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Dynamic Games and Rational-Expectations Models of Macroeconomic Policies

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Abstract We consider a linear-quadratic differential game with two decision makers which is interpreted as a model of the interactions between the government and the private sector. The open-loop Stackelberg equilibrium solution of this game is determined analytically. We then formulate a linear dynamic continuous-time model with rational expectations. We show that under some assumptions, the problem of determining optimal policies for a government with an economy given by the rational-expectations model is equivalent to the problem of determining the leader's open-loop Stackelberg equilibrium strategy for the differential game. Consequences are briefly discussed for the time inconsistency of optimal policies and for the problem of the non-uniqueness of the solutions of rational-expectations models.

Keywords Differential games · macroeconomics · Stackelberg games · rational expectations · stability · linear differential equations.

1 Introduction

After the discovery of the mathematical tools of modern optimal control theory (in particular, Bellman's or Isaacs's dynamic programming and Pontryagin's maximum principle), and after the great success of these techniques in control engineering, optimal control theory started to attract the attention of

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economists in the late 1960s and early 1970s. The first applications, mostly of a theoretical nature, took place in the areas of operations research and economic growth theory. To bring the ideas of dynamic optimization more down to earth, macroeconomists started to apply them to numerically specified models of an economy, mostly of the econometric type. One of the first works in this field was Pindyck's PhD dissertation ((Pindyck 1973)), which derived optimal policy measures for stabilizing the US economy. Later on, (Chow 1975, Chow 1981) and (Kendrick 1981), among others, extended this framework and initiated important new developments in the study of optimal macroeconomic policies. Hopes were high at that time that optimal control theory could provide policy makers with an excellent tool to achieve goals such as full employment, high and steady economic growth, price stability, external balance, a balanced budget, and much more. These attempts rested mainly on Keynesian economic models, which implied strong and predictable reactions by economic objective variables to discretionary changes in policy instruments (the policy maker's control variables).

This heyday of economic policy applications of optimal control theory ended with the oil price shocks and the resulting slowdown in economic growth from the mid-1970s onwards. Disappointment with the results of demand side macroeconomic policies based on simulations with econometric models and calculations of optimal policies led to a revival of monetarism and, in particular, to the emergence of new classical macroeconomics, which is very critical of discretionary policies. One of its core assumptions is the hypothesis of rational expectations. It means that private agents (households and companies) form their expectations about economic variables, including those relating to economic policies, in a forward-looking way, using all the information available at each point in time.

The consequences of the rational-expectations revolution, as it is sometimes called, on the theory of economic policy were considered profound: Under rational expectations, macroeconomic stabilization policies may become ineffective because private agents, who modify their behavior according to the expected policies, correctly anticipate their systematic elements. Moreover, according to the famous (Lucas 1976) critique, policy simulations and optimal control experiments with macroeconometric models may become meaningless because systematic policy changes in general alter the structure of the economic system, which results from the aggregation of forward-looking optimizing agents' behavior. Finally, as shown by (Kydland and Prescott 1977), optimal government policies in a rational-expectations environment may be time-inconsistent, thus providing policy makers with an incentive to deviate from the originally optimal time path of policy variables later on. Although most of these propositions from new classical macroeconomics require further assumptions in addition to that of rational expectations, often the conclusion is drawn that discretionary stabilization policies derived by optimum control methods are not feasible and should be replaced by "fixed rules", such as a constant money growth rule.

The rational-expectations hypothesis is now standard in mainstream macroeconomic models and theories. Nevertheless, as recent extensions of the theory of economic policy ((Acocella *et al.* 2013)) have shown, there is still some room for active government macroeconomic policy making. However, this requires explicit consideration of strategic interactions between the public and the private sector. If private agents anticipate future developments and, in particular, government policies, these policies will have to take private-sector agents' reactions into account. Therefore, a government designing its policy measures will have to follow different guidelines when confronted with a private sector holding rational expectations than in the case of a "passive" economic system. Dynamic game theory instead of optimum control theory is the appropriate tool to deal with the policy problem in a situation with rationally and strategically acting private agents.

Strategic interactions between the government and the aggregate private sector do not imply that they are on a par because the private sector is composed of a large number of "small" agents who cannot act together as easily as the government. For such a situation of asymmetry, a Stackelberg game can be an adequate model. Several authors (e.g., (Miller and Salmon 1985*b*, Cohen and Michel 1988, Petit 1990)) have already pointed out close connections between the Stackelberg equilibrium solutions for a dynamic game and the optimal policy of a government facing a private sector with rational expectations. Here we extend this idea to a more general class of policy problems. In particular, we consider a rather general linear-quadratic differential game with two decision makers, who can be viewed as the government and the aggregate of private agents. We determine analytically the open-loop Stackelberg equilibrium solution of that game and show that under some additional assumptions, the government's strategies are equivalent to the optimal policies of a government for the linear rational-expectations model in continuous time. By providing additional insights into the problem of the time-inconsistency of optimal government policies, heuristics in developing remedies for the time-inconsistency of optimal stabilization policies can be obtained from the correspondence between the dynamic Stackelberg game and the rational-expectations model.

2 A Differential Game

In this section, we consider a linear-quadratic differential game with two decision makers with the following characteristics: the objective functions are quadratic; the system is linear; the model is deterministic; there are no inequality restrictions on any variable; the coefficients of the system and the objective functions are time-invariant and common knowledge; no exogenous non-controllable variables are present; the objective functions exhibit an infinite time horizon and discounting, where both decision makers apply the same rate of discount.

