Electoral Campaign Attacks: Theory and Evidence

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This paper studies the determinants of electoral campaign attacks. We first propose a model to examine the main factors that influence candidates' decisions to attack. Our theoretical analysis yields a number of predictions which we test using information from "right of reply" lawsuits filed in Brazil. Our empirical analysis exploits a regression discontinuity design based on virtual ties between 2nd and 3rd place candidates to show that candidates with an electoral advantage are more likely to receive an attack. We then exploit another discontinuity to show that the patterns of campaign attacks differ significantly under single and dual ballot plurality. (JEL D72, D79, C29)

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1 Introduction

Electoral campaigns are at the core of democracy. Campaigns matter because they provide voters with an opportunity to obtain information about candidates, their proposals, and past performances. The acquisition of information is essential for well-informed decision-making and has a direct impact on policy outcomes, political selection, and accountability (Dewan and Shepsle, 2011; Ashworth, 2012). The campaign strategies adopted by candidates are, therefore, crucial since they affect both the content and quality of the information transmitted to voters. However, while campaigns may facilitate access to true and relevant information, they are often carried out with the deliberate intent to distort and mislead, often focusing on personal and defamatory attacks.[1]

"Dirty campaigning" and "mudslinging" are by no means new phenomena in politics, but their potential negative impact has increased significantly in recent years with the advent of social media and the ability of campaigns to target specific constituencies and reach ever larger audiences (Allcott and Gentzkow, 2017). Indeed, there is an increasing concern that campaign attacks and the spread of misinformation may lead to suboptimal policies, voter demobilization, mistrust in politics and, ultimately, pose serious threats to democracy (Ansolabehere et al., 1994; Hochschild and Einstein, 2015; Grossman and Helpman, 2020). However, in spite of the importance of these issues, a more systematic understanding of the main political and institutional factors that influence the candidates' decisions to "go dirty" is still needed.

This paper studies the main determinants of electoral campaign attacks, both theoretically and empirically. We first propose a model of electoral contests with "impressionable" voters (Baron, 1994; Grossman and Helpman, 1996), where the probability of winning an election depends on the candidates' initial levels of political support plus their binary decisions to attack or not their opponents. A campaign attack is assumed to be costly and to cause a fraction of the targeted candidate's support to switch to her rivals. We provide a general characterization of the equilibrium in races with two and three candidates. Overall, under single ballot plurality, we show that the 2nd place candidate is always the most aggressive candidate, while candidates with an electoral advantage are relatively more likely to receive an attack.

Our model also yields the prediction that, in three-candidate races, the two front-runners become less aggressive towards each other when the support for the 3rd place candidate increases. Intuitively, a campaign attack benefits not only the attacking candi-

[1] A recent example of how the incentives to distort and attack may dominate an entire campaign comes from the 2020 US presidential elections. The first Biden-Trump debate provides a concrete case in point. According to CNN, the debate was "*rancorous and chaotic (...) full of insults, slashing interruptions, and callous attacks*".

date but also all other opponents of the targeted candidate. Our analysis shows that the first two candidates are particularly concerned about this spillover effect when the 3rd candidate is strong, which makes them less willing to attack each other in this case. Next, we show that the pattern of campaign attacks differs in significant ways under single and dual ballot plurality systems. Specifically, focusing on the behavior of candidates in the first round of elections, we show that the 3rd place candidate is always the most aggressive under a dual ballot system. Moreover, we also show that campaign attacks between 2nd and 3rd place candidates become more likely to occur and we derive conditions under which all candidates become more aggressive under dual ballot plurality.

Empirical studies on campaign strategies are usually made difficult by the lack of available data about how campaigns are actually carried out.[2] To overcome these difficulties, we collect detailed information contained in all "right of reply" lawsuits filed in Brazil (with the exception of three states) during the 2012 and 2016 municipal elections. The Brazilian electoral legislation protects candidates against slanderous, defamatory, and false accusations, granting the victim of an attack the right to respond to the offense at the offender's cost. "Right of reply" lawsuits are simple, inexpensive, and must be decided by the local electoral judge within 72 hours. Based on information retrieved from these lawsuits, we construct a unique dataset of 69,252 ordered pairs of candidates containing the precise directions of campaign attacks in 10,461 distinct electoral races.

We begin our empirical analysis by examining the general patterns of campaign attacks in Brazilian municipal elections. Focusing on electoral races with two and three effective candidates held under single ballot plurality, and controlling for a rich set of candidates' characteristics and both municipality and election-year fixed effects, we find a pattern of attacks that closely matches our main theoretical predictions. Specifically, we show that the 2nd place candidate is always the most aggressive candidate. Moreover, in three-candidate races, we find that candidates are always more likely to target their highest-ranked opponent, with the front-runner being the most attack. Finally, we show that an increase in the electoral strength of the 3rd place candidate significantly reduces the likelihood of an attack between the two front-runners.

Next, we exploit two different research designs to investigate how certain specific aspects of the political and institutional environment affect campaign attacks. A robust prediction of our model is that candidates with an electoral advantage are more likely to receive an attack. To test this hypothesis, we exploit quasi-experimental variation in electoral support arising from virtual ties between 2nd and 3rd place candidates. Our

[2]According to Lau and Rovner (2009), "*the most fundamental problem in the study of political campaigns involves data: the lack, until very recently, of any good evidence on exactly what candidates actually do when they are running for office.*"

approach follows Anagol and Fujiwara (2016) who used data from municipal elections in Brazil, India, and Canada to show that close runner-ups are substantially more likely to run in and win the subsequent elections. Importantly, they provide ample evidence suggesting that these results come from simply being labeled "the runner-up". We exploit a similar regression discontinuity design to show that close runner-ups are about 2 percentage points more likely to receive an attack in the next elections, which corresponds to a striking 160% increase relatively to the 3rd place candidates' mean. Moreover, following a procedure proposed by Lee (2009) and adapted by Anagol and Fujiwara (2016), we show that selection into candidacy alone is unlikely to explain these results.

Finally, we investigate whether the pattern of campaign attacks differs under single and dual ballot plurality systems. To do so, we exploit quasi-experimental variation arising from the fact that in Brazil municipalities with less than $200,000$ registered voters must use single ballot plurality, while those above this threshold must use dual ballot plurality. Consistently with the predictions of our model, we find that 3rd place candidates become significantly more aggressive under dual ballot plurality. Moreover, we show that the frequency of attacks between 2nd and 3rd place candidates increases substantially and we find suggestive evidence that campaigns become generally more aggressive under dual ballot plurality.

Our paper contributes to the political economy and institutional design literatures in several ways. First, our work relates to a theoretical literature on negative campaigning in elections (Skaperdas and Grofman, 1995; Harrington and Hess, 1996) and sabotage in contests (Lazear, 1989; Konrad, 2000; Chen, 2003).[3] While these papers are primarily interested in examining the amount of effort allocated between positive and negative activities, our analysis focuses on the candidates' binary decisions to attack or not each of their opponents.[4] Our model turns out to be very tractable under certain conditions. In particular, we are able to provide a detailed characterization of the equilibrium for any distribution of initial electoral support in races with two and three candidates. Moreover, we derive novel comparative static results showing how the pattern of campaign attacks varies with the competitiveness of races and under different electoral systems.

We also contribute to a large empirical literature on the determinants of the decision to "go negative" (Theilmann and Wilhite, 1998; Kahn and Kenney, 1999; Lau and Rovner, 2009; Dowling and Krupnikov, 2016). In particular, we take advantage of a

[3]Contrarily to negative campaigning, campaign attacks are often illegitimate and illegal and, as such, tend to be "extreme" events. Jamieson et al. (2000) and Lau and Rovner (2009) emphasize the distinction between "negative" and "dirty" campaigning noting that, while most political analysts condemn baseless attacks, many view legitimate criticism as essential for democracy.

[4]In doing so, our approach is related to papers that study network patterns in the presence of enmity and antagonism, particularly Hiller (2017).

unique feature of the Brazilian electoral legislation, which allows us to use detailed information contained in "right of reply" lawsuits to construct an objective measure of campaign attacks. Importantly, we are able to systematically identify the precise direction of attacks in a large number of electoral races held across the country. Our study establishes novel and robust stylized facts about the patterns of campaign attacks in two and three-candidate races.

More specifically, we contribute to a literature on negative campaigning in multi-candidate elections (Hansen and Pedersen, 2008; Elmelund-Praestekaer, 2008). In particular, Ghandi et al. (2016) showed that electoral races with three or more candidates are associated with fewer negative ads than two-candidate races. Moreover, in a large field experiment, Galasso et al. (2020) found causal evidence for the existence of a positive spillover effect on the third main candidate (neither the target nor the attacker) arising from negative campaigning. Our analysis complements these studies by showing that an increase in the strength of the 3rd place candidate reduces the frequency of attacks between the two front-runners.

Our paper is also related to a strand of the literature which studies sabotage in contests with heterogeneous agents. A central finding in this literature is that "abler" contestants are expected to receive more attacks (Skaperdas and Grofman, 1995; Chen, 2003; Münster, 2007; Chowdhury and Gürtler, 2015). This prediction has been corroborated by a number of experimental studies (Harbring et al., 2007; Gürtler et al., 2013; Charness et al., 2014) and observational studies using field data from sports (Balafoutas et al., 2012; Deutscher et al., 2013). To the best of our knowledge, our paper is the first to provide quasi-experimental evidence on the effect of an electoral advantage on the likelihood of receiving a campaign attack exploiting virtual ties between 2nd and 3rd place candidates. In doing so, we provide a novel application of the approach proposed by Anagol and Fujiwara (2016). More generally, our paper also relates to the literature on rank-based decision-making in politics (Folke et al., 2016; Meriläinen and Tukiainen, 2018; Fujiwara and Sanz, 2020; Pons and Tricaud, 2020).

Finally, we also contribute to a literature which examines the properties of runoff electoral systems (Duverger, 1954; Fujiwara, 2011; Bouton, 2013; Pons and Tricaud, 2018; Bouton et al., 2019). While most papers focus on the effects of single versus dual ballot plurality systems on the behavior of voters, Bordignon et al. (2016) showed that electoral rules also affect the strategies of parties. In particular, they found that runoff systems allow moderate candidates to run alone – without having to form coalitions with extreme parties – leading to more moderate policies in equilibrium. Our paper adds to this literature by showing that campaign strategies also adjust to the electoral rule. In particular, our findings suggest that politicians, particularly 3rd place candidates, may have an incentive to campaign harder and more aggressively under dual ballot plurality. Our results differ from the conventional view that the two-party majority system adopted in the US creates the most incentive for negative campaigning. Our theoretical analysis highlights the fact that certain characteristics of the races, such as the degree of competition, interact with the electoral rule to shape the incentives to attack. In doing so, our analysis also contributes to the literature on constitutional design (Aghion et al., 2004; Persson and Tabellini, 2005).

