The Flexible Inverse Logit Model

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The Flexible Inverse Logit (FIL) model, a structural inverse demand model for products that are differentiated in a way that is both observed and unobserved by the modeller. The FIL model has three main attractive features: (i) it is easy to estimate by linear IV regression; (ii) it provides rich substitution patterns; (iii) it is consistent with utility maximization by heterogeneous consumers.

General Setting

J differentiated products ($j = 1, \dots, J$) and 1 outside good ($j = 0$).

- $\mathbf{x} = (x_1, \dots, x_J)$: vector of observed characteristics,
- $\boldsymbol{\xi} = (\xi_1, \dots, \xi_J)$: vector unobserved characteristics terms,
- $\mathbf{p} = (p_1, \dots, p_J)$: vector of prices,
- $\mathbf{s} = (s_1, \dots, s_J) \in \Delta_J^+$: vector of non-zero market shares.

Linear Index Restriction (Berry and Haile, 2014). Set $\mathbf{x} = \left(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}\right)$

- $\mathbf{x}^{(1)}$ will enter the (inverse) demand function through an index

$$\delta_j = \mathbf{x}_j^{(1)} \boldsymbol{\beta} - \alpha p_j + \xi_j, \quad j > 0, \quad \text{and} \quad \delta_0 = 0. \qquad (1)$$

- $\mathbf{x}^{(2)}$ will enter in an unrestricted way.

Flexible Inverse Logit Model

The FIL model is the inverse demand function $\boldsymbol{\sigma}^{-1} : \Delta_J^+ \to \mathbb{R}^J$

$$\sigma_j^{-1}(\mathbf{s}; \boldsymbol{\mu}) \equiv \ln\left(\frac{s_j}{1 - \sum_{k=1}^J s_k}\right) - \sum_{i \neq j} \mu_{ij} \ln\left(\frac{s_j}{s_i + s_j}\right) = \delta_j, \quad j > 0. \qquad (2)$$

Economic Restrictions. Imply that the FIL model (2) is invertible, i.e., defines a demand function (rather than a correspondence).

(R1) $\sum_{i \neq j} \mu_{ij} < 1$ for all $j > 0$,

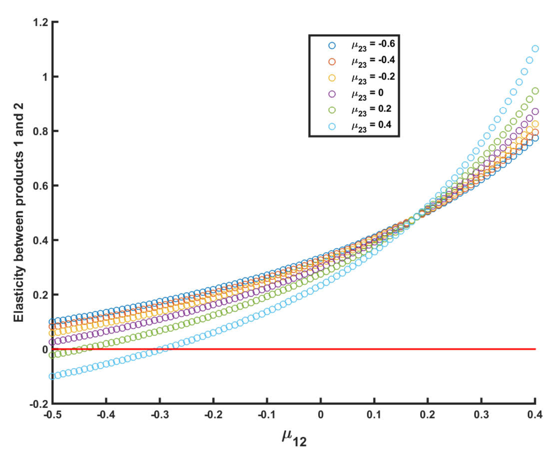
(R2) $\mu_{ij} = \mu_{ji}$ for all $i, j > 0$, $i \neq j$.

Motivations. The FIL model

1. allows deviations from IIA thanks to its parameters μ_{ij} (it reduces to the logit model when all $\mu_{ij} = 0$).
2. is a member of Fosgerau, Monardo and de Palma (2020)'s class of closed-form inverse demand models based on nesting (with a nest for each pair (i, j) of products and a nest for $j = 0$).
3. is specific instance of the large class of models of consumer heterogeneity studied by Allen and Rehbeck (2019): $\boldsymbol{\mu}$ parametrizes the distribution of preferences.

Substitution Patterns.

- The FIL model is flexible in the sense of Diewert (1974) in a large class of well-defined inverse demand functions.
- (R1) and (R2) imply that (i) σ_j is strictly increasing in p_j, and (ii) does not restrict products to be substitutes in demand.
- μ_{ij} governs the substitution between products i and j. Example: $J = 3$, $s_1 = 0.15$, $s_2 = 0.25$, $s_3 = 0.20$, $p_1 = p_2 = p_3 = 1$ and $\mu_{23} = 0.2$.



$\Rightarrow$ Higher μ_{12} implies a higher cross-price elasticity.