The dynamic economic system is given by a system of first-order linear differential equations:

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t), \tag{1}$$

where $t \in [0, \infty)$ denotes time, $x(t) \in \mathbb{R}^n$ is the vector of state variables, $u_i(t) \in \mathbb{R}^{m_i}, i = 1, 2$, is the vector of control (instrument) variables of the i-th decision maker (player), $u(t) \equiv [u_1'(t)u_2'(t)]' \in \mathbb{R}^m, m = m_1+m_2$, A is an $(n \times n)$-matrix, and the $B_i, i = 1, 2$, are $(n \times m_i)$-matrices. The initial state $x(0) = x_0$ is assumed to be known. There are two output (observation) equations, defining the objective ("output") variables of the two decision makers:

$$y_i(t) = D_i x(t) + E_i u_i(t) + F_i u_j(t), i, j = 1, 2, i \neq j, \tag{2}$$

where $y_i(t) \in \mathbb{R}^{k_i}$ denotes the vector of objective variables of the i-th decision maker, D_i are $(k_i \times n)$-matrices, E_i are $(k_i \times m_i)$-matrices, and F_i are $(k_i \times m_j)$-matrices, $i, j = 1, 2, i \neq j$. E_i and F_i are assumed to be of full rank.

We assume quadratic objective functions for both players, which are cost functions to be minimized by decision maker $i, i = 1, 2$:

$$J_i = \int_0^\infty \exp(-rt)[(1/2)y_i'(t)W_i y_i(t) + w_i' y_i(t) + v_i]dt, \tag{3}$$

where $W_i, i = 1, 2$, are symmetric positive definite $(k_i \times k_i)$-matrices. If the W_i were not symmetric, similar results could be obtained by considering $(1/2)(W_i + W_i')$ instead of W_i. Moreover, $w_i \in \mathbb{R}^{k_i}$ and $r, v_i \in \mathbb{R}$; $r \geq 0$ is the common rate of discount. A well-known special case of (3) is an objective function penalizing deviations of state and own control variables from constant "ideal" values, but the formulation (3) is more general; for instance, it also covers the external effects of one decision maker's controls on those of the other.

The above game model may be interpreted in terms of the theory of economic policy in several ways. For instance, it may be a model of stabilization policies on a national level where, for instance, the government and the central bank are considered as players (e.g. (Neck and Dockner 1987)), or it may be a model of stabilization policies on an international level, where the governments of two countries may play against each other (e.g. (Buiter and Marston 1985, Plasmans *et al.* 2006)), or it may combine both like, for instance, in the context of a monetary union (e.g. (Plasmans *et al.* 2009, Neck and Blueschke 2016)). In the present context, we will interpret player 1 as the government and player 2 as the private sector (the aggregate of private agents) in a similar way to (Cohen and Michel 1988), where a discussion of the microfoundations of the private sector's behavior can also be found.

In order to determine a solution for our model, we have to specify the solution concept and the strategy spaces available to the players (the information pattern of the differential game). Symmetric feedback solutions, namely the no-memory feedback Nash equilibrium solution and no-memory Pareto-optimal solutions as well as Pareto-optimal non-cooperative equilibrium solutions for the same game were derived by (Dockner and Neck 1990). Here we consider an asymmetric mood of play, namely a Stackelberg equilibrium solution: Player 1 is the leader and has the power to declare his (her) strategy first and to enforce it upon player 2, the follower, who reacts rationally to the leader's strategy.

Moreover, we assume an open-loop information pattern for both players: At the initial time, each player chooses once and for all a time path as his (her) optimal policy. These time paths of the control variables can be interpreted as generalized "fixed rules"; such a choice requires unilateral pre-commitment by both decision makers. Thus, we determine the open-loop Stackelberg equilibrium solution of the differential game under consideration; cf. (Basar and Olsder 1995) or (Dockner *et al.* 2000) for the terminology used. For an empirical economic application of the Stackelberg dynamic-game concept, see, e.g., (Wirl 1991).

3 The Open-Loop Stackelberg Equilibrium Solution

We start with the optimum control problem for the follower (player 2): It consists in minimizing, with respect to the trajectory $\{u_2(t)\}$, J_2, subject to the system (1) with (2), regarding the leader's control trajectory $\{u_1(t)\}$ as given. The follower's current-value Hamiltonian is

$$\begin{aligned} H^2 &= (1/2)y_2'(t)W_2y_2(t) + w_2'y_2(t) + v_2 + \\ &+ \lambda_2'(t)[Ax(t) + B_1u_1(t) + B_2u_2(t)], \end{aligned} \tag{4}$$

where $\lambda_2(t) \in \mathbb{R}^n$ is the current-value costate variable of player 2, which is determined according to the costate equation

$$\dot{\lambda}_2(t) = r\lambda_2(t) - dH^2/dx(t). \tag{5}$$

Necessary conditions for the follower's optimization problem include minimization of H^2 over all $u_2(t)$ at each instant in time t, the system equation (1) with initial condition $x(0) = x_0$ and the adjoint equation (5). A sufficient transversality condition for the infinite-horizon problem requires

$$\lim_{t\to\infty} x'(t)\lambda_2(t)\exp(-rt) = 0. \tag{6}$$

Differentiating H^2 with respect to $u_2(t)$, we obtain

$$E_2'W_2D_2x(t) + E_2'W_2F_2u_1(t) + E_2'W_2E_2u_2(t) + E_2'w_2 + B_2'\lambda_2(t) = 0. \tag{7}$$