2 Model

This section proposes a theoretical framework to study the incentives behind campaign attacks in electoral races with two and three candidates.

2.1 Two-Candidate Races

Setup. Consider an electoral race with two candidates $i \in \{1, 2\}$, each with initial support $s_i^o \in \mathbb{R}_+$ with $s_1^o > s_2^o$. The electoral support of a candidate can be interpreted as a measure of her political strength and is assumed to be common knowledge.[5] Candidates decide simultaneously whether to attack or not each other, with $a_i \in \{0, 1\}$ representing i's binary decision to attack. For convenience, let $n_i = a_{-i}$ indicate whether candidate i received or not an attack.

We assume that a campaign attack allows a candidate to "steal" a fraction $\phi \in (0, 1)$ of her opponent's initial support. Given both players' decisions, the final support of candidate i is given by:

$$x_i(n_i, n_j) = (1 - \phi n_i)\, s_i^o + n_j \phi s_j^o + \epsilon_i \tag{1}$$

where ϵ_i is an iid shock with Type I Extreme Value distribution which is realized after the players' decisions have been made and captures all uncertainty associated with the electoral process in a reduced form fashion.[6] We define $s_i(n_i, n_j) := (1 - \phi n_i)\, s_i^o + n_j \phi s_j^o$.

Elections are held by simple majority and the candidate with the largest final support

[5]Our analysis assumes that the initial support of candidates is exogenously given at the moment when they make their decisions to attack or not. We think of this initial support as being determined in a previous stage of the electoral competition game, where it can be influenced by the choice of platforms, campaign expenditures, candidates' characteristics, among other factors. For simplicity, we do not model these various potential factors explicitly.

[6]In Appendix B.2, we show that our results are robust to considering a version of the model where campaign attacks lead to the demobilization of voters of both the attacked and attacking candidates. In particular, our main conclusions remain unchanged even if an attack serves only to demobilize voters of rival candidates.

wins. Following McFadden (1974), the probability that candidate i wins the election is:

$$p_i(n_i, n_j) = \frac{\exp(s_i(n_i, n_j))}{\exp(s_i(n_i, n_j)) + \exp(s_j(n_i, n_j))} \tag{2}$$

This particular functional form is often referred to as the Logit contest success function (CSF).[7]

The cost of an attack is given by a constant $c \in \mathbb{R}_+$, which is assumed to be common knowledge and captures all expenses associated with undertaking a campaign attack, including those related to the ensuing litigation. We suppose that candidates seek to maximize their probability of winning net of attacking costs:

$$u_i(a_i, a_j) = p_i(a_j, a_i) - a_i c \tag{3}$$

Equilibrium Analysis. We now proceed to characterize the Nash equilibrium of the game. Conditional on whether candidate i is being attacked or not, $n_i \in \{0, 1\}$, the benefit obtained by i when she attacks j is:

$$\Delta_{ij}(n_i) = p_i(n_i, 1) - p_i(n_i, 0), \tag{4}$$

i.e. the difference between the probability of winning the election when she attacks and when she does not attack her opponent. The following proposition establishes some basic properties of the function $\Delta_{ij}(n_i)$.

Proposition 1. *The benefit function $\Delta_{ij}(n_i)$ satisfies the following properties:*

i. For candidate 1, the benefit of an attack is larger when she is attacked:

$$\Delta_{12}(0) < \Delta_{12}(1)$$

ii. For candidate 2, the benefit of an attack is larger when she is not attacked:

$$\Delta_{21}(1) < \Delta_{21}(0)$$

iii. Candidate 2 is more aggressive than candidate 1 in the sense that:

$$\Delta_{12}(1) < \Delta_{21}(1)$$

Observe that candidate 1 is more willing to attack when she is attacked. Intuitively,

[7] In Appendix B.4, we show that our basic qualitative results are robust to using an alternative widely used class of functions, the so-called Tullock CSF proposed by Tullock (1980).

receiving an attack reduces the front-runner's lead, which in turn makes it more likely that an attack against 2 may be decisive for the election. The opposite result holds for candidate 2. Note that receiving an attack reduces her support, which in turn makes it less likely that an attack against candidate 1 is decisive for the outcome of the election. We also show that candidate 2 is the most aggressive candidate in that she always benefits more from an attack. In particular, observe that from Proposition 1 it follows that:

$$\Delta_{12}(0) < \Delta_{12}(1) < \Delta_{21}(1) < \Delta_{21}(0)$$

Given this structure of incentives, the next proposition provides a complete characterization of the unique equilibrium of the game.[8]

Proposition 2. *There exists a unique Nash equilibrium with the following characteristics:*

i. Both candidates attack if, and only if, $c \leq \Delta_{12}(1)$.

ii. Only candidate 2 attacks, if and only if, $\Delta_{12}(1) < c \leq \Delta_{21}(0)$.

iii. No candidate attacks if, and only if, $\Delta_{21}(0) < c$.

Figure A.1 depicts the region of parameters where each class of equilibrium exists. Note that, as the cost of attacking increases, we move through three different parameter regions where the following equilibria exist: (*i*) an equilibrium where both candidates attack, (*ii*) an equilibrium where only candidate 2 attacks, and (*iii*) an equilibrium where nobody attacks. Thus, our analysis suggests that the candidate in the lead is always the one most likely to receive a campaign attack in two-candidate races.

2.2 Three-Candidate Races

Setup. We now consider the case of a race with three candidates, with $s_1^o > s_2^o > s_3^o > 0$. As before, players decide simultaneously whether to attack each opponent, with $a_{ij} \in \{0, 1\}$ representing candidate i's binary decision to attack j. For simplicity, we suppose that each candidate may target at most one rival.[9] Let $n_i = \sum_{k \neq i} a_{ki}$ represent the number of attacks received by candidate i and define $n = (n_1, n_2, n_3) \in \mathcal{N}$, where $\mathcal{N}$ represents the set of all possible profiles of attacks. The final support of candidate i is given by:

[8]For simplicity, we assume that a candidate attacks when indifferent.

[9]This assumption considerably simplifies the analysis, allowing a more direct and concise exposition of the main results. In Appendix B.3, we consider the case where candidates are allowed to target multiple opponents, showing that our main qualitative results remain largely unchanged.

$$x_i(n) = (1 - \phi n_i)s_i^o + \sum_{j \neq i} \frac{n_j \phi s_j^o}{2} + \epsilon_i \tag{5}$$

where we suppose that an attack against candidate j benefits both of her rivals equally so that each gets $\phi s_j^o/2$ regardless of who attacked.[10] As before, ϵ_i is an iid shock with Type I EV distribution. We define $s_i(n) := (1 - \phi n_i)s_i^o + \sum_{j \neq i} \frac{n_j \phi s_j^o}{2}$.

Under the assumption that elections are held by single ballot plurality, the probability of winning is given by:

$$\widetilde{p}_i(n) = \frac{\exp(s_i(n))}{\sum_{k=1}^{3} \exp(s_k(n))} \tag{6}$$

Finally, as before, we assume that candidates seek to maximize their probability of winning net of attacking costs, $u_i(a) = \widetilde{p}_i(n) - (a_{ij} + a_{ik})c$.

Equilibrium Analysis. Let $\mathcal{N}_i \subset \mathcal{N}$ denote the set of all possible values which the vector n may assume when we impose the restriction that player i is not attacking anyone, i.e. when $a_{ij} = 0$ for $j \neq i$. For any $n \in \mathcal{N}_i$, the benefit obtained by candidate i when she attacks j is:

$$\widetilde{\Delta}_{ij}(n) = \widetilde{p}_i(n_i, n_j + 1, n_k) - \widetilde{p}_i(n_i, n_j, n_k) \tag{7}$$

i.e. the difference between the probability of winning the election when i attacks j and when she does not attack j.

With three candidates, the expression for the benefit function $\widetilde{\Delta}_{ij}$ becomes considerably less tractable. In what follows, we provide a characterization of the equilibrium focusing on the case where the impact of an attack, as captured by the parameter ϕ, is "sufficiently" small. This assumption is consistent with empirical evidence suggesting that the effect of negative campaigning tends to be small in general (Lau and Rovner, 2009). Moreover, from a technical point of view, it makes our model more tractable by reducing the dependence of each candidate's incentives on other players' strategies.[11] The next proposition establishes some basic properties of the function $\widetilde{\Delta}_{ij}(n)$ when ϕ is small.

Proposition 3. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then we have:*

i. For candidate 1, the benefit of an attack on 2 is larger than that of an attack on 3:

$$\widetilde{\Delta}_{13}(n) < \widetilde{\Delta}_{12}(n') \qquad \textit{for any } n, n' \in \mathcal{N}_1$$

[10] The main qualitative results of our analysis are robust to allowing for an asymmetric division of the benefits of an attack.

[11] Intuitively, as ϕ decreases, the benefit of an attack for candidate i becomes less dependent on whether her opponents are engaging or not in campaign attacks, and who exactly they are targeting. In other words, $\widetilde{\Delta}_{ij}(n)$ becomes less sensitive to changes in $n \in \mathcal{N}_i$.

ii. For candidate 2, the benefit of an attack on 1 is larger than that of an attack on 3:

$$\widetilde{\Delta}_{23}(n) < \widetilde{\Delta}_{21}(n') \qquad \textit{for any } n, n' \in \mathcal{N}_2$$

iii. For candidate 3, the benefit of an attack on 1 is larger than that of an attack on 2:

$$\widetilde{\Delta}_{32}(n) < \widetilde{\Delta}_{31}(n') \qquad \textit{for any } n, n' \in \mathcal{N}_3$$

iv. Candidate 2 is the most aggressive candidate in the sense that:

$$\max\{\widetilde{\Delta}_{12}(n_1), \widetilde{\Delta}_{31}(n_3)\} < \widetilde{\Delta}_{21}(n_2) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

Thus, each candidate prefers to target her highest-ranked opponent. Moreover, candidate 2 is always the most aggressive candidate, while the relationship between $\widetilde{\Delta}_{12}$ and $\widetilde{\Delta}_{31}$ is ambiguous in general. Interestingly, it is possible to show that if s_2^o is close enough to s_1^o, then $\widetilde{\Delta}_{12}(n_1) > \widetilde{\Delta}_{31}(n_3)$ for any $n_i \in \mathcal{N}_i$; whereas if s_2^o is sufficiently close to s_3^o, then the opposite holds (see Proposition B.1 in Appendix B.1). Intuitively, the candidate who is closer to candidate 2 in terms of initial support inherits her more aggressive behavior. In Appendix B.1, for completeness, we extend the characterization of the function $\widetilde{\Delta}_{ij}(n)$ to all other pairs of candidates (see Propositions B.2 and B.3 and Corollary B.1).