- Use the distance-metric of Pinkse, Slade and Brett (2002) to obtain substitution patterns that depend on $\mathbf{x}^{(2)}$ directly:
  - Closer products in $\mathbf{x}^{(2)}$ tends to be more substitutable.
  - Example: projection into $x \in [0, 1]$ with similarity measure $d_{ij} = 1 - |x_i - x_j|$: specify $\mu\left(d_{ij}; \gamma\right) = \sum_{k=0}^M \gamma_k \left(d_{ij}\right)^k$.

Estimation and Identification

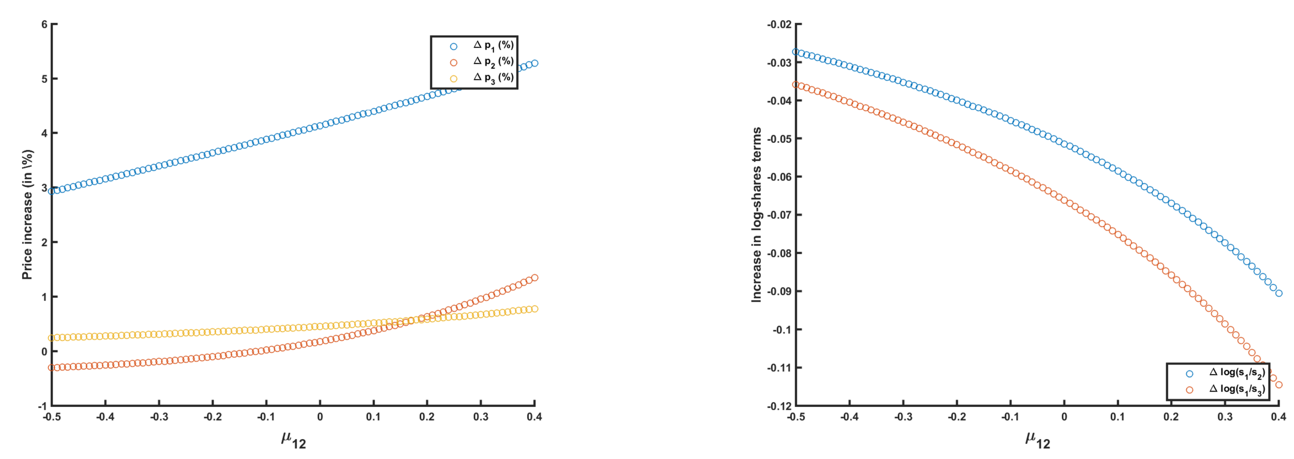
Estimation by Linear IV Regression with Aggregate Data.

$$\ln\left(\frac{s_j}{s_0}\right) = \mathbf{x}_j^{(1)} \boldsymbol{\beta} - \alpha p_j + \sum_{i \neq j} \mu_{ij} \ln\left(\frac{s_j}{s_i + s_j}\right) + \xi_j, \quad j > 0 \qquad (3)$$

- Assumption: prices and log-shares are endogenous (i.e., correlated with $\boldsymbol{\xi}$), but product characteristics are exogenous .

Identification = Identification of the parameters.

- Main identification assumption: existence of instruments $\mathbf{z}$.
- **Identification of $\boldsymbol{\delta}$.** Easy!
  - Higher market shares implies higher utility indexes: given $\boldsymbol{\mu}$, there is a one-to-one mapping between $\boldsymbol{\delta}$ and $\mathbf{s}$.
  - Parameters in $\boldsymbol{\delta}$: dealing with price endogeneity thanks to valid supply-side instruments (cost shifters and/or markup shifters).
- **Identification of μ_{ij}'s.** More tricky (as random coefficients).
  - Requires exogenous variation in the relative share of product j with respect to product i, $\ln(s_j/(s_i + s_j)) = -\ln(s_i/s_j + 1)$.
  - Need instruments that reveal about the substitution patterns: variables that generate exogenous variation in the choice set (including changes in prices) are good candidates.
  - Stylized example: $J = 3$, $s_1 = 0.15$, $s_2 = 0.25$, $s_3 = 0.20$, $p_1 = p_2 = p_3 = 1$ and $\mu_{13} = \mu_{23} = 0.2$.
    - ⋆ Variation in cost shifters: $\Delta c_1 = 10\%$.