The sufficient second-order condition for a minimum of H^2,

$$E_2'W_2E_2 > 0, \tag{8}$$

is fulfilled as W_2 is positive definite. Under our assumptions, $E_2'W_2E_2$ has an inverse, and we obtain for the optimum (equilibrium) control of the follower, to be denoted by superscript S:

$$u_2^S(t) = -(E_2' W_2 E_2)^{-1}[E_2' W_2 D_2 x(t) + \\ + E_2' W_2 F_2 u_1(t) + E_2' w_2 + B_2' \lambda_2(t)], \tag{9}$$

which gives the reaction function of the follower to policies $u_1(t)$ announced by the leader. The adjoint equation is determined from (5) and (9) as

$$\begin{aligned}\dot{\lambda}_2(t) = & -D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 D_2 x(t) - \\ & - D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] W_2 F_2 u_1(t) + \\ & + [rI - A' + D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2'] \lambda_2(t) - \\ & - D_2'[I - W_2 E_2 (E_2' W_2 E_2)^{-1} E_2'] w_2, \end{aligned} \tag{10}$$

where I denotes an identity matrix of the appropriate dimension.

Next, consider the optimization problem of the leader. He (she) has to minimize, with respect to $\{u_1(t)\}$, J_1, subject to the constraints (from (1) and (9))

$$\begin{aligned}\dot{x}(t) = & [A - B_2 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2] x(t) + \\ & + [B_1 - B_2 (E_2' W_2 E_2)^{-1} E_2' W_2 F_2] u_1(t) - \\ & - B_2 (E_2' W_2 E_2)^{-1} B_2' \lambda_2(t) - \\ & - B_2 (E_2' W_2 E_2)^{-1} E_2' w_2 \end{aligned} \tag{11}$$

with initial condition $x(0) = x_0$, and (10) with transversality condition (6). The leader's objective variable becomes

$$\begin{aligned}y_1(t) = & [D_1 - F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 D_2] x(t) + \\ & + [E_1 - F_1 (E_2' W_2 E_2)^{-1} E_2' W_2 F_2] u_1(t) - \\ & - F_1 (E_2' W_2 E_2)^{-1} B_2' \lambda_2(t) - \\ & - F_1 (E_2' W_2 E_2)^{-1} E_2' w_2. \end{aligned} \tag{12}$$

Note that the costate variable $\lambda_2(t)$ of the follower becomes a state variable for the leader. The leader's current-value Hamiltonian is given by

$$\begin{aligned}H^1 = & (1/2) y_1'(t) W_1 y_1(t) + w_1' y_1(t) + v_1 + \\ & + \lambda_{11}'(t) \dot{x}(t) + \lambda_{12}'(t) \dot{\lambda}_2(t). \end{aligned} \tag{13}$$

The necessary conditions for optimality of the leader's strategy demand minimization of H^1 with respect to $u_1(t)$ for all t, together with the system and the adjoint equations:

$$\dot{x}(t) = \partial H^1 / \partial \lambda_{11}(t), \tag{14}$$

$$\dot{\lambda}_2(t) = \partial H^1/\partial\lambda_{12}(t), \tag{15}$$

$$\dot{\lambda}_{11}(t) = r\lambda_{11}(t) - \partial H^1/\partial x(t), \tag{16}$$

$$\dot{\lambda}_{12}(t) = r\lambda_{12}(t) - \partial H^1/\partial\lambda_2(t), \tag{17}$$

where the leader's current-value costate variables $\lambda_{11}(t)$ and $\lambda_{12}(t)$ correspond to $x(t)$ and $\lambda_2(t)$, respectively. As boundary conditions, we have transversality conditions for $\lambda_2(t)$, namely (6), and for $\lambda_{11}(t)$, namely

$$\lim_{t\to\infty} x'(t)\lambda_{11}(t)\exp(-rt) = 0, \tag{18}$$

and initial conditions for $x(t)$, namely $x(0) = x_0$, and for $\lambda_{12}(t)$, namely (cf. (Simaan and Cruz Jr. 1973); (Basar and Olsder 1995)):

$$\lambda_{12}(0) = 0. \tag{19}$$

The system of differential equations (14) – (17) constitutes a modified Hamiltonian system with $4n$ variables, namely $2n$ "states" $s(t) \equiv [x'(t)\lambda_2'(t)]'$ and $2n$ "costates" $\lambda(t) \equiv [\lambda_{11}'(t)\lambda_{12}'(t)]'$. Denoting the minimized current-value Hamiltonian of the leader by H^{1^*}, the Jacobian matrix of that system is given by

$$\begin{bmatrix} \frac{\partial}{\partial s(t)}\left[\frac{\partial H^{1^*}}{\partial\lambda(t)}\right] & \frac{\partial}{\partial\lambda(t)}\left[\frac{\partial H^{1^*}}{\partial\lambda(t)}\right] \\ -\frac{\partial}{\partial s(t)}\left[\frac{\partial H^{1^*}}{\partial s(t)}\right] & rI - \frac{\partial}{\partial\lambda(t)}\left[\frac{\partial H^{1^*}}{\partial s(t)}\right] \end{bmatrix}. \tag{20}$$