Given such structure of incentives, we are able to provide a complete characterization of the unique equilibrium of the game (see Proposition B.4 in Appendix B.1 and Figure A.2). Overall, our analysis shows that the most likely directions of attacks are, respectively: (*i*) from candidate 2 against 1 and (*ii*) either from candidate 1 against 2 or from candidate 3 against 1. Moreover, candidates with an electoral advantage are always more likely to receive a campaign attack.[12]

Comparative Statics. We now use our basic model to examine two comparative statics questions. First, we investigate how the incentives for candidates 1 and 2 to attack each other change when we vary the initial support of candidate 3. Our main result is summarized in the next proposition.

Proposition 4. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then we have:*

$$\frac{\partial \widetilde{\Delta}_{12}(n_1)}{\partial s_3^o} < 0 \quad \textit{and} \quad \frac{\partial \widetilde{\Delta}_{21}(n_2)}{\partial s_3^o} < 0 \qquad \textit{for any } n_i \in \mathcal{N}_i$$

Thus, candidates 1 and 2 become less aggressive towards each other when the electoral

[12]Given any set of parameter values, candidate 1 is (weakly) more likely to receive a campaign attack than candidate 2, who is in turn (weakly) more likely to receive an attack than candidate 3.

strength of candidate 3 increases. Note that the mere presence of a third candidate "dilutes" the benefit of a campaign attack for candidates 1 and 2, given that now the resulting electoral gains have to be split with a rival (see Proposition B.5 in Appendix B.1). Interestingly, the above result shows that this "dilution effect" is amplified when the initial level of support of candidate 3 increases, in which case she poses a larger competitive threat to 1 and 2.

Next, we examine how the incentives to attack vary under different electoral systems, focusing on the comparison between single and dual ballot plurality systems. When elections are held under dual ballot (runoff) plurality, the probability that a candidate advances to the second round is given by:

$$\widetilde{p}_i^{DB}(n) = \widetilde{p}_i(n) + \widetilde{p}_j(n) \frac{\exp(s_i(n))}{\exp(s_i(n))+\exp(s_k(n))} + \widetilde{p}_k(n) \frac{\exp(s_i(n))}{\exp(s_i(n))+\exp(s_j(n))}, \tag{8}$$

with $i, j, k \in \{1, 2, 3\}$, where $\widetilde{p}_i(n)$ represents the likelihood that candidate i ranks first (see equation (6)). The above expression, thus, gives the probability that candidate i finishes either in first or second place. The benefit function is now defined as $\widetilde{\Delta}_{ij}^{DB}(n) = p_i^{DB}(n_i, n_j + 1, n_k) - p_i^{DB}(n_i, n_j, n_k)$. As before, candidates maximize their probability of advancing to the second round net of attacking costs.[13]

Our analysis highlights the fact that the pattern of campaign attacks differs in significant ways under single and dual ballot plurality systems. In particular, we show that, under dual ballot plurality, candidate 3 is the most aggressive candidate followed by 2 and 1, respectively.

Proposition 5. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ then under dual ballot plurality candidate 3 is the most aggressive candidate followed respectively by candidates 2 and 1, in the sense that:*

$$\max\{\widetilde{\Delta}_{12}^{DB}(n_1), \widetilde{\Delta}_{13}^{DB}(n_1^{'})\} < \max\{\widetilde{\Delta}_{21}^{DB}(n_2), \widetilde{\Delta}_{23}^{DB}(n_2^{'})\} < \max\{\widetilde{\Delta}_{31}^{DB}(n_3), \widetilde{\Delta}_{32}^{DB}(n_3^{'})\}$$

for any $n_i, n_i^{'} \in \mathcal{N}_i$.

Thus, if in equilibrium candidate 1 attacks an opponent, then 2 and 3 must attack someone as well; while if candidate 2 attacks an opponent, then 3 must attack someone as well. Intuitively, under dual ballot plurality, candidate 3 is the one most fiercely competing for a spot in the second round and therefore has the largest incentives to attack. Note that while Duverger's Law (Duverger, 1954) states that voters have an incentive to behave differently under single and dual ballot plurality, our analysis highlights the

[13] Our analysis focuses on characterizing the behavior of candidates in the first round of elections, given that the incentives in the second round are the same as those in two-candidate races.

fact that candidates also have an incentive to follow different campaign strategies under these two systems. Interestingly, we show that under dual ballot plurality candidates 2 and 3 may prefer to target each other rather than the front-runner (see Proposition B.6 in Appendix B.1). Finally, we also show that if the race is sufficiently close in the sense that $s_1^o - s_3^o > 0$ is small enough, then all three candidates become more aggressive under dual ballot plurality (see Proposition B.7 in Appendix B.1).

2.3 Discussion and Extensions

We conclude this section with a discussion of the welfare implications of greater aggressiveness in campaigns. Our model highlights the idea that campaign attacks can cause important distortions in political outcomes, leading to the choice of bad politicians and platforms, by interfering with the process of aggregation of information and preferences via elections. Moreover, our model can be amended to show that the results derived above also hold under an alternative setting where campaign attacks lead to voter demobilization (see Appendix B.2). Indeed, aggressive campaigning and mudslinging have been shown to reduce turnout and political participation by alienating voters and increasing mistrust in politics and electoral institutions (Ansolabehere et al., 1994; Kahn and Kenney, 1999; Chong et al., 2015). Finally, our analysis also suggests that the impact of campaign attacks may fall disproportionately on the more "impressionable" groups of society, particularly the poorer and more disillusioned voters, thus causing further distortions in the political representation system.[14] In the long run, all these elements can pose serious threats to democracy.

Throughout this section we have made a few assumptions which we now discuss. In Appendix B.3, we show that our main qualitative results remain largely unchanged when we extend the model to allow candidates to target multiple opponents. Next, in Appendix B.4, we show that our basic results are robust to considering an alternative functional form for the probability of winning based on a Tullock CSF. Finally, in Appendix B.5, we simulate the model for specific parameter values to show that our results hold for reasonable (i.e. not exceedingly small) values of the parameter ϕ. Overall, our model provides a flexible framework for studying the incentives behind campaign attacks. Importantly, our analysis yields a number of specific predictions about the behavior of candidates which we can test using data from real world campaigns.

[14] The recent rise in populism around the world is often associated with voters' disillusionment with traditional parties and political institutions. In Europe, Guiso et al. (2020) show that recent shocks to economic insecurity lead to a significant reduction in turnout and to an increase in the willingness to support right-wing populist parties.

3 Campaign Attacks in Brazil

3.1 Institutional Background

Municipal elections in Brazil are held every four years, with each municipality electing a single mayor. The Brazilian Constitution establishes that municipalities with less than 200,000 registered voters must use a single ballot plurality system, while those above this threshold must use a dual ballot system. The first round of elections occurs simultaneously in the entire country − usually on the first Sunday of October − and campaigning is allowed only during a specific period.[15] For example, in 2012 campaigns were allowed between July 6 and October 6 (day before Election Day), while in 2016 this period was shortened to the interval between August 16 and October 2 (day before Election Day).[16]

During the campaign period, candidates have the chance to advertise themselves and present their opinions and proposals. They may do so by holding campaign rallies and displaying ads on traditional media (e.g. TV and radio) and social media (e.g. Facebook and Instagram). Electoral campaigns must be conducted in strict compliance with the electoral legislation. For instance, the Brazilian law prohibits anyone from offering goods and services in exchange for political support. Furthermore, an incumbent is not allowed to increase certain categories of spending or raise public employees' salaries during an election year.

Brazil has one of the world's most stringent legislations regarding offensive and dirty campaigning. Any candidate or party targeted by a slanderous, defamatory, or false accusation (i.e. "campaign attack") is guaranteed the right to respond to the offense, using the same media employed for the attack, at the offender's cost. Thus, for instance, if the attack occurred through a TV ad, then the offender is obliged to provide some of his own TV time for a reply of the same length as the attack. Alternatively, if the attack took place via a Facebook post, then the offender is required to provide space on his own page for a reply of a similar size and number of characters, which must remain visible for a period at least twice as long as the offense.[17]

Importantly, "right of reply" lawsuits are simple and inexpensive, requiring only that the plaintiff provides adequate proof of the occurrence of the attack. These lawsuits are

[15]In municipalities where a second round is needed, the runoff is usually held on the last Sunday of October.

[16]This reduction in the campaign period was the result of a reform aiming to limit campaign spending in Brazil (Avis et al., 2021). As we shall describe later in more detail, our analysis includes election-year fixed effects to control for time-specific shocks, such as legislative changes, that affected all municipalities simultaneously.

[17]As in other countries, media exposure is a crucial resource for candidates in Brazil. Using data from Brazilian gubernatorial elections, da Silveira and Mello (2011) provide quasi-experimental evidence showing that an increase in a candidate's TV time leads to a significant increase in her vote share.

processed by local electoral courts on a fast track in order to ensure an immediate response to the offended candidate and avoid irreparable damages to the election. According to the legislation, a decision on a right of reply lawsuit must be issued by the local electoral judge within 72 hours of the filing of the complaint, and there is abundant anecdotal evidence suggesting that such deadline is strictly adhered to.[18]

3.2 Data

In order to examine the incentives of candidates, we construct a unique dataset of campaign attacks based on detailed information collected from Brazil's Regional Electoral Courts' databases.[19] Specifically, we recover all "right of reply" lawsuits (henceforth, RR) filed in the entire country during the municipal elections of 2012 and 2016.[20] For each RR lawsuit, we retrieve information about the identities of the plaintiff and the defendant, the date in which the complaint was filed, and the municipality where the attack took place. We then construct a dataset of ordered pairs of candidates for each municipality and election-year. Our measure of campaign attack is an indicator variable Y_{ijmt} which equals one if candidate i "attacked" j in municipality m and election-year t, and zero otherwise.

Our measure of campaign attack captures a particularly aggressive type of negative campaigning, involving slander, defamation, and false accusations. Relatively to previous studies, our measure has the advantage of being based on an objective criterion, which allows us to systematically collect data throughout the country. Due to limitations related to the Electoral Courts' databases, we were able to recover information about the courts' rulings (i.e. whether the decisions were favorable or not to the plaintiff) only for a limited number of cases. However, a careful inspection of the lawsuits shows that the majority of them were adequately supported by evidence that an "attack" indeed occurred. Importantly, our results are robust to using a stricter definition where we consider that an attack took place if, and only if, we find a RR lawsuit with a decision favorable the plaintiff.[21]

Our measure of campaign attack captures the discrete decision of candidates to attack or not. Our focus on the "extensive margin" is due in part to data limitations, given that the same RR lawsuit may receive different identification numbers in the Electoral Courts'

[18] In Appendix E, we provide an in-depth discussion of two cases of right of reply lawsuits.

