Estimating and Forecasting Long-Horizon Dollar Return Skewness

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Introduction

While huge theoretical literature studies how the **physical skewness of an asset's dollar return** (i.e., the ratio of later value to earlier value) affects investor behavior and asset prices, only few empirical studies convincingly test the main predictions of that literature. The reason for this gap is that **it is challenging to empirically estimate skewness with realistic amounts of data, especially over the long return horizons** the theoretical literature focuses on. Only recently, a handful of studies including Fama and French (2018) and Farago and Hjalmarsson (2019) have started addressing the estimation issue head on, proposing estimators relying on fewer data. Yet, those estimators often deliver biased estimates under realistic assumptions.

This paper develops a new parametric estimator of the skewness of an asset's return over an arbitrary horizon under the assumption that the asset's value can be modelled using a stochastic process from the affine stochastic volatility (ASV) model class.

The simulation exercise shows that our estimator is close to unbiased and efficient. In a further contrast to other estimators, it is also able to forecast skewness.

Theoretical Framework

We first define ASV models in a way consistent with Duffie et al. (2000, 2003), Chernov et al. (2003), etc. In our definition:

1. the joint conditional moment generating function (MGF)
$$M_t(u, w, h; V_t) \equiv \mathrm{E}[e^{r_{t,h}u + \langle V_{t+h}, w\rangle} | X_t, V_t],$$
where $t \geq 0$, $(u, w, h) \in \mathbb{C} \times \mathbb{C}^d \times \mathbb{R}_+$, X_t is the observed log-value process, V_t is the associated d-dimensional variance process, and $r_{t,h} \equiv X_{t+h} - X_t$;
2. the unconditional MGF also exists, and
$$M(u, w, h) \equiv \mathrm{E}[M_t(u, w, h; V_t)];$$
3. the third moments of X_t do not explode over any finite horizon h.

Denote dollar return as $R_{t,h}$. By definition, $R_{t,h} = e^{r_{t,h}}$, thus $\mathrm{E}_t[R^k_{t,h}] = \mathrm{E}_t[e^{kr_{t,h}}] = M_t(k, 0, h)$ and $\mathrm{E}[R^k_{t,h}] = M(k, 0, h)$. Let $M_t(u, h) \equiv M_t(u, 0, h)$ and $M(u, h) \equiv M(u, 0, h)$. Substitute them for the expected moments of dollar return in the skewness coefficient formula and simplify:

$$\mathrm{Skew}_t[R_{t,h}] = \frac{M_t(3, h) - 3M_t(1, h)M_t(2, h) + 2M_t(1, h)^3}{[M_t(2, h) - M_t(1, h)^2]^{3/2}};$$
$$\mathrm{Skew}[R_{t,h}] = \frac{M(3, h) - 3M(1, h)M(2, h) + 2M(1, h)^3}{[M(2, h) - M(1, h)^2]^{3/2}}.$$

Since the MGFs depend on the parameter values of the stochastic process, we can then obtain consistent estimates of the two skewness versions from consistent estimates of the parameters, assuming that the MGFs are smooth functions of the parameter vector.

Example: the Heston (1993) Model

The model assumes the following processes:

$$dX_t = (\mu - \frac{1}{2}V_t)dt + \sqrt{V_t}dW_t,$$
$$dV_t = \kappa(\alpha - V_t)dt + \xi\sqrt{V_t}dB_t,$$

where μ, κ, α, and ξ respectively denote drift, mean reversion, long-run variance, and volatility-of-volatility. W_t and B_t are Brownian motions with $[W, B]_t = \rho t$. When $\rho < 0$, leverage effect appears. The model is simple yet flexible, popular, and has closed-form MGFs available in papers like Bates (2006), Andersen (2008), etc.

The Estimator for Heston Parameters

We propose a simple GMM estimator for Heston parameters, where we match the first four central moments and two central cross-moments of dollar return with the corresponding MGFs and cross-MGFs. The two cross-moments respectively capture leverage effects and volatility clustering effects.

