kappa+1}, z^{\kappa}, \ldots, z^{\kappa}\right)}_{=s(z)} f(z) \right] B\varepsilon_t$$

- Parametrize the number of zeros inside the unit circle
- Unique

# Identifiability

## Connect External Characteristics Uniquely to Internal Characteristics

- Is there an **injective function** from the (deep) parameters to the observed aspects of the stochastic process?
- Is it possible to **deduce the internal characteristics from the external ones**?

## Here

- Second moment information (spectral density) does not allow for identification of root location and the static shock transmision matrix.
- Higher order spectral densities do (under assumptions)!

# Assumptions on Input Shocks

## (a) Non-zero cumulants

Components of $\varepsilon_t$ are **mutually independent** (not necessarily i.i.d.), have a **non-zero cumulant** of order $r \geq 3$ and **finite moments up to order** $\tau$, where $\tau > r$ is even.

## (b) Non-Gaussian i.i.d.

Components of $\varepsilon_t$ are i., **identically,** d., and **non-Gaussian**.

## Theorem (Chan, Ho, Tong (2004, 2006))

*Under (a) or (b), the deep **parameters** in $(a(z), p(z)s(z)f(z), B, \Sigma)$ are **identifiable up to signed permutations of** $B$.*

# Outline

# Literature: Univariate

- Deconvolution using entropy methods: Wiggins (1978), Donoho (1981), Gassiat (1993)
- Andrews, Breidt, Davis, Lii, Rosenblatt (1982-2007+): Deconvolution, MLE for non-Gaussian, non-invertible ARMA, MLE for non-causal AR, rank-based estimation of all-pass models etc
  - Rosenblatt's 2000 book "Gaussian and Non-Gaussian Linear Time Series and Random Fields"
- Gouriéroux, Zakoian (2015, JTSA): On uniqueness of MA representations of heavy-tailed stationary processes
- Velasco, Lobato (2019, Annals): Non-causal and non-invertible ARMA using polyspectra

# Literature: Multivariate

- Chan, Ho, Tong (2004, 2006, Biometrika): Uniqueness of two-sided multivariate non-Gaussian moving average processes
- Lanne, Saikkonen (2013, ET): Non-causal VAR
- GMR = Gouriéroux, Monfort, Renne (2019, ReStud): Identification and Estimation in Non-Fundamental SVARMA Models
- Velasco (2020): Non-invertible SVARMA - Polyspectra approach
  - Similar problems as GMR but still a working paper
  - Multivariate version of the univariate objective function in VL19

# Funovits (2020, arxiv): Comment on GMR

- Bivariate VARMA(p,1)
- $b_0 = I_2$ excludes zeros at zero (corresponding to delays)
- Estimates $2^{n \cdot q}$ models via root flipping, similar to the approach presented at TU Dortmund in Jan 2017
  - WHF approach estimates $n \cdot q$ models
- Ignores deliberately complex-conjugated roots
  - Leaves real-valued parameter space
  - Non-trivial problem, see Scherrer, Funovits (2020, arxiv)

# Outline

# External Characteristics: Second Moment Information

## Autocovariance Function $\gamma(s)$ of $(y_t)$

$$\gamma(s) = Cov\left(y_{t+s} y_t'\right)$$

## Spectral density of $(y_t)$

$$f(z) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \gamma(s) z^s, \quad z = e^{-i\lambda}$$

$$= \frac{1}{2\pi} a(z)^{-1} b(z) b'\left(\frac{1}{z}\right) a^{-1'}\left(\frac{1}{z}\right), \quad z = e^{-i\lambda}$$

Same information as autocovariances $\gamma(s) = \mathbb{E}\left(y_{t+s} y_t'\right)$, but sometimes easier to manipulate

# Overview: From External to Internal Characteristics

1. From (full rank) rational **spectral density** $f(z)$ to **true spectral factor** $l(z)$ such that $f(z) = l(z)l'\left(\frac{1}{z}\right)$ holds.
2. From (normalized) spectral factor $k(z)$ to **polynomial matrix fraction description** $(a(z), b(z))$ such that $l(z) = a(z)^{-1}b(z)B\Sigma$ with
3. From **MA polynomial matrix** $(b(z), B, \Sigma)$ to **Wiener-Hopf factorization** $\left((p(z), s(z), f(z)), B, \Sigma\right)$

# Observationally Equivalent Spectral Factors

For a given (observed) rational spectral density, there are many spectral factors:

**MA(1): $y_t = \varepsilon_t + 3\varepsilon_{t-1}$ or $y_t = \tilde{\varepsilon}_t + \frac{1}{3}\tilde{\varepsilon}_{t-1}$?**

$$
\begin{aligned}
f(z) &= (1+3z)\sigma^2\left(1+\frac{3}{z}\right) = \\
&= \left(\frac{1}{3z}+1\right)3z\sigma^2\frac{3}{z}\left(\frac{z}{3}+1\right) \\
&= \left(1+\frac{z}{3}\right)\left[9\sigma^2\right]\left(1+\frac{1}{3z}\right)
\end{aligned}
$$

# Solution of Dynamic Identifiability Problem

## "Time-equivalent" of Orthogonal Matrices: All-Pass Filters



In the same way as $QQ' = I_n$ for orthogonal matrices $Q$ it holds for all-pass filters that

$$T(z)\,T'\left(\frac{1}{z}\right) = I_n$$

- $t(z)t\left(\frac{1}{z}\right) = \left[\frac{1-3z}{z-3}\right]\frac{1-3\frac{1}{z}}{\frac{1}{z}-3} = \left[\frac{1-3z}{z-3}\right]\frac{\frac{1}{z}(z-3)}{\frac{1}{z}(1-3z)} = 1$
- All-pass filters are "dynamic rotations"