[19] Our searches were performed on "*Sistema de Acompanhamento de Documentos e Processos*" (SADP), which is a database specific to the Brazilian Electoral Justice.

[20] We were unable to recover data for the states of Alagoas, Espirito Santo, and Rondonia, which together amount to less than 5% of the Brazilian population.

[21] The results of the analysis using this alternative definition of campaign attack are available upon request.

databases as it progresses through the Brazilian judicial system. This feature of the data severely limits our ability to count the number of different attacks between candidates in a given election.[22] Most importantly, however, the focus on the extensive margin is particularly appropriate in the context of our study because it allows us to directly test the main predictions of the model.

Our main dataset is thus composed of all ordered pairs of candidates who run in a given municipality m and election-year t (electoral race), where for each ordered pair we have information about whether an attack took place or not.[23] We complement this dataset with detailed information obtained from Brazil's Tribunal Superior Eleitoral (TSE) about the characteristics of candidates, such as gender, educational level, party affiliation and campaign expenditures, and electoral races' characteristics, such as number of registered voters and final vote shares. Finally, from the 2010 Population Census, we obtain various municipal characteristics, such as population, income per capita, share of urban population, illiteracy rate, and Gini index.

3.3 Sample and Summary Statistics

Our basic dataset consists of $69,252$ ordered pairs of candidates, comprising $10,461$ distinct electoral races which took place during the 2012 and 2016 municipal elections.[24] Table 1 provides summary statistics for the main variables in our dataset at various levels of aggregation. Panel A reports descriptive statistics for some selected socioeconomic characteristics of the municipalities in our sample. The average population of a municipality in 2010 was $33,807$ and the average monthly per capita income was R$ 496.50 (or approximately US$ 275.00 in 2010). Panel B, in turn, provides general information about the electoral races in our sample. The average turnout rate was 86% and the percentage of valid votes (excluding blank and null votes) was 92%.[25] The average vote shares of winners and runner-ups were 55% and 38%, respectively. Finally, the average number of candidates per race was 2.83, with the average number of candidates who received at least 10% and 15% of the valid votes being 2.23 and 2.14, respectively.

Next, Panel C reports descriptive statistics for some selected individual characteristics of the candidates in our sample. The proportion of female candidates is 13% and the fraction of candidates who have a college degree is 51%. The average campaign expen-

[22] As we show below, RR lawsuits occur relatively infrequently, potentially due to the disincentives created by the electoral legislation itself. Therefore, the more interesting dimension for the analysis does seem to be the extensive margin.

[23] For example, if there are 4 candidates in a given municipality m and election-year t, then our dataset will feature 12 observations consisting of all possible ordered pairs of candidates associated with that particular race.

[24] Our sample excludes a small number of uncontested races, i.e. those with a single candidate.

[25] Vote is mandatory in Brazil for all citizens between 18 and 70 years old.

diture is R$ $119,401$ (or approximately US$ $36,000$ in 2016). Finally, Panel D provides summary statistics for our sample of ordered pairs of candidates. The overall fraction of ordered pairs for which an attack was observed is 2.6%. Moreover, conditional on the race having "2 candidates", the frequency of attacks is 5.4%; while conditional on it having "3 candidates" or "4 or more candidates", the frequency falls to 2.8% and 1.8%, respectively. Lastly, the likelihood of an attack is 2.5% under single ballot plurality and 4.2% under dual ballot plurality.[26]

4 Hypotheses and Empirical Strategies

Our model yields a number of specific predictions which we can test using our dataset on RR lawsuits. In this section, we discuss our main hypotheses and empirical strategies. Throughout, we denote by "$i \to j$" an attack from candidate i against j. Whenever necessary, candidates are ordered according to their final position in the race and we refer to the k^{th} place candidate simply as "candidate k".[27]

4.1 Campaign Attacks: General Patterns

Our theoretical analysis yields several implications about the general pattern of campaign attacks under single ballot plurality. Specifically, in two-candidate races, we expect $2 \to 1$ to be more likely than $1 \to 2$. Moreover, in three-candidate races, we expect each candidate to be more likely to attack her highest-ranked opponent, with $2 \to 1$ being more likely than $1 \to 2$ and $3 \to 1$.

We begin our empirical investigation by examining whether these general theoretical predictions find support in the data, without necessarily attempting to recover causal effects. To do so, we consider only races held under single ballot plurality and we restrict the sample to include only "effective candidates". An effective candidate is defined as one who obtains more than a certain share of the votes. The idea here is to exclude candidates who have no real chances of winning and, therefore, may be subject to different electoral incentives. To check the robustness of our results, we perform our analysis using various different thresholds.

[26] In Figure A.4 we show that campaign attacks tend to be more frequent in larger, richer and more urban municipalities, where electoral races tend to be more competitive. Interestingly, we find no relationship between frequency of attacks and income inequality, as measured by the gini index.

[27] Note that the initial level of support of candidates is generally unobserved. Thus, following a standard approach in the literature, we use the final ("ex-post") ranks of candidates as a proxy for their initial ("ex-ante") positions. This procedure seems appropriate in the context of our study. Indeed, based on a sample of 78 electoral races (comprising 464 candidates), for which we were able to recover opinion polls conducted at the beginning of the campaign period, we find that the correlation between the initial voting intentions and the final vote shares are 0.87.

We, first, restrict the sample to include only races with two effective candidates and consider only the pairs formed by the first two candidates. We then estimate the following regression:

$$Y_{ijmt} = \alpha + \beta_{21} D_{21mt} + X_{imt}\gamma + X_{jmt}\zeta + \delta_m + \delta_t + \epsilon_{ijmt} \tag{9}$$

where Y_{ijmt} represents whether candidate i attacked j in municipality m and election-year t, D_{21mt} is a dummy which equals one if candidate i is the 2nd place candidate and j is the 1st place candidate, and X_{imt} and X_{jmt} are vectors of candidate-specific characteristics.[28] We also include municipality fixed effects δ_m and election-year fixed effects δ_t. Standard errors are clustered at the municipality level. Our coefficient of interest here is β_{21}, which captures the likelihood of an attack $2 \rightarrow 1$ relatively to $1 \rightarrow 2$ (omitted category). Consistently with our theory, we expect $\beta_{21} > 0$.

Next, we restrict the sample to include only races with three effective candidates and consider only the pairs formed by the first three candidates. We then estimate the following regression:

$$\begin{aligned} Y_{ijmt} = \alpha &+ \beta_{12} D_{12mt} + \beta_{13} D_{13mt} + \beta_{21} D_{21mt} + \beta_{31} D_{31mt} \\ &+ \beta_{32} D_{32mt} + X_{imt}\gamma + X_{jmt}\zeta + \delta_m + \delta_t + \epsilon_{ijmt} \end{aligned} \tag{10}$$

where we include the dummies D_{12mt}, D_{13mt}, D_{21mt}, D_{31mt}, and D_{32mt}, defined similarly as before. Note that the omitted category here is D_{23mt} so that all coefficient estimates should be interpreted relatively to the frequency of attacks $2 \rightarrow 3$.[29] In line with our theoretical predictions, we expect: (i) $\beta_{12} > \beta_{13}$, $\beta_{21} > 0$ and $\beta_{31} > \beta_{32}$, and (ii) $\beta_{21} > \beta_{12}$ and $\beta_{21} > \beta_{31}$.[30] Or to put it in another way, we expect: (i) each candidate to be more likely to target her highest-ranked opponent and (ii) candidate 2 to be the most aggressive candidate. As before, standard errors are clustered at the municipality level.

Finally, our model predicts that the first two candidates should become less aggressive towards each other when the electoral support for the 3rd place candidate increases. To test this hypothesis, we restrict the sample to include only races with three or more effective candidates and focus only on the pairs formed by the first two candidates.[31] We

[28] We control for gender, age, age squared, marital status, high school and college degrees, the logarithm of campaign spending, incumbency status, and affiliation to three main parties, PT, PSDB, and PMDB. We also include a dummy indicating whether the pair was formed by candidates from the PT and PSDB since Brazilian politics was polarized between these two parties during the period of our analysis.

[29] Indeed, as shown in Corollary B.1, our model predicts $2 \rightarrow 3$ to be the least likely direction of attack in three-candidate races.

[30] Formally, we test the joint null hypotheses $H_o^1 : \beta_{12} = \beta_{13}$, $\beta_{21} = 0$, $\beta_{31} = \beta_{32}$, and $H_o^2 : \beta_{21} = \beta_{12}$, $\beta_{21} = \beta_{31}$, which amount to testing the predictions derived in Proposition 3, items *i*-*iii* and item *iv*, respectively.

[31] Our results are robust to restricting the sample to include only races with exactly three effective

then estimate the following regression:

$$\begin{aligned} Y_{ijmt} = \alpha + \beta_{21} D_{21mt} + \pi_{12} D_{12mt} Share3rd_{mt} \\ + \pi_{21} D_{21mt} Share3rd_{mt} + X_{imt}\gamma + X_{jmt}\zeta + \delta_m + \delta_t + \epsilon_{ijmt} \end{aligned} \tag{11}$$

where $Share3rd_{mt}$ is the vote share of the 3rd place candidate in municipality m and election-year t. Our coefficients of interest are π_{12} and π_{21}, which capture how the frequency of attacks $1 \rightarrow 2$ and $2 \rightarrow 1$ vary when the vote share of 3rd candidate increases, respectively. Consistently with our theoretical predictions, we expect $\pi_{12} < 0$ and $\pi_{21} < 0$. As before, standard errors are clustered at the municipality level.

4.2 Electoral Advantage Effect on Campaign Attacks

Our model also predicts that better-ranked candidates should be more likely to be targets of campaign attacks. The ideal experiment to test this hypothesis would be to take pairs of identical candidates across several municipalities, randomly assign an "electoral advantage" to a member of each pair and then compare the frequency of attacks received by candidates in treatment and control groups. Naturally, such an experiment would be totally unfeasible. In practice, the best one could hope to achieve would be to exploit an exogenous source of variation in candidates' support.

Our analysis exploits quasi-experimental variation in electoral support arising from virtual ties between 2nd and 3rd place candidates. Our approach follows Anagol and Fujiwara (2016) who used data from local elections in Brazil, India, and Canada to compare the subsequent electoral performances of candidates who finished almost tied for 2nd and 3rd places. They find that runner-ups are significantly more likely to run in and win the next elections relatively to close 3rd place candidates. Importantly, they show that the "runner-up effect" is not driven by systematic differences between candidates, such as distinct degrees of media coverage. Anagol and Fujiwara (2016) argue that simply being labeled "the runner-up" makes a candidate more salient and, therefore, more likely to be coordinated on by voters.