Innovatively, we use the **short-time-increment power-expansion approximations** of the MGFs and cross-MGFs (see Proposition 3 in paper) instead of their original forms to increase computational efficiency and improve parameter identification.

Simulation Setup

We simulate the Heston model with 10,000 replications of 10-year daily observations, using various parameter settings (e.g., $\mu = 0.10$, $\kappa = 3.00$, $\alpha = 0.09$, $\xi = 0.30$, and $\rho = -0.90$).

Apart from our skewness estimator, we also apply Fama and French's (2018) bootstrap estimator (F&F), Farago and Hjalmarsson's (2019) closed-form estimator (F&H), and the sample skewness estimator for comparison.

To see how model misspecification affect the performance of our estimator, we further simulate data from the double-Heston process. Parameters are set in a way that distinguishes the process as much as possible from a Heston process.

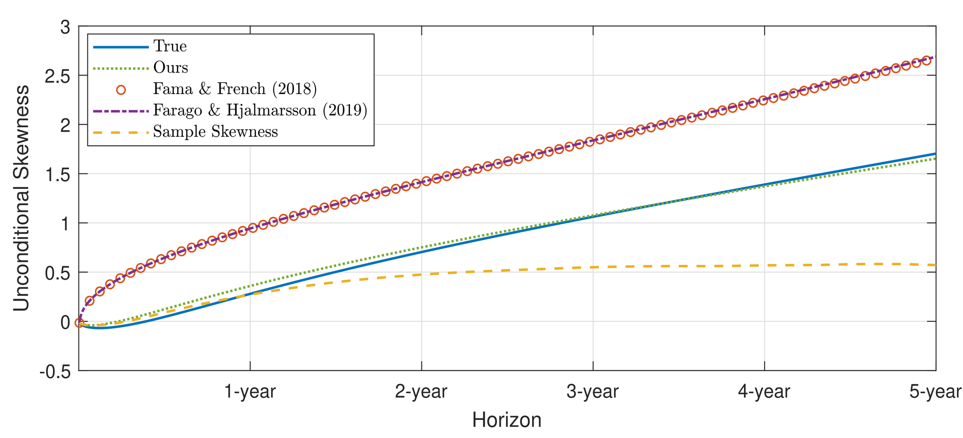
Results: Strong Leverage Effect

Our estimator always yields the smallest mean absolute percent error and MSE regardless of horizon, and the outperformance is economically large.

Horizon	True		Ours	F&F	F&H	Sample
1 Week	-0.040	Mean	-0.030	0.094	0.095	-0.028
		%|Bias|	**24%**	336%	338%	29%
		MSE	**0.001**	0.020	0.019	0.011
1 Month	-0.066	Mean	-0.041	0.244	0.245	-0.041
		%|Bias|	**38%**	471%	472%	38%
		MSE	**0.005**	0.097	0.097	0.035
1 Year	0.279	Mean	0.359	0.941	0.943	0.271
		%|Bias|	29%	237%	238%	**3%**
		MSE	**0.026**	0.444	0.444	0.154
3 Years	1.063	Mean	1.077	1.832	1.838	0.550
		%|Bias|	**1%**	72%	73%	48%
		MSE	**0.032**	0.629	0.619	0.501
5 Years	1.704	Mean	1.653	2.665	2.689	0.572
		%|Bias|	**3%**	56%	58%	66%
		MSE	**0.054**	1.095	1.027	1.611

Unconditional Skewness Estimation ($\rho = -0.9$)

Assuming that returns are i.i.d., F&F and F&H overshoot skewness over all horizons and fail to capture possible decline in skewness over shorter horizons. The sample skewness estimator seems to be reliable only for horizons less than 1-year.