$\Rightarrow$ Monotonic relationships: the way prices and relative shares change with Δc_1 drives the estimate of μ_{12}.

Comparison to BLP

Simulated DGP based on Armstrong(2016).

- Simulate a fully structural static model of demand and supply.
  - Demand: RCL model with utility linear in income and prices and with one normally distributed coefficient on an exogenous continuous characteristic $x^{(2)}$: $\beta_n \sim \mathcal{N}(3, 6)$.
  - Supply: price competition model with multi-product firms.

Results.

	Own-Elasticities	Cross-Elasticities	Markups	Merger ($\Delta p\%$)			New Product ($\Delta p\%$)
DGP with $J = 25$ and $T = 100$				All Firms	Merging Firms	Others	All Firms
True	-4.065	0.161	0.335	3.349	7.170	0.775	3.253
	[-4.095 ; -4.035]	[0.160 ; 0.163]	[0.329 ; 0.341]	[3.300 ; 3.400]	[7.082 ; 7.258]	[0.766 ; 0.784]	[3.200 ; 3.305]
FIL	-3.869	0.159	0.363	3.611	7.680	0.872	3.158
	[-4.437 ; -3.300]	[0.136 ; 0.182]	[0.303 ; 0.424]	[3.531 ; 3.691]	[7.528 ; 7.831]	[0.848 ; 0.896]	[3.042 ; 3.274]
BLP	-4.076	0.162	0.335	3.310	7.126	0.739	2.602
	[-4.471 ; -3.681]	[0.146 ; 0.178]	[0.302 ; 0.368]	[3.241 ; 3.377]	[7.009 ; 7.243]	[0.708 ; 0.770]	[2.542 ; 2.662]
DGP with $J = 50$ and $T = 200$							
True	-4.157	0.081	0.329	3.266	7.009	0.777	2.883
	[-4.173 ; -4.141]	[0.080 ; 0.082]	[0.325 ; 0.332]	[3.246 ; 3.286]	[6.976 ; 7.042]	[0.773 ; 0.780]	[2.860 ; 2.905]
FIL	-4.009	0.080	0.341	3.389	7.277	0.804	2.584
	[-4.287 ; -3.731]	[0.074 ; 0.085]	[0.318 ; 0.365]	[3.361 ; 3.417]	[7.222 ; 7.331]	[0.797 ; 0.810]	[2.563 ; 2.605]
BLP	-4.138	0.080	0.330	3.284	7.046	0.782	2.430
	[-4.333 ; -3.942]	[0.076 ; 0.084]	[0.314 ; 0.347]	[3.258 ; 3.309]	[6.999 ; 7.093]	[0.775 ; 0.789]	[2.404 ; 2.457]
DGP with $J = 100$ and $T = 20$							
True	-4.207	0.0401	0.325	3.207	6.890	0.774	2.602
	[-4.242 ; -4.173]	[0.040 ; 0.042]	[0.318 ; 0.333]	[3.151 ; 3.263]	[6.806 ; 6.973]	[0.764 ; 0.785]	[2.548 ; 2.656]
FIL	-4.3410	0.042	0.356	3.531	7.410	0.968	3.342
	[-4.889 ; -3.794]	[0.037 ; 0.048]	[0.308 ; 0.403]	[3.454 ; 3.607]	[7.278 ; 7.543]	[0.949 ; 0.987]	[3.272 ; 3.412]
BLP	-4.242	0.040	0.324	3.182	6.846	0.762	2.251
	[-4.705 ; -3.779]	[0.036 ; 0.045]	[0.288 ; 0.360]	[3.116 ; 3.248]	[6.733 ; 6.958]	[0.744 ; 0.779]	[2.189 ; 2.312]

Notes: Summary statistics across 100 Monte Carlo replications. For each replication, I compute the average. Numbers are averages over replications; numbers in brackets are the bounds of the 95% CI.

Conclusion. Simulations show that the FIL is able to match the substitution patterns of the RCL model pretty well and to obtain quite right predictions of a merger/new product's price effects.

Perspectives. The FIL model allows for complementarity in demand: (i) simulation with Genztkow (2007)'s DGP; (ii) revisit his work on the substitution between offline and online channels with an application to the hotel industry.