The following results are known about the stability of such a system (cf. Brock, 1987): For $r = 0$, if μ is an eigenvalue of (20), then $-\mu$ is also an eigenvalue of (20). For $r \neq 0$, the eigenvalues are symmetric around $r/2$. For $r \neq 0$, the system either exhibits saddle-point stability, or it is completely unstable. Not all eigenvalues of (20) can be purely imaginary ((Kurz 1968)). For $r = 0$, if the leader's problem is strictly concave in $s(t)$, then the system is saddle-point stable ((Levhari and Leviatan 1972)). Saddle-point stability means that exactly half ($2n$) of the eigenvalues have negative real parts and half of the eigenvalues have positive real parts. In this case, the transversality conditions imply that the initial values of the variables ($\lambda_{11}(0)$ and $\lambda_2(0)$, in our case) must be chosen so that the system converges to the steady state for the given initial values of the other variables ($x(0)$ and $\lambda_{12}(0)$, in our case); see also (Brock and Haurie 1976). As (Scheinkman 1976) has shown, the stable manifold of the Hamiltonian system is continuous in r at $r = 0$, which means that the global asymptotic stability of that system also holds for r near zero. Moreover, global asymptotic stability can be shown ((Brock and Scheinkman 1976)) for $r^2 < 4\alpha\beta$, where α and β are the smallest eigenvalues of $-(\partial/\partial s(t))[\partial H^{1^*}/\partial s(t)]$ and $(\partial/\partial\lambda(t))[\partial H^{1^*}/\partial\lambda(t)]$, respectively. In the sequel we will assume that the system (14) – (17) exhibits saddle-point stability;

although this need not be true for all possible parameter constellations, the above discussion shows that it holds at least for sufficiently small values of the rate of discount.

For notational convenience, we introduce the $(k_1 \times m_1)$-matrix G as

$$G \equiv E_1 - F_1(E_2' W_2 E_2)^{-1} E_2' W_2 F_2. \tag{21}$$

We assume that $G'W_1G$ has full rank (m_1). Then, differentiating H^1 with respect to $u_1(t)$ results in the following expression for the optimal (equilibrium) control of the leader:

$$\begin{aligned} u_1^S(t) = &-(G'W_1G)^{-1}G'W_1[D_1 - F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2]x(t) + \\ &+ (G'W_1G)^{-1}G'W_1F_1(E_2'W_2E_2)^{-1}B_2'\lambda_2(t) - \\ &- (G'W_1G)^{-1}[B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}B_2']\lambda_{11}(t) + \\ &+ (G'W_1G)^{-1}F_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2'D_2\lambda_{12}(t) - \\ &- (G'W_1G)^{-1}G'[w_1 - W_1F_1(E_2'W_2E_2)^{-1}E_2'w_2]. \end{aligned} \tag{22}$$

The second-order condition for a minimum of H^1 requires that $G'W_1G$ be positive definite. Substituting from (22) for $u_1(t)$ into (9), we obtain the equilibrium control of the follower:

$$\begin{aligned} u_2(t) = &(E_2'W_2E_2)^{-1}E_2'W_2\{F_2(G'W_1G)^{-1}G'W_1[D_1 - \\ &- F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2] - D_2\}x(t) - \\ &- (E_2'W_2E_2)^{-1}[I + E_2'W_2F_2(G'W_1G)^{-1}G'W_1F_1 \cdot \\ &\cdot (E_2'W_2E_2)^{-1}]B_2'\lambda_2(t) + \\ &+ (E_2'W_2E_2)^{-1}E_2'W_2F_2(G'W_1G)^{-1} \cdot \\ &\cdot [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}B_2']\lambda_{11}(t) - \\ &- (E_2'W_2E_2)^{-1}E_2'W_2F_2(G'W_1G)^{-1}F_2' \cdot \\ &\cdot [I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2D_2\lambda_{12}(t) + \\ &+ (E_2'W_2E_2)^{-1}E_2'W_2F_2(G'W_1G)^{-1}G'w_1 - \\ &- (E_2'W_2E_2)^{-1}[I + E_2'W_2F_2(G'W_1G)^{-1} \cdot \\ &\cdot G'W_1F_1(E_2'W_2E_2)^{-1}]E_2'w_2. \end{aligned} \tag{23}$$

For the canonical system (14) – (17), we get

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}_{12}(t) \\ \dot{\lambda}_{11}(t) \\ \dot{\lambda}_2(t) \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda_{12}(t) \\ \lambda_{11}(t) \\ \lambda_2(t) \end{bmatrix} + \begin{bmatrix} H_{15} & H_{16} \\ H_{25} & H_{26} \\ H_{35} & H_{36} \\ H_{45} & H_{46} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \tag{24}$$

where the $H_{ij}, i = 1, ..., 4, j = 1, ..., 6$, are given as follows:

$$H_{11} = A - B_2(E_2'W_2E_2)^{-1}E_2'W_2D_2 - [B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2] \cdot \\ \cdot \ (G'W_1G)^{-1}G'W_1[D_1 - F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2], \tag{25}$$

$$H_{12} = [B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2](G'W_1G)^{-1}F_2' \cdot \\ \cdot \ [I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2D_2, \tag{26}$$

$$H_{13} = -[B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2](G'W_1G)^{-1} \cdot \\ \cdot \ [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}B_2'], \tag{27}$$

$$H_{14} = -\{B_2 - [B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2] \cdot \\ \cdot \ (G'W_1G)^{-1}G'W_1F_1\}(E_2'W_2E_2)^{-1}B_2', \tag{28}$$

$$H_{15} = -[B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2](G'W_1G)^{-1}G', \tag{29}$$