Our study examines whether runner-ups are also more likely to receive campaign attacks in the following elections.[32] Our analysis is performed at the candidate-municipality-election-year level. Let x_{jmt} be the running variable for candidate j in municipality m

candidates.

[32] An alternative strategy based on elections decided by a close margin, comparing winners and runner-ups, would be problematic in the context of our study because incumbency affects a number of dimensions, other than just electoral support, that may directly impact campaign attacks. For instance, an incumbent may receive more attacks simply because more issues can be brought up against her, in which case candidates in treatment and control groups would be systematically different.

and election-year t. For a 2nd place candidate, x_{jmt} is defined as the candidate's own vote share minus the vote share of the 3rd place candidate. Similarly, for a 3rd place candidate, this variable is defined as her vote share minus the vote share of the 2nd place candidate. The main outcome variable Y_{jmt} is a dummy that equals one if candidate j received at least one attack in municipality m and election-year t.

Under the usual continuity assumption on conditional expectations, the effect of being the close runner-up rather than the close 3rd place candidate on the likelihood of receiving a campaign attack in the following elections is given by:

$$\tau = \lim_{x \downarrow 0} \mathbb{E}[Y_{jmt+1} | x_{jmt} = x] - \lim_{x \uparrow 0} \mathbb{E}[Y_{jmt+1} | x_{jmt} = x] \tag{12}$$

To avoid selection problems, we perform our analysis unconditionally on the candidates' decisions to run again in the subsequent elections.[33]

We estimate the treatment effect by restricting the sample to include only races held under single ballot plurality and running the following local linear regression using only observations within a bandwidth h of the threshold:

$$Y_{jmt+1} = \alpha + \tau \mathbb{1}[x_{jmt} > 0] + \beta_1 x_{jmt} + \beta_2 x_{jmt} \mathbb{1}[x_{jmt} > 0] + \delta_t + \epsilon_{jmt}, \tag{13}$$

where τ is our parameter of interest. The model includes election-year fixed effects to control for certain features of the electoral competition that are specific to races held in the same year, e.g. the national political environment and legislative changes. Standard errors are clustered at the municipality level. Our preferred specification uses a linear polynomial fully interacted with the treatment indicator, with a triangular kernel and the Calonico et al. (2014b) optimal bandwidth. We check the robustness of our results by using alternative bandwidth sizes and different polynomial orders, as well as controlling for state fixed effects and a number of candidate characteristics. Our results are robust to using a rectangular kernel.

4.3 Single vs. Dual Ballot Plurality Systems

Finally, our model predicts that the pattern of campaign attacks should differ systematically under single and dual ballot plurality systems. In particular, our analysis suggests that 3rd place candidates should become more aggressive relatively to other candidates under dual ballot plurality, while 2nd and 3rd place candidates may become more likely to attack each other. Furthermore, it is ultimately an empirical question whether cam-

[33] As discussed below, we provide bounds on the conditional treatment effect following the approach proposed by Anagol and Fujiwara (2016).

paigns are more aggressive under single or dual ballot systems. To test these hypotheses, we exploit quasi-experimental variation arising from the fact that, in Brazil, municipalities with less than $200,000$ registered voters are obliged to adopt a single ballot plurality system, while those above this threshold must use a dual ballot system.

Our analysis is performed at the candidate pair-municipality-election-year level, where the main outcome Y_{ijmt} is a dummy representing whether candidate i attacked j in municipality m and election-year t. The running variable v_{mt} is the number of registered voters in municipality m and election-year t. Under the usual continuity assumption on conditional expectations, the effect of a change from single to dual ballot plurality on the likelihood of a campaign attack between any pair of candidates i and j is:

$$\lambda = \lim_{v \downarrow 200{,}000} \mathbb{E}[Y_{ijmt} | v_{mt} = v] - \lim_{v \uparrow 200{,}000} \mathbb{E}[Y_{ijmt} | v_{mt} = v] \tag{14}$$

Moreover, in order to directly test our main theoretical predictions, we examine whether the treatment effect is heterogeneous across different types of pairs. We are particularly interested in investigating whether the effect varies with the rank of the attacking candidate. Formally, the effect of a change in the electoral rule on the probability of an attack by a candidate in the k^{th} position is given by:

$$\lambda^{(k)} = \lim_{v \downarrow 200{,}000} \mathbb{E}[Y^{(k)}_{ijmt} | v_{mt} = v] - \lim_{v \uparrow 200{,}000} \mathbb{E}[Y^{(k)}_{ijmt} | v_{mt} = v] \tag{15}$$

where we use the superscript (k) to indicate that the analysis is restricted to consider only pairs where the attacking candidate was in the k^{th} position, with $k \in \{1, 2, 3\}$. Consistently with our theoretical predictions, we expect the treatment effect to be largest for 3rd place candidates.

To estimate these effects, we restrict the sample to include only races with three or more effective candidates and consider only the pairs formed by the first three candidates. We then estimate the following local linear regression using only observations within a bandwidth h of the threshold:

$$Y_{ijmt} = \alpha + \lambda \mathbb{1}[v_{mt} > 200,000] + \beta_1 v_{mt} + \beta_2 v_{mt} \mathbb{1}[v_{mt} > 200,000] + \delta_t + \epsilon_{ijmt} \tag{16}$$

where λ is our parameter of interest. As before, our analysis includes election-year fixed effects and the standard errors are clustered at the municipality level. Our preferred specification uses a linear polynomial fully interacted with the treatment indicator, with a triangular kernel and the Calonico et al. (2014b) optimal bandwidth. Our main heterogeneity analysis is performed by running separate regressions for subsamples of pairs where the attacking candidate placed in 1st, 2nd, and 3rd. We also examine whether the estimated effects vary across pairs formed by 1st and 2nd, 1st and 3rd, and 2nd and 3rd place candidates. Finally, we check the robustness of our results by using alternative bandwidth sizes and different polynomial orders, as well as controlling for state fixed effects and a number of candidate characteristics. Our results are robust to using a rectangular kernel.

5 Main Results

5.1 Campaign Attacks: General Patterns

We begin our analysis by reporting in Figure 1 the frequency of campaign attacks in races with two and three effective candidates, using a 15% threshold for the definition of effective candidate and focusing only on races held under single ballot plurality. Observe that in races with two effective candidates the frequency of attacks $2 \rightarrow 1$ is 6.7 percentage points (pp), while the frequency of attacks $1 \rightarrow 2$ is 5.4 pp (diff = 1.3 pp, p-value < 0.01). Moreover, in races with three effective candidates, the most likely directions of attacks are $2 \rightarrow 1$ (5.2 pp), $1 \rightarrow 2$ (4.5 pp), and $3 \rightarrow 1$ (3.1 pp).[34] These figures are generally consistent with the main predictions of the model. In what follows, we perform a detailed regression analysis in order to examine the robustness of these findings.

Table 2 reports coefficient estimates for equation (9) using a sample of electoral races with two effective candidates. We consider four different thresholds for the definition of effective candidate, 5%, 10%, 15%, and 20%, and the results obtained for each separate regression are reported in columns $1-4$.[35] Note that the point estimates are very stable across specifications, implying that the likelihood of an attack $2 \rightarrow 1$ is about 1.2 pp larger than that of an attack $1 \rightarrow 2$. These results are consistent with our prediction that 2nd place candidates are the most aggressive ones in two-candidate races. In all cases, we are able to reject the null hypothesis that $\beta_{21} = 0$ at 1% significance level.

Next, Table 3 reports coefficient estimates for equation (10) using a sample of electoral races with three effective candidates. As before, the results show that the most likely directions of attack are, in order, $2 \rightarrow 1$, $1 \rightarrow 2$, and $3 \rightarrow 1$.[36] Specifically, according

[34] The difference between the frequency of attacks $2 \rightarrow 1$ and $1 \rightarrow 2$ is 0.7 pp (p-value = 0.36), the difference between the frequency of attacks $2 \rightarrow 1$ and $3 \rightarrow 1$ is 2.0 pp (p-value < 0.01) and the difference between the frequency of attacks $1 \rightarrow 2$ and $3 \rightarrow 1$ is 1.3 pp (p-value = 0.05).

[35] Moreover, in order to guarantee that our results are not being influenced by measurement error in the classification of candidates, given that we use their final ranks as proxy for their initial positions, we perform an additional robustness check where we exclude from the sample all races where the final vote shares of any two effective candidates are too "close" (e.g. within 5 pp distance). We show that all results reported in this subsection remain unchanged. Additional details are available upon request.

[36] Note that the estimates associated with these three directions of attack are always statistically significant, while those associated with $1 \rightarrow 3$ and $3 \rightarrow 2$ are never significant.

to the estimates reported in column 1, based on a 5% threshold, an attack $2 \rightarrow 1$ is 4.0 pp more likely to occur than $2 \rightarrow 3$ (omitted category), while $1 \rightarrow 2$ and $3 \rightarrow 1$ are, respectively, 3.3 pp and 0.8 pp more likely. Furthermore, joint hypothesis tests provide support for our predictions that candidates are more likely to target their highest-ranked opponents (H_o^1) and that 2nd place candidates are the most aggressive in three-candidate races (H_o^2).[37]

In columns $2-4$, we show that our main qualitative results are robust to using alternative thresholds for the definition of effective candidate. Interestingly, note that as the threshold increases the magnitudes of the estimates associated with attacks $1 \rightarrow 2$ and $2 \rightarrow 1$ gradually go down, meaning that candidates 1 and 2 become less aggressive towards each other. This pattern is consistent with the "dilution effect" identified in our theoretical analysis. Indeed, note that as the threshold increases the vote shares of 3rd place candidates included in the sample necessarily go up, so that we are progressively focusing on three-candidate races where the 3rd candidate is stronger.

We provide a more direct test of the "dilution effect" by estimating equation (11).[38] The results reported in Table 4 show that the frequency of attacks between the first two candidates decreases as the 3rd candidate becomes stronger. This effect is particularly pronounced for attacks originating from the 2nd place candidate, with the estimates for the interaction between $2 \rightarrow 1$ and the 3rd candidate's vote share being always negative and statistically significant. According to the results reported in column 1, based on a 0% threshold, a 10 pp increase in the vote share of the 3rd candidate is associated with a reduction in the frequency of attacks $2 \rightarrow 1$ by 2.3 pp. In columns $2-4$, we increase the threshold used for the definition of effective candidate to 2%, 5%, and 10%. Note that as the threshold goes up the magnitudes of the estimates for both interactions increase substantially. These results suggest that the "dilution effect" becomes stronger as the vote share of the 3rd place candidate increases.