Mean Unconditional Skewness Estimates ($\rho = -0.9$)

Results: Model Misspecification

Our estimator always gets the smallest MSE that comprehensively reflects bias and standard error of an estimate. Hence, realistic deviations between the asset value process assumed by our estimator and the true process only marginally affect our performance.

Horizon	True		Ours	F&F	F&H	Sample
1 Week	-0.018	Mean	-0.009	0.104	0.104	-0.011
		%|Bias|	48%	679%	680%	**40%**
		MSE	**0.002**	0.017	0.016	0.014
1 Month	-0.012	Mean	0.008	0.260	0.260	0.001
		%|Bias|	161%	2214%	2215%	**109%**
		MSE	**0.006**	0.075	0.075	0.045
1 Year	0.527	Mean	0.537	0.998	1.000	0.401
		%|Bias|	**2%**	89%	90%	24%
		MSE	**0.017**	0.228	0.227	0.194
3 Years	1.406	Mean	1.321	1.967	1.974	0.603
		%|Bias|	**6%**	40%	40%	57%
		MSE	**0.033**	0.361	0.343	0.888
5 Years	2.167	Mean	1.975	2.911	2.932	0.606
		%|Bias|	**9%**	34%	35%	72%
		MSE	**0.084**	0.808	0.654	2.750

Unconditional Skewness Estimation (Double-Heston)

Results: Forecasts

True conditional skewness varies over sample paths, but it is on average close to unconditional ones. The performance measures suggest that our estimates are generally close to conditional skewness.

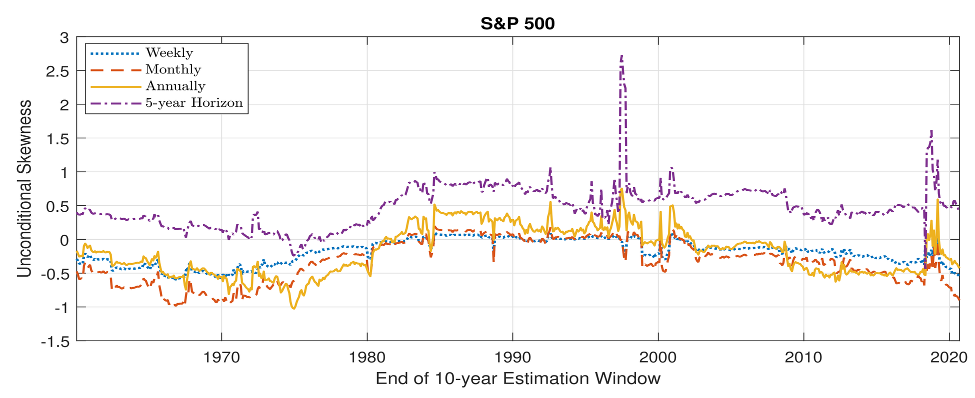
Horizon	True	Mean	%|Bias|	MSE
1 Year	0.266	0.369	39%	0.032
3 Years	1.059	1.086	3%	0.032
5 Years	1.702	1.662	2%	0.051
10 Years	3.372	3.116	8%	0.248

Our Conditional Skewness Estimation ($\rho = -0.9$)

Empirical Results: S&P 500

Based on its 1950-2020 daily returns and a rolling 10-year estimation window, our application suggests:

- The evolutions of the skewness estimates for different return horizons are highly correlated over time.
- Leverage effect is strong (weekly estimates often $>$ monthly estimates) and time-varying.
- The skewness of long-horizon dollar returns is probably much lower than suggested in recent works.



Estimated Trends of Unconditional Skewness

Conclusions

Assuming that asset values can be modelled by ASV models, we derive a novel parametric estimator of the skewness of dollar returns. The simulation exercise based on the Heston process shows that our estimator strongly outperforms the others in most parameter settings and even under model misspecification. Empirical application on stock indexes shows an important time-varying leverage effect in that asset class and refutes the idea that the skewness of long-horizon returns is usually too high to be useful.