$$H_{16} = -\{B_2 - [B_1 - B_2(E_2'W_2E_2)^{-1}E_2'W_2F_2] \cdot \\ \cdot \ (G'W_1G)^{-1}G'W_1F_1\}(E_2'W_2E_2)^{-1}E_2', \tag{30}$$

$$H_{21} = B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'] \cdot \\ \cdot \ W_1[D_1 - F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2], \tag{31}$$

$$H_{22} = A - B_2(E_2'W_2E_2)^{-1}\{I - F_1'W_1G(G'W_1G)^{-1}F_2' \cdot \\ \cdot \ [I - W_2E_2(E_2'W_2E_2)^{-1}E_2']\}W_2D_2, \tag{32}$$

$$H_{23} = B_2(E_2'W_2E_2)^{-1}\{I - F_1'W_1G(G'W_1G)^{-1} \cdot \\ \cdot \ [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}\}B_2', \tag{33}$$

$$H_{24} = -B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'] \cdot \\ \cdot \ W_1F_1(E_2'W_2E_2)^{-1}B_2', \tag{34}$$

$$H_{25} = B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'], \tag{35}$$

$$\begin{aligned} H_{26} = &-B_2(E_2'W_2E_2)^{-1}F_1'[I - W_1G(G'W_1G)^{-1}G'] \cdot \\ &\cdot\ W_1F_1(E_2'W_2E_2)^{-1}E_2', \end{aligned} \tag{36}$$

$$\begin{aligned} H_{31} = &-[D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1'][I - W_1G(G'W_1G)^{-1}G'] \cdot \\ &\cdot\ W_1[D_1 - F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2], \end{aligned} \tag{37}$$

$$\begin{aligned} H_{32} = &\{D_2' - [D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1']W_1G \cdot \\ &\cdot\ (G'W_1G)^{-1}F_2'\}[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2D_2, \end{aligned} \tag{38}$$

$$\begin{aligned} H_{33} = &\, rI - A' + D_2'W_2E_2(E_2'W_2E_2)^{-1}B_2' + \\ &+ [D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1']W_1G(G'W_1G)^{-1} \cdot \\ &\cdot\ [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}B_2'], \end{aligned} \tag{39}$$

$$\begin{aligned} H_{34} = &\,[D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1'] \cdot \\ &\cdot\ [I - W_1G(G'W_1G)^{-1}G']W_1F_1(E_2'W_2E_2)^{-1}B_2', \end{aligned} \tag{40}$$

$$\begin{aligned} H_{35} = &-[D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1'] \cdot \\ &\cdot\ [I - W_1G(G'W_1G)^{-1}G'], \end{aligned} \tag{41}$$

$$\begin{aligned} H_{36} = &\,[D_1' - D_2'W_2E_2(E_2'W_2E_2)^{-1}F_1'] \cdot \\ &\cdot\ [I - W_1G(G'W_1G)^{-1}G']W_1F_1(E_2'W_2E_2)^{-1}E_2', \end{aligned} \tag{42}$$

$$\begin{aligned} H_{41} = &-D_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2 \cdot \\ &\cdot\ \{D_2 - F_2(G'W_1G)^{-1}G'W_1[D_1 - F_1(E_2'W_2E_2)^{-1}E_2'W_2D_2]\}, \end{aligned} \tag{43}$$

$$\begin{aligned} H_{42} = &-D_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2F_2(G'W_1G)^{-1} \cdot \\ &\cdot\ F_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2D_2, \end{aligned} \tag{44}$$

$$H_{43} = D_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2F_2(G'W_1G)^{-1} \cdot \\ \cdot\ [B_1' - F_2'W_2E_2(E_2'W_2E_2)^{-1}B_2'], \tag{45}$$

$$H_{44} = rI - A' + D_2'\{W_2E_2 - [I - W_2E_2(E_2'W_2E_2)^{-1}E_2'] \cdot \\ \cdot\ W_2F_2(G'W_1G)^{-1}G'W_1F_1\}(E_2'W_2E_2)^{-1}B_2', \tag{46}$$

$$H_{45} = D_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2']W_2F_2(G'W_1G)^{-1}G', \tag{47}$$

$$H_{46} = -D_2'[I - W_2E_2(E_2'W_2E_2)^{-1}E_2'][I + W_2F_2 \cdot \\ \cdot\ (G'W_1G)^{-1}G'W_1F_1(E_2'W_2E_2)^{-1}E_2']. \tag{48}$$

We introduce the following notation:

$$H_1 \equiv \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix}, H_2 \equiv \begin{bmatrix} H_{15} & H_{16} \\ H_{25} & H_{26} \\ H_{35} & H_{36} \\ H_{45} & H_{46} \end{bmatrix}, \tag{49}$$

$$k(t) \equiv \begin{bmatrix} x(t) \\ \lambda_{12}(t) \\ \lambda_{11}(t) \\ \lambda_2(t) \end{bmatrix}, w \equiv \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{50}$$