5.2 Electoral Advantage Effect on Campaign Attacks

In this subsection, we estimate the effect of an electoral advantage on the likelihood of receiving a campaign attack by exploiting quasi-experimental variation in electoral support arising from virtual ties between 2nd and 3rd place candidates. We begin our analysis by discussing two important assumptions required for the validity of our research design. First, 2nd and 3rd place candidates should be comparable in terms of their individual

[37]We reject both null hypotheses in all cases, with the exception of H_o^2 in the specification reported in column 4, based on a 20% threshold.

[38]For this analysis, we employ smaller thresholds for the definition of effective candidate, since doing so allows us to more fully exploit the variation in the vote shares of 3rd place candidates.

characteristics around the discontinuity. We provide evidence for this assumption by showing in Table A.1 that a number of pre-determined characteristics, such as gender, age, marital status, level of schooling, campaign expenditures, and party affiliation, vary smoothly around the cutoff − with the possible exception of affiliation to the Labor Party (PT).[39] As we show below, our results are robust to controlling for all these variables in the regressions.

Second, our research design depends crucially on the existence of the "runner-up effect" in our study context. In Figure 2, we examine this question by plotting binned averages of the candidates' chances of running again (Panel A), their vote shares (Panel B) and their chances of winning the next elections (Panel C) against the vote share differences between 2nd and 3rd place candidates, together with a second-order polynomial fitted separately on each side of the discontinuity.[40,41] Consistently with Anagol and Fujiwara (2016), we find a significant jump at the discontinuity in all three variables. In particular, close runner-ups are about 12 pp more likely to run again and 10 pp more likely to win the next elections.

We complement the graphical analysis above by presenting estimation results in Panel A of Table 5. For each dependent variable, we report the sample mean to the left of the threshold (3rd place candidates' mean) and the optimal bandwidth. According to our preferred specification (column 1), which is based on a local linear regression, close runner-ups are about 11.6 pp more likely to run again, their vote shares are 4.2 pp larger and they are 6.6 pp more likely to win the next elections.[42] We check the robustness of these findings by controlling for candidates' characteristics and state fixed effects (column 2), fitting a quadratic polynomial on each side of the discontinuity (column 3), and estimating the difference in means for a narrow bandwidth of 5 pp (column 4). All our results are robust to these various specifications.

Next, turning to our main question, we examine the effect of being the close runner-up on the likelihood of receiving a campaign attack. In Panel D of Figure 2, we plot the frequency of attacks in the next elections against our running variable, with a second-order polynomial fitted separately on each side of the discontinuity. The graph displays a clear discontinuous jump at the cutoff, suggesting that barely 2nd place candidates are

[39]The point estimates reported in column 1 of Table A.1 are obtained by estimating the local linear regression specified in equation (13). In column 2, we report estimates for an additional specification using a quadratic polynomial, with the Calonico et al. (2014b) optimal bandwidth.

[40]As noted before, our analysis is performed unconditionally on running again, so that both average vote shares and proportion of candidates who win the next elections are calculated taking into account all candidates, including those who did not to run again.

[41]The binned averages are computed within quantile-spaced bins of the vote share difference, where the number of bins is chosen optimally according to Calonico et al. (2014a).

[42]These estimated effects are substantial and represent an increase of about 47%, 48%, and 70% respectively, relatively to the 3rd place candidates' means.

about 2 pp more likely to receive an attack relatively to close 3rd place candidates.

We complement our graphical investigation by reporting in Panel B of Table 5 the results of a detailed regression analysis. According to our preferred specification (column 1), close runner-ups are about 2 pp more likely to receive a campaign attack in the next elections. This effect is quite substantial and corresponds to an increase of approximately 160% relatively to the 3rd place candidates' mean. Our results are robust to a number of different specifications (columns 2−4), with estimates ranging from 1.3 pp to 2.1 pp. Furthermore, in Figure A.5 we plot point estimates obtained from local linear regressions using a variety of bandwidths. The graph shows that the estimated effects are very stable at around 2 pp, with the point estimates being generally statistically significant at 10% confidence level.[43]

Placebo and Heterogeneous Effects. Next, we report the results of a placebo test, where we compare the likelihood of receiving an attack between close 3rd and 4th place candidates. Anagol and Fujiwara (2016) showed that these candidates do not differ systematically in terms of their probabilities of running again and winning the next elections. In Panel A of Table A.2, we confirm those findings in our sample by showing that there are, indeed, no significant differences in subsequent electoral outcomes between close 3rd and 4th place candidates. Then, consistently with these results, in Panel B we find no significant effect on the likelihood of receiving an attack in the subsequent elections. These results are presented graphically in Figure A.6.[44]

We also investigate whether the electoral advantage effect varies with the share of votes received by almost tied 2nd and 3rd place candidates. The idea is that a larger electoral strength should make voters more likely to coordinate on the runner-up, which in turn should increase the chances that she receives an attack in the subsequent elections. Following Anagol and Fujiwara (2016), we perform separate analyses for two subsamples, one containing races in which the sum of the shares of 2nd and 3rd place candidates is larger than the share of the winner, $s_2 + s_3 \geq s_1$, and another containing races where the opposite holds, $s_2 + s_3 < s_1$. The results reported in Table A.4 show that the runner-up effect comes entirely from the subsample of "strong" 2nd and 3rd place candidates. Most importantly, according to our preferred specification (Panel A, column 1), "strong"

[43] In a complementary analysis, we employed a fuzzy RDD approach to estimate the contemporaneous effect of an increase in the vote share on the likelihood that a candidate receives a campaign attack, using the cutoff between 2nd and 3rd place candidates in the previous election as instrument for vote shares. We find that a 10 pp increase in a candidate's vote share raises the probability that she receives an attack by about 4.9 pp. Additional details are available upon request.

[44] Furthermore, in Table A.3 we perform an additional placebo test where we compare close 2nd and 3rd place candidates in terms of their likelihood of receiving an attack in the *same* election (at period t). As expected, we find no significant differences between them.

runner-ups are about 3.3 pp more likely to receive a campaign attack in the subsequent elections, which correspond to a 270% increase relatively to the 3rd place candidates' mean. Conversely, the corresponding effect for "weak" runner-ups is negative and statistically insignificant (Panel B, column 1).

Bounds on Conditional Effects. Our basic analysis was performed unconditionally on the candidates' decisions to run again in the next elections. We now complement our study by investigating whether the previous results cannot be simply attributed to the fact that barely 2nd place candidates choose to run again more often and are, therefore, "mechanically" more likely to receive an attack in the next elections. To investigate this question, we follow Anagol and Fujiwara (2016), who adapted an approach proposed by Lee (2009), to provide bounds for the runner-up effect on the probability of receiving an attack conditional on running again. Their methodology rests on the standard assumption that there are no "defiers", i.e. candidates who choose to run again if they place 3rd but not if they place 2nd. Crucially, the approach requires an assumption on the probability of being attacked after finishing in 3rd place for a "complier", which by definition cannot be observed since a "complier" never runs again in this case.[45] The lower bound on the conditional effect can then be obtained by assuming a "conservatively" large value for this probability.

A reasonable assumption to calculate the lower bound is to suppose that 3rd place compliers are attacked with the same probability as runner-ups who choose to run again, which we estimate to be 7.3 pp. Under this assumption, the lower bound would be 3.7 pp (s.e. = 2.3), which is a quite substantial effect.[46] Moreover, we calculate that for the conditional effect to be zero, a 3rd place complier would have to be attacked with a probability of 20 pp, which is an implausibly large probability.

5.3 Single vs. Dual Ballot Plurality

In this subsection, we turn to the analysis of the impact of a change in the voting rule on the pattern of campaign attacks. To do so, we focus on a sample of electoral races with three or more effective candidates, using a 5% threshold for the definition of effective candidate.[47] We begin our investigation by discussing the assumptions required for the validity of our research design, which exploits a discontinuity in the assignment of electoral rules at $200,000$ registered voters. First, municipalities should not be able to

[45] Formally, a "complier" chooses to run again if she places 2nd but not if she places 3rd.

[46] The upper bound can be obtained by assuming that 3rd place compliers are attacked with probability zero, which yields an estimated upper bound of 5.9 pp (s.e. = 2.8).

[47] Our results are robust to using alternative thresholds.

systematically sort themselves around the discontinuity. Figure A.7 plots the distribution of the number of registered voters across Brazilian municipalities, pooling data from 2012 and 2016. As expected, we find no evidence of strategic manipulation around the cutoff.

Second, the validity of our research design relies on the assumption that municipalities just above and just below the threshold are comparable in terms of their general characteristics. We provide evidence for this requirement by documenting in Table A.5 that a number of pre-determined characteristics of the municipalities in our sample, namely latitude, longitude, income per capita, share of urban population, share of population living under extreme poverty and Gini coefficient, vary smoothly around the threshold.[48]

We now turn to our main question of how a change from single to dual ballot plurality affects the likelihood of campaign attacks among pairs formed by the first three effective candidates. Panel A of Figure 3 plots the frequency of attacks against our running variable, together with a second-order polynomial fitted on each side of the discontinuity. The graph displays a slight positive jump in the likelihood of a campaign attack at the cutoff, suggesting that dual ballot plurality tends to exacerbate the general level of aggressiveness of the campaigns.

In Panel A of Table 6, we refine the graphical analysis above by means of a more detailed regression analysis. According to our preferred specification (column 1), which is based on a local linear regression with the optimal bandwidth, the likelihood of a campaign attack increases by about 21.6 pp under dual ballot plurality. Our results are robust to a number of different specifications (columns 2−4), with estimates ranging from 15.1 pp to 24.7 pp. Furthermore, in Panel A of Figure A.9, we show that the point estimates obtained from local linear regressions are always statistically significant at 10% level for a variety of bandwidth choices. Overall, our results suggest that dual ballot plurality has a positive local effect on the likelihood of campaign attacks.

Next, we examine whether the impact of a change in the electoral rule varies with the position of the attacking candidate. In Panels B−D of Figure 3, we plot the frequency of attacks against our running variable separately for pairs where the attacking candidate was placed in 1st, 2nd and 3rd, respectively. Consistently with the predictions of the model, we find a large positive jump in the frequency of attacks coming from 3rd place candidates at the cutoff (Panel D), suggesting that these candidates become more aggressive under dual ballot plurality. Moreover, we observe a positive but relatively smaller effect on attacks coming from 2nd place candidates (Panel C) and no discernible impact on the behavior of 1st place candidates (Panel B).

In Panel B of Table 6, we complement our investigation by presenting the results of a regression analysis performed separately for attacking candidates in different positions.

[48] All socioeconomic variables used here were taken from the 2010 Brazilian Population Census.