# Solution of Dynamic Identifiability Problem

## Third Order Cumulants

### Cumulant Function

$$\gamma^{(3)}(r,s) = Cumu\left(y_{t+r}y_{t+s}y_t\right)$$

### Cumulant based bispectral density

$$f^{(3)}\left(e^{-i\lambda_1}, e^{-i\lambda_2}\right) = \left(\frac{1}{2\pi}\right)^2 \sum_{r,s=-\infty}^{\infty} \gamma^{(3)}(r,s) e^{-i(r\lambda_1 + s\lambda_2)}$$

# Solution of Dynamic Identifiability Problem

## Example: Blaschke factor $t(z) = \frac{1-3z}{z-3}$

### Spectral density of $y_t = t(z)\varepsilon_t$ is constant!

$$f\left(e^{-i\lambda}\right) = \frac{1-3e^{-i\lambda}}{e^{-i\lambda}-3}\frac{1-3e^{i\lambda}}{e^{i\lambda}-3} \equiv 1$$

### Bispectral density of $y_t = t(z)\varepsilon_t$

$$f^{(3)}\left(e^{-i\lambda_1}, e^{-i\lambda_2}\right) = \frac{1-3e^{-i\lambda_1}}{e^{-i\lambda_1}-3}\frac{1-3e^{-i\lambda_2}}{e^{-i\lambda_2}-3}\frac{1-3e^{i(\lambda_1+\lambda_2)}}{e^{i(\lambda_1+\lambda_2)}-3}$$

- Cumulant of $\varepsilon_t$: $\kappa_3 = 1 \neq 0$

# Generic Partial Indices

**Theorem (Gohberg, Krein (1960))**

*In an open and dense (aka. generic) set in the parameter space, the difference between the largest and the smallest partial index is smaller than two.*

- $\kappa_1 = \cdots = \kappa_k = \kappa + 1$ *and* $\kappa_{k+1} = \cdots = \kappa_n = \kappa$

- Weaker assumption than Velasco's root separation.
- Not mentioned in GMR.

# (Non-) Uniqueness of WHF

## Theorem (Clancey, Gohberg (1981))

*The partial indices of $b(z)$ are unique.*

- In the case $(\kappa, 0)$, the WHF can be made unique by requiring that $p(0) = I_n$.
- In the case $(\kappa, k)$, $k \neq 0$, the equivalence class of WHFs is described by unimodular matrices of the form

$$u(z) = u_0 + \begin{pmatrix} 0 & \tilde{u}_1 \\ 0 & 0 \end{pmatrix} z.$$

  - $\mathring{p}(z) = p(z)u(z)$, $\mathring{f}(z) = s(z)^{-1}u(z)^{-1}s(z)f(z)$.
    - transformation does not change the row degrees of $f(z)$ or $s(z)f(z)$.

## Theorem (Funovits 2020)

A unique representative among all tuples $(p(z), s(z), f(z))$ such that $b(z) = p(z)s(z)f(z)$ can be chosen and is of the described form.

# Outline

# Remarks

- Asymptotics are standard, similar to Rosenblatt (2000, Chapter 8)
- Main difficulty is identifiability
  - From identifiability it follows that information matrix is non-singular (main difficulty in "Lii, Rosenblatt 1996: MLE for non-Gaussian non-minimum phase ARMA sequences")
- Invertibility of $f_0$ is essential
  - $f_0 = I_n$ simplifies Jacobian
- Formulas are complicated in the multivariate case
  - Equality restrictions in the case $(\kappa, k)$

# R-Packages

- Building on two packages written by Wolfgang Scherrer and myself
  - **rationalmatrices**
    - Implements matrix polynomials, left- and right-matrix fraction description of rational matrices, state space representations of rational matrices, Hankel matrices as representations
    - Conversions between these (S3-) classes, Kronecker normal forms, etc
    - Matrix factorizations: Smith-form, Wiener-Hopf factorization, column reduction
  - **RLDM (Rational Linear Dynamic Models)**
    - Mainly estimation algorithms, optimization of different (structural) models
    - Simulations, visualization of IRF, FEVD, prediction
    - Templates for different realizations (Kronecker form for VARMA, state space system, etc)

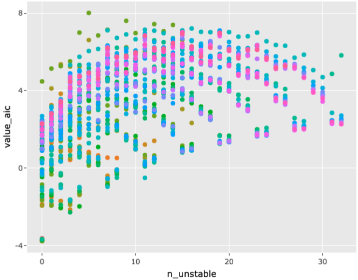
# Feasibility

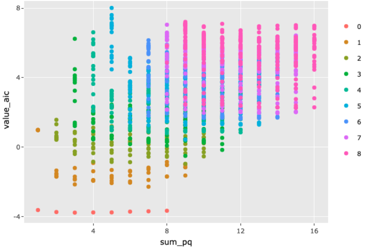
- Seems that the WHF parametrisation is the only feasible approach
  - Easy to check location of roots of $p(z)$ and $f(z)$
  - Easy to provide useful initial values
  - No root flipping required
  - Very costly to evaluate cumulant spectra (sample size $200 \Rightarrow 200^3$ frequency to integrate over)

# Outline

# Monetary Model + RER

- unemployment, FFR, CPI inflation, real exchange rate
- AR models clearly best, no unstable roots



# Outline

# Summary and Conclusion

- New feasible parametrisation for estimating andn analyzing non-invertible SVARMA
  - Exists on a topologically large set in the parameter space
  - Parametrises the number of MA zeros inside the unit circle
- Allows data-driven evaluation of DSGE models
- Makes “incredible restrictions” testable (and unnecessary to impose them apriori)