Note that the matrix H_1 can be obtained from the Jacobian matrix (20) by a permutation of its elements. We assume that H_1 is nonsingular. Denoting steady-state values of $k(t)$ and its elements by an asterisk, $k^* = [x^{*'} \lambda_{12}^{*'} \lambda_{11}^{*'} \lambda_2^{*'}]'$, we have

$$0 = H_1k^* + H_2w \tag{51}$$

and hence

$$k^* = -H_1^{-1}H_2w. \tag{52}$$

The system (24) can be written as

$$\dot{k}(t) = H_1k(t) + H_2w = H_1[k(t) - k^*]. \tag{53}$$

As already mentioned, H_1 has the same properties as the modified Hamiltonian matrix (20). We assume that it exhibits saddle-point stability with different eigenvalues. Let us define by M the diagonal matrix of the eigenvalues of H_1:

$$M = \mathrm{diag}(\mu_1, ..., \mu_{4n}), \tag{54}$$

and

$$\mu^1 = \begin{bmatrix} \mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \mu_{2n} \end{bmatrix}, \mu^2 = \begin{bmatrix} \mu_{2n+1} \\ \cdot \\ \cdot \\ \cdot \\ \mu_{4n} \end{bmatrix}, \mu^1, \mu^2 \in \mathbb{C}^{2n}. \tag{55}$$

are the vectors of stable and unstable eigenvalues of H_1, respectively. Then we have

$$\begin{gathered} Re(\mu_i) = r/2 - Re(\xi_i) < 0, i = 1, ..., 2n, \\ Re(\mu_j) = r/2 + Re(\xi_i) > 0, j = 2n+1, ..., 4n, \end{gathered} \tag{56}$$

for some $\xi_1, ..., \xi_{2n} \in \mathbb{C}$. There exists a nonsingular $(4n \times 4n)$-matrix V such that

$$M = V^{-1} H_1 V, \tag{57}$$

where V is the matrix of column eigenvectors of H_1:

$$H_1 V = VM. \tag{58}$$

We introduce the $4n$-vector of canonical variables

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ \varphi_1(t) \\ \varphi_2(t) \end{bmatrix}, \tag{59}$$

which is defined by

$$k(t) - k^* \equiv Vz(t). \tag{60}$$

Then we have

$$\dot{k}(t) = H_1[k(t) - k^*] = V\dot{z}(t) \tag{61}$$

and hence

$$\dot{z}(t) = V^{-1} H_1 V z(t) = Mz(t). \tag{62}$$

The solution of this system is given by

$$z(t) = Sz(0), \tag{63}$$

$$S \equiv \mathrm{diag}[\exp(\mu_1 t), ..., \exp(\mu_{4n} t)]. \tag{64}$$

The initial conditions for $z(t)$ are determined by

$$k(0) - k^* = Vz(0). \tag{65}$$

Here we have $x(0) = x_0$ and $\lambda_{12}(0) = 0$, and $\lambda_{11}(0)$ and $\lambda_2(0)$ are chosen such that the system starts within its $2n$-dimensional stable manifold by setting

$$\varphi_1(0) = 0, \varphi_2(0) = 0. \tag{66}$$

The solution of the canonical system is given by

$$z(t) = SV^{-1}[k(0) - k^*], \tag{67}$$

$$\begin{aligned} k(t) = Vz(t) + k^* = VSV^{-1}z(0) + k^* = \\ = VSV^{-1}k(0) + [I - VSV^{-1}]k^*. \end{aligned} \tag{68}$$

This completes the analytical characterization of the open-loop Stackelberg equilibrium solution of our differential game.

4 A Dynamic Economic Rational Expectations Model

In formulating our dynamic game model, we have assumed that the economic system is given by the linear differential equation (1) with initial conditions $x(0) = x_0$. This means that we have assumed all n state variables contained in $x(t)$ to be predetermined, as is usual in dynamic systems theory. On the other hand, in economic rational-expectations models it is necessary to distinguish between predetermined and non-predetermined variables. Several possibilities exist for classifying economic variables among these two categories, but the most useful one is the definition suggested by (Buiter 1982). According to it, a variable is non-predetermined or forward-looking iff its current value is a function of current anticipations of future values of endogenous and/or exogenous (including policy instrument) variables. Such a variable can therefore respond instantaneously to changes in expectations due to news and shocks (including policy shocks). By contrast, a variable $x(t)$ is predetermined iff $x(t)$ is not a function of expectations, formed at time t, of future endogenous and/or exogenous variables, that is, if its current value is determined by the past. This distinction forms the basis for the formulation of the linear dynamic deterministic continuous-time rational-expectations model according to (Buiter 1984), which we adopt here with some minor modifications.

In this model, there exists a vector of predetermined state variables $x(t) \in \mathbb{R}^n$, with n initial conditions given by $x(0) = x_0$ as before. In addition, there is a vector of non-predetermined state variables, to be called $v(t) \in \mathbb{R}^{n_1}$, which can respond to changes in the information set conditioning expectations formed at time t. For $v(t)$, in a rational-expectations model there are n_1 boundary conditions given in the form of transversality conditions. Moreover, there exists a vector of exogenous or forcing variables $b(t) \in \mathbb{R}^i$. Then the linear deterministic first-order differential equations rational-expectations model with constant coefficients can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{v}^e(t) \end{bmatrix} = K \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + Lb(t) + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{69}$$

where K and L are constant $((n+n_1) \times (n+n_1))$- and $((n+n_1) \times l)$-matrices, respectively, $c_1 \in \mathbb{R}^n, c_2 \in \mathbb{R}^{n_1}$ are constant vectors, and superscript e denotes the value of the variable concerned expected by the private sector, given the information available at time t. This formulation covers most linear deterministic continuous-time rational-expectations models occurring in macroeconomics and open-economy macroeconomics, as shown by (Buiter 1984).