Consistently with the graphical evidence presented above, we find that 3rd place candidates are the ones whose campaign strategies are most impacted by a change from single to dual ballot plurality. Specifically, according to our preferred specification (column 1), the likelihood that a 3rd place candidate attacks an opponent increases by about 29.3 pp under dual ballot plurality, which represents a 300% increase relatively to the single ballot's mean.[49] Our results are robust to a number of different specifications (columns 2−4). In Panel D of Figure A.9, we show that the estimated effects obtained from local linear regressions are always statistically significant for various bandwidth choices. [50]

Moreover, the estimates presented in Panel B of Table 6 also suggest that 2nd place candidates become more aggressive under dual ballot plurality, with the probability of an attack coming from them increasing by about 18.7 pp (column 1).[51] Finally, note that the estimated effect for 1st placed candidates is much smaller in magnitude and very imprecisely estimated. As before, these results are robust to a number of different specifications (columns 2−4) and bandwidth choices (Panels B and C of Figure A.9).

We complement our analysis by investigating whether the impact of a change in the electoral rule varies across different types of pairs, focusing on those formed by 1st and 2nd, 1st and 3rd, and 2nd and 3rd place candidates. The graphical evidence reported in Figure A.8 shows the existence of a particularly large jump in the likelihood of attacks between 2nd and 3rd place candidates (Panel C). These results are confirmed by a regression analysis reported in Table A.6. According to our preferred specification (column 1), the likelihood of attacks between 2nd and 3rd place candidates increases by about 23.6 pp under dual ballot plurality, which represents a substantial increase relatively to the single ballot's mean. Finally, we show that our findings are robust to a number of different specifications (columns 2−4) and bandwidth choices (Figure A.10). These results are consistent with the implications of the model.

Placebo and Alternative Mechanisms. To further assess the robustness of our results, we perform a placebo test where we estimate local treatment effects at false thresholds, above and below the true discontinuity at 200,000 registered voters. For conciseness, Figure A.11 reports results only for our main outcomes variables, namely frequency of

[49] Interestingly, our results suggest that 3rd place candidates become the most aggressive candidates under dual ballot plurality. Based on the point estimates reported in column 1, and adding them to the corresponding single ballot's mean, we estimate that the frequency of attacks coming from 3rd place candidates is 38.8%, while the frequency of attacks coming from 2nd and 1st place candidates are 32.8% and 17.3%, respectively.

[50] As before, we perform an additional robustness check where we exclude from the sample all races where the final vote shares of any two effective candidates are too "close" (e.g. within 5 pp distance). We show that all our results remain unchanged. Additional details are available upon request.

[51] Although this particular point estimate is not statistically significant, note that the estimates reported in columns 2 and 4 are similar in magnitude and statistically significant.

attacks among the first three effective candidates (Panel A), frequency of attacks coming from 3rd place candidates (Panel B), and frequency of attacks between 2nd and 3rd place candidates (Panel C). Note that the point estimates obtained at placebo thresholds are much smaller in magnitude and almost always statistically insignificant. These results are in sharp contrast with the treatment effects obtained at the real cutoff, which are always positive and very precisely estimated.

Finally, we conclude the analysis by discussing alternative mechanisms that might be driving our results. Indeed, electoral races held under single and dual ballot plurality might differ in dimensions other than just those considered in our model, such as turnout, number of candidates, campaign spending and distribution of support across candidates. In Table A.7, we investigate whether these characteristics of electoral competition are similar in races held just above and just below the threshold. Interestingly, we find no systematic differences in turnout, number of candidates, maximum and average campaign spending, and the final vote shares of the first three candidates.[52] Moreover, we find no evidence of discontinuity in the sum of the vote shares of the 3rd and lower-placed candidates and the sum of the shares of the 4th and lower-placed candidates.

6 Discussion and Conclusion

This paper studied both theoretically and empirically the determinants of electoral campaign attacks. We first proposed a model to study the main factors that influence the candidates' decisions to attack. Our theoretical analysis yielded a number of predictions which we tested using detailed information obtained from all "right of reply" lawsuits filed in Brazil during the 2012 and 2016 municipal elections. As our model suggests, campaign attacks and the spread of misinformation may have potentially large consequences on political outcomes, leading to the choice of bad politicians and platforms, to a generalized sense of mistrust in politics and institutions, and to a decrease in political participation and voter turnout.[53]

While a thorough investigation of the welfare implications of campaign attacks is beyond the scope of this paper, we conclude our study with a brief discussion of the *potential* impacts of aggressive campaigning and false information on electoral outcomes, focusing on the case of the 2018 Brazilian presidential elections.[54] To do so, we take advantage of

[52]For races held under dual ballot plurality, campaign expenditures refer only to the amount spent in the first round of elections.

[53]The recent evidence suggesting Russian interference in elections in the US and other western democracies (e.g. France, Germany, and UK) shows that these concerns are of utmost importance.

[54]The 2018 presidential election was sharply polarized between Jair Bolsonaro in the far-right and Fernando Haddad, former president Luiz Inacio Lula da Silva's protégé, in the left. According to The Guardian, this was "*the bitterest and most polarized election in recent history*".

a data set from a nationally representative survey that interviewed voters after the first round of the 2018 elections focusing specifically on questions about campaign attacks received via WhatsApp. The survey revealed some interesting yet disturbing facts. First, among the respondents, 25% claimed to have received at least one message attacking one of the candidates in the week preceding the elections.[55] Moreover, conditional upon receiving one or more of these messages, 23% declared that their content actually influenced their choice of whom to vote for.[56] Taken at face value, these numbers imply that at least 36.8 million Brazilian voters were exposed to campaign attacks via WhatsApp in the week before the first round of elections, and approximately 8.5 million of them (or about 5.7% of the electorate) were influenced to some degree by these messages. While these numbers were not large enough to overturn the outcome of the election, they highlight the magnitude and extent of the problem. A better understanding of the impact of campaign attacks on political outcomes constitutes an important topic for future research.

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[55] While it is not possible to guarantee that these messages contained slander or false information, they were most certainly aggressive in tone and malicious in intent. Indeed, Bolsonaro later faced numerous accusations of spreading misinformation and fake news via WhatsApp during his campaign. See https://nyti.ms/3p8jqoP.

[56] In Appendix F we provide a more detailed description of the data. Moreover, in a complementary regression analysis we show that receiving an attack − which was most likely directed at either Haddad or Bolsonaro − increases the likelihood that an individual votes for a third candidate in the first round of elections and, at the same time, raises the probability that she declares an intention to cast a null vote in the second round. Interestingly, we find that these effects are stronger among the more disillusioned groups of voters, particularly those pessimistic about the future of the country.

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Figures and Tables

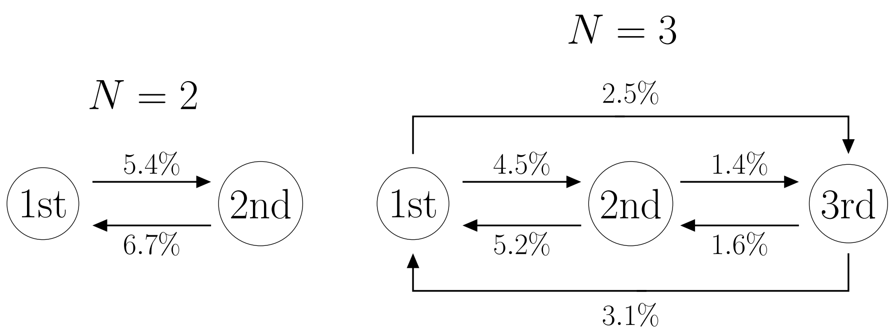


Figure 1: Frequency of Campaign Attacks

Notes: This figure shows the frequency of campaign attacks in races with two (left panel) and three effective candidates (right panel), using a 15% threshold for the definition of effective candidate and focusing only on races held under single ballot plurality.

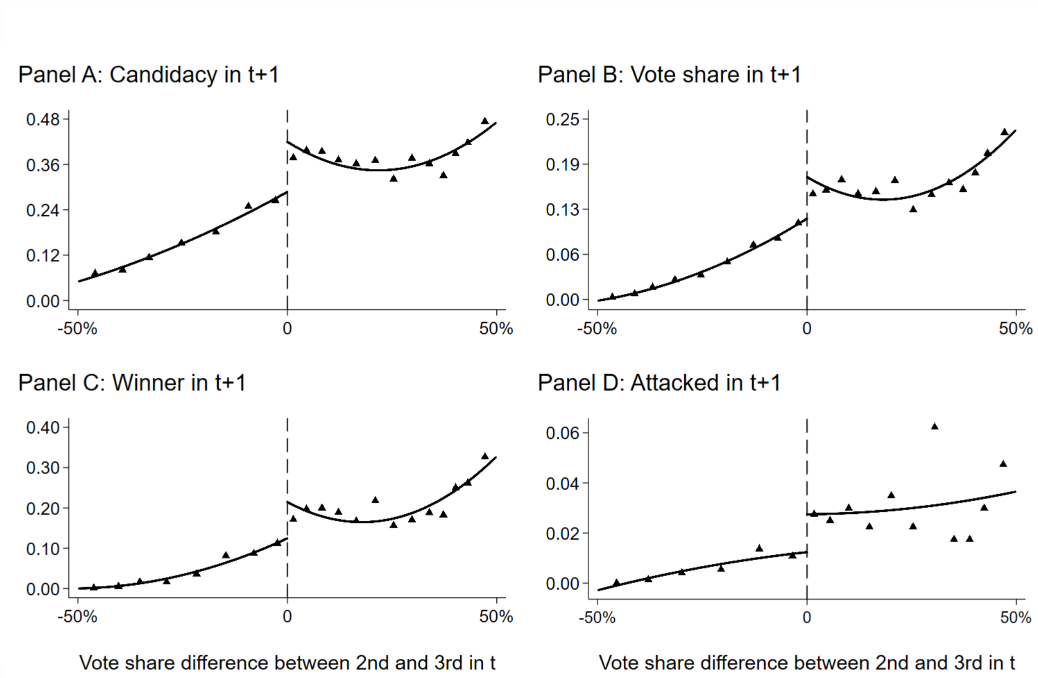


Figure 2: Runner-Up Effect and Electoral Advantage Effect on Campaign Attacks

Notes: This figure plots local averages of the following variables: (i) a dummy indicating whether the candidate ran in the next election at t+1 (panel A), (ii) the candidate's vote share at t+1 (panel B), (iii) a dummy indicating whether the candidate won the election at t+1 (panel C) and (iv) a dummy indicating whether the candidate received a campaign attack at t+1 (panel D). Averages are calculated within quantile-spaced bins of the vote share difference (running variable), where the number of bins is chosen optimally according to Calonico et al. (2014b). A quadratic polynomial is fitted over the original data on each side of the discontinuity. The sample includes only candidates who placed 2nd and 3rd in the election at t, focusing only on races held under single ballot plurality.