Several assumptions are usually made for this class of models:

(A) The information set $l(t)$ conditioning expectations at time t is given by $l(t) = \{x(s), v(s), b(s), s \leq t; K, L\}$, which implies $v^e(s) = v(s)$ for $s \leq t$, meaning perfect hindsight for $s < t$ and weak consistency for $s = t$.

(B) $l(t) \supseteq l(s)$ for $t > s$.

(C) $b^e(s)$ is a bounded function of s on $[t, \infty)$ and is continuous almost everywhere, which means that the exogenous variables do not explode too fast. Assumptions **(A)** – **(C)** imply that actual equal anticipated rates of change of predetermined variables, $\dot{x}(t) = \dot{x}^e(t)$; this is not necessarily true for the non-predetermined variables, whose expectations may be revised in the presence of shocks.

(D) K is diagonalizable by a similarity transformation:

$$K = U^{-1} \Lambda U \quad \text{or} \quad UKU^{-1} = \Lambda, \tag{70}$$

where U is an $((n + n_1) \times (n + n_1))$-matrix whose rows are the linearly independent left-eigenvectors of K, and $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_{n+n_1})$, where the $\lambda_i, i = 1, ..., n + n_1$ are the eigenvalues of K.

(E) K has n stable eigenvalues, i.e., characteristic roots with nonpositive real parts, and n_1 unstable eigenvalues, i.e., characteristic roots with positive real parts.

Based upon these assumptions, (Buiter 1984) shows how the dynamic rational-expectations model (69) can be solved analytically. This procedure consists of the following steps:

1. K, L, U, U^{-1}, and Λ are partitioned conformably with $x(t)$ and $v(t)$; for instance:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \tag{71}$$

   with K_{11} an $(n \times n)$-, K_{12} an $(n \times n_1)$-, K_{21} an $(n_1 \times n)$-, K_{22} an $(n_1 \times n_1)$-, L_1 an $(n \times l)$-, and L_2 an $(n_1 \times l)$-matrix.

2. Canonical variables $p(t) \in \mathbb{R}^n$, $q(t) \in \mathbb{R}^{n_1}$ are defined by

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \equiv U \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}. \tag{72}$$

3. $\dot{q}^e(t)$ is expressed as a linear function of $q^e(t)$ and $b^e(t)$, and $\dot{q}^e(s)$ as a linear function of $q^e(s)$ and $b^e(s)$ for $s > t$.
4. The forward-looking solution for $q^e(s)$ for $s \geq t$ is determined by integrating the linear differential equations obtained in step 3. This can be done only if a transversality condition is imposed, which constrains the initial values of the non-predetermined variables to lie on the stable manifold of the system. Hence, n_1 boundary conditions are required for the convergence of the system. In most macroeconomic models, these transversality conditions are introduced in an ad-hoc way. For instance, when the non-predetermined variables are asset prices determined in efficient markets, then it is argued that optimal intertemporal speculation rules out anticipated future jumps in these variables. However, if these non-predetermined variables can be regarded as costates of Hamiltonian dynamics, as we shall argue below, then the transversality conditions can be justified as characterizing an optimal intertemporal plan in a model with an infinitely-lived private sector.
5. Weak consistency implies $q(t) = q^e(t)$. Then the solution for $v(t)$ can be obtained from that of $q(t)$. Here the current values of non-predetermined variables depend on the current values of the predetermined variables $x(t)$ and on the current anticipations (rational expectations) of the future values of the exogenous variables $b(t)$.
6. Finally, a backward-looking solution can be obtained for the predetermined variables $x(t)$ with initial conditions $x(0) = x_0$. The values of these variables at time t depend on the initial conditions x_0, on the actual values of the exogenous variables $b(s)$ between $s = 0$ and $s = t$, and on the rational expectations, formed at each $s \in [0, t]$, of all values of the exogenous variables beyond s.
7. Buiter also suggests some modifications for cases where assumption **(E)** above is not satisfied. If there are more stable eigenvalues than predetermined variables, then the transversality conditions no longer suffice to ensure a unique solution. However, additional linear boundary conditions may guarantee uniqueness, such as linear restrictions on the state variables at $t = 0$, or linear restrictions on the state variables across initial and future times. On the other hand, if there are fewer stable eigenvalues than predetermined variables, no convergent solution exists for arbitrary initial values of the predetermined variables.

Consider now the problem of a government designing optimal stabilization policies over an infinite time horizon, faced with a dynamic rational-expectations economic system of the form (69). With (71), (69) can be written as:

$$\dot{x}(t) = K_{11}x(t) + K_{12}v(t) + L_1 b(t) + c_1, \tag{73}$$

$$\dot{v}^e(t) = K_{21}x(t) + K_{22}v(t) + L_2 b(t) + c_2, \tag{74}$$

with initial conditions $x(0) = x_0$ for the predetermined variables and transversality conditions for the non-predetermined variables $v(t)$. We assume assumptions **(A)** $-$ **(E)** above to hold; especially as there are as many stable eigenvalues as predetermined variables. Then, from (73), (74) and (10), (11), it is easy to see that the rational-expectations state system (69) is equivalent to the state system of the leader in the open-loop Stackelberg game of the previous section if the following additional assumptions hold:

(F) $n = n_1$, that is, there is exactly the same number of predetermined and non-predetermined variables.

(G) The matrices K and L and the vectors c_1 and c_2 are of the following form:

$$K_{11} = A - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 D_2, \tag{75}$$

$$K_{12} = -B_2(E_2' W_2 E_2)^{-1} B_2', \tag{76}$$

$$K_{21} = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] W_2 D_2, \tag{77}$$

$$K_{22} = rI - A' + D_2' W_2 E_2 (E_2' W_2 E_2)^{-1} B_2', \tag{78}$$

$$L_1 = B_1 - B_2(E_2' W_2 E_2)^{-1} E_2' W_2 F_2, \tag{79}$$

$$L_2 = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] W_2 F_2, \tag{80}$$

$$c_1 = -B_2(E_2' W_2 E_2)^{-1} E_2' w_2, \tag{81}$$

$$c_2 = -D_2'[I - W_2 E_2(E_2' W_2 E_2)^{-1} E_2'] w_2. \tag{82}$$

(H) The exogenous variables $b(t)$ are policy instruments of the government, i.e., $b(t) = u_1(t)$ with $l = m_1$, and there are no further exogenous influences in the rational-expectations model. This assumption could be relaxed if, in the dynamic game model, non-controlled exogenous variables were included.

(I) The non-predetermined rational-expectations variables $v(t)$ of the private sector are the optimum costate variables $\lambda_2(t)$ of the follower; their transversality condition is given by (6).

(J) The objective function of the government is J_1 from (3), with the objective variable $y_1(t)$ defined as in (12), i.e., as a linear function of all (predetermined and non-predetermined) state variables $x(t)$ and $v(t)$ and of the government's instrument variable $b(t)$.

Since in the dynamic game model the state variable $x(t)$ is predetermined and the variable $\lambda_2(t)$, which is the costate variable of the follower but a state variable from the point of view of the leader, is forward-looking, the problem of the leader can be regarded as analogous to an optimization problem with a dynamic system where agents have rational expectations. Under the assumptions **(F)** − **(J)**, optimal economic policies for a single decision maker (the government) with an economic system characterized by rational expectations are equivalent to the policies for the leader within an open-loop Stackelberg equilibrium solution.

It must be noted that the open-loop Stackelberg equilibrium solution interpretation refers to a special case of a rational-expectations system. Among the assumptions **(F)** − **(J)**, **(F)** seems to be the most restrictive one because it implies the uniqueness of the solution while non-uniqueness is a generic property of dynamic rational-expectations models. Several selection criteria for choosing among the multiple solutions of these models are proposed in the literature, such as using an infinite-horizon stochastic control problem whose unique stationary solution provides a solution to the linear rational expectations model ((Basar 1989)), the minimal state variable (MSV) criterion by (McCallum 1983, McCallum 1999), the expectational stability (E-stability) criterion by (Evans 1986), or the finite-horizon criterion by (Driskill 2002); see (Driskill 2006) for a review of these and further criteria. In terms of the government optimization problem vis-à-vis a private sector with rational expectations, this leads to the topic of the stabilizability of an unstable uncontrolled dynamic system; see (Acocella *et al.* 2013) (chapter 11) for a development along these lines for the discrete-time case.

The other assumptions are less problematic. Admittedly, assumption **(G)** implies a special structure of the system matrices involved; for instance, K_{12} and K_{21} have to be symmetric. But note that the matrices on the right-hand sides of (75) – (82) come from the dynamic system (1) and from the objective function J_2, and the objective variables $y_2(t)$ only. They do not depend on the objectives of the leader (the government), but only on the predetermined system and on the objectives of the follower (the private sector), i.e., on the non-predetermined system. By choosing these matrices and the objective variables of the leader properly, a large number of rational-expectations models fulfilling **(F)** can be converted to the open-loop Stackelberg formulation. The equivalence also provides a justification for the transversality condition in the rational-expectations model because the condition that $\lambda_2(t)$ must converge for a stable open-loop Stackelberg equilibrium to exist corresponds to the convergence of the non-predetermined variable $v(t)$ in the rational-expectations model.

For the class of rational-expectations models where the equivalence applies, the solution procedure for the open-loop Stackelberg equilibrium solution given in the previous section can be directly applied to determine the optimal stabilization policies. These policies are time-inconsistent, as becomes clear from the initial condition (19) for the costate variable $\lambda_{12}(t)$; thus they require pre-commitment and credibility of the government (cf. (Miller and

Salmon 1985*a*, Dockner and Neck 2008)). The game-theoretic interpretation of the problem of optimal stabilization policies in the presence of rational expectations may also be helpful to suggest remedies for the time-inconsistency of such policies. If pre-commitment is not feasible, for instance due to a lack of credibility of the government, other equilibrium solution concepts for the same dynamic game may be applied, such as the solution proposed by (Cohen and Michel 1988) or the feedback Stackelberg equilibrium solution ((Dockner and Neck 1990)). Thus the equivalence between rational-expectations and dynamic game models may also become useful for obtaining further theoretical insights into the interactions between a government and a sophisticated private sector.

5 Conclusions

We formulated a linear-quadratic differential game with two decision makers meant to model the interactions between the government and the private sector. The open-loop Stackelberg equilibrium solution of this game was characterized analytically. Then, a linear dynamic continuous-time model with rational expectations was presented. We showed that under some assumptions, the problem of determining optimal policies for a government confronted with an economy given by the rational-expectations model is equivalent to the problem of determining the leader's open-loop Stackelberg equilibrium strategy for the differential game under consideration. Consequences for the time inconsistency of optimal policies were briefly discussed. A major problem for the correspondence between the Stackelberg equilibrium solution of the game and the rational-expectations model concerns the well-known problem of the generic non-uniqueness of solutions to rational-expectations models.

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