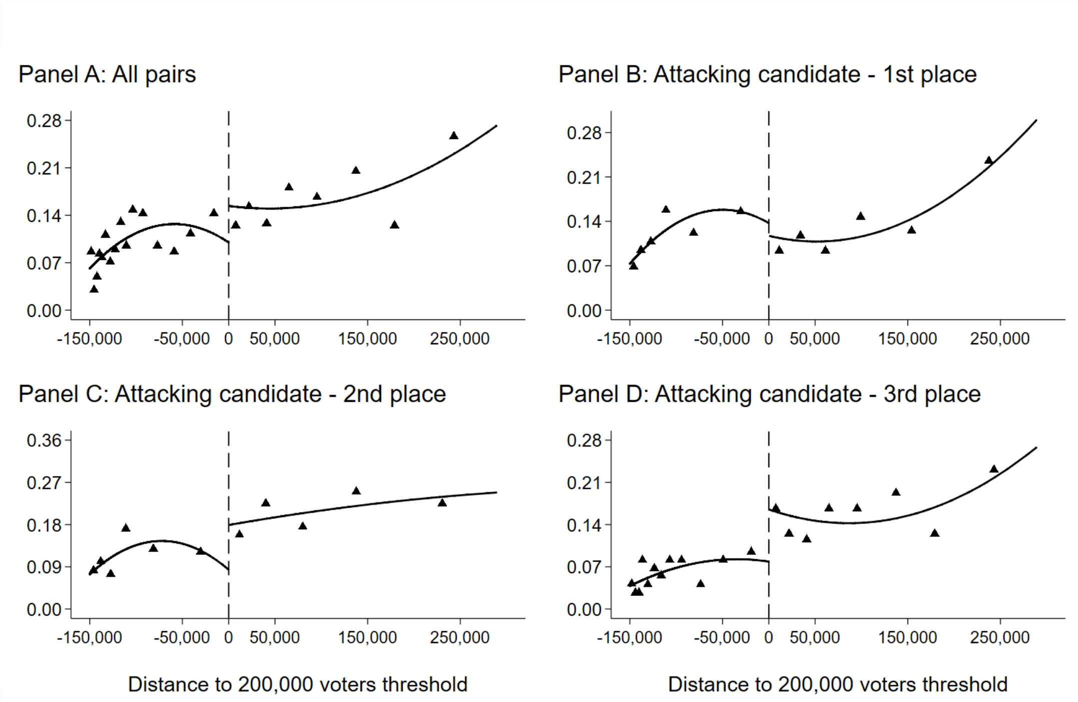


Figure 3: Single vs Dual Ballot Plurality

Notes: This figure plots local averages of a dummy indicating whether a campaign attacked took place between candidates of an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with three or more effective candidates, using a 5% threshold for the definition of effective candidate. Panel A considers all ordered pairs formed by the first three effective candidates. Panel B focuses on a subsample of ordered pairs where the attacking candidate placed 1st; panel C considers a subsample where the attacking candidate placed 2nd; and panel D focuses on a subsample where the attacking candidate placed 3rd. Averages are calculated within quantile-spaced bins of the distance to the 200,000 voters threshold (running variable), where the number of bins is chosen optimally according to Calonico et al. (2014b). A quadratic polynomial is fitted over the original data on each side of the discontinuity.

Table 1: Summary Statistics

	Mean	Std. Dev.	N
Panel A: Municipalities (2010 Census)			
- Population	33,807	203,999	5,323
- Percentage of Urban Population	0.64	0.22	5,323
- Monthly Income per Capita (in Reais)	496.5	243.6	5,323
- Illiteracy Rate	0.16	0.10	5,323
- Poverty Rate	0.11	0.12	5,323
- Gini Index	0.49	0.07	5,323
Panel B: Electoral Races			
- No. Registered Voters	25,804	156,617	10,461
- Turnout Rate	0.86	0.06	10,461
- Percentage of Valid Votes	0.92	0.08	10,461
- Vote Share: Winner (% Valid Votes)	0.55	0.11	10,461
- Vote Share: Runner-up (% Valid Votes)	0.38	0.10	10,461
- No. of Candidates	2.83	1.19	10,461
- No. of Cands. with More than 10% of the Votes	2.23	0.56	10,461
- No. of Cands. with More than 15% of the Votes	2.14	0.46	10,461
Panel C: Candidates			
- Female	0.13	0.34	29,654
- Age	49.2	10.6	29,651
- Married	0.72	0.45	29,654
- High School	0.84	0.37	29,654
- College	0.51	0.50	29,654
- Campaign Expenditures (in Reais)	119,401	522,877	28,559
- Party Filiation: PMDB	0.14	0.35	29,654
- Party Filiation: PSDB	0.11	0.31	29,654
- Party Filiation: PT	0.09	0.29	29,654
Panel D: Ordered Pairs of Candidates			
- Campaign Attack	0.026	0.160	69,252
- Campaign Attack | Races with 2 Cands.	0.054	0.225	10,808
- Campaign Attack | Races with 3 Cands.	0.028	0.165	18,018
- Campaign Attack | Races with 4+ Cands.	0.018	0.134	40,426
- Campaign Attack | Single Ballot	0.025	0.155	62,298
- Campaign Attack | Dual Ballot	0.042	0.201	6,954

Notes: This table reports summary statistics for some of the main variables considered in our analysis. Panel A reports summary statistics for some socioeconomic characteristics of the municipalities in our sample based on data from the 2010 Population Census. Panel B provides descriptive statistics for some selected characteristics of the electoral races in our sample based on information from the Brazil's Tribunal Superior Eleitoral (TSE). Panel C reports summary statistics for some individual characteristics of the candidates in our sample based on information from TSE. Panel D provides descriptive statistics for our sample of ordered pairs of candidates based on information retrived from right of reply lawsuits. The variable "campaign attack" is a dummy indicating whether a RR lawsuit was observed between candidates of an ordered pair. Descriptive statistics for this dummy are reported for the whole sample and for a few selected subsamples.

Table 2: Patterns of Campaign Attacks: Two-Candidate Races

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0121***	0.0123***	0.0118***	0.0110***
	[0.0035]	[0.0033]	[0.0032]	[0.0031]
Effective Candidate Threshold	5%	10%	15%	20%
Candidate Characteristics	Yes	Yes	Yes	Yes
Municipality and Election-Year FEs	Yes	Yes	Yes	Yes
Dep. Var. Mean	0.055	0.059	0.060	0.059
No. of Electoral Races	6,726	7,455	8,015	8,487
Observations	13,452	14,910	16,030	16,974
Adj. R-squared	0.300	0.300	0.290	0.266

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates of an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with two effective candidates and considers only the pairs formed by the first two candidates. In columns $1-4$, we report estimates using four different thresholds for the definition of effective candidate, namely 5%, 10%, 15% and 20%, respectively. Candidate characteristics for each member of the pair include: gender, age, age squared, marital status, high school and college degrees, incumbency status, the logarithm of campaign expenditures and party affiliation to PT, PMDB and PSDB. We also control for a dummy indicating whether the pair was formed by candidates from PT and PSDB. All regressions include municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Table 3: Patterns of Campaign Attacks: Three-Candidate Races

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
1st against 2nd (β_{12})	0.0329***	0.0316***	0.0223***	0.0168**
	[0.0056]	[0.0058]	[0.0063]	[0.0075]
1st against 3rd (β_{13})	0.0022	0.0040	0.0074	0.0057
	[0.0028]	[0.0036]	[0.0046]	[0.0058]
2nd against 1st (β_{21})	0.0406***	0.0364***	0.0296***	0.0220***
	[0.0059]	[0.0061]	[0.0067]	[0.0077]
3rd against 1st (β_{31})	0.0081**	0.0129***	0.0141***	0.0123*
	[0.0040]	[0.0046]	[0.0053]	[0.0067]
3rd against 2nd (β_{32})	−0.0021	0.0009	0.0013	−0.0005
	[0.0032]	[0.0035]	[0.0040]	[0.0050]
Joint Hypothesis Tests (p-values)				
$H_o^1 : \beta_{12} = \beta_{13},\ \beta_{21} = 0,\ \beta_{31} = \beta_{32}$	0.000	0.000	0.000	0.012
$H_o^2 : \beta_{21} = \beta_{12},\ \beta_{21} = \beta_{31}$	0.000	0.000	0.069	0.450
Effective Candidate Threshold	5%	10%	15%	20%
Candidate Characteristics	Yes	Yes	Yes	Yes
Municipality and Election-Year FEs	Yes	Yes	Yes	Yes
Dep. Var. Mean	0.033	0.031	0.031	0.030
No. of Electoral Races	2,467	2,017	1,582	995
Observations	14,202	11,686	9,156	5,766
Adj. R-squared	0.165	0.144	0.144	0.170

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates of an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with three effective candidates and considers only the pairs formed by the first three candidates. In columns 1−4, we report estimates using four different thresholds for the definition of effective candidate, namely 5%, 10%, 15% and 20%, respectively. Candidate characteristics for each member of the pair include: gender, age, age squared, marital status, high school and college degrees, incumbency status, the logarithm of campaign expenditures and party affiliation to PT, PMDB and PSDB. We also control for a dummy indicating whether the pair was formed by candidates from PT and PSDB. All regressions include municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Table 4: Patterns of Campaign Attacks: The Dillution Effect

	Dependent Variable: Campaign Attack			
	(1)	(2)	(3)	(4)
2nd against 1st (β_{21})	0.0364***	0.0434***	0.0278**	0.0224
	[0.0072]	[0.0093]	[0.0126]	[0.0201]
1st against 2nd × Share 3rd (π_{12})	−0.0887	−0.1221	−0.1916*	−0.3393**
	[0.0694]	[0.0803]	[0.1083]	[0.1601]
2nd against 1st × Share 3rd (π_{21})	−0.2321***	−0.3035***	−0.3004***	−0.4148**
	[0.0733]	[0.0853]	[0.1159]	[0.1686]
Effective Candidate Threshold	0%	2%	5%	10%
Candidate Characteristics	Yes	Yes	Yes	Yes
Municipality and Election-Year FEs	Yes	Yes	Yes	Yes
Dep. Var. Mean	0.063	0.064	0.062	0.054
No. of Electoral Races	4,463	3,805	3,021	2,245
Observations	8,926	7,610	6,042	4,490
Adj. R-squared	0.238	0.240	0.267	0.288

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether an attack took place between candidates of an ordered pair. The sample is restricted to include only observations (ordered pairs of candidates) from races with three or more effective candidates and considers only the pairs formed by the first two candidates. In columns 1−4, we report estimates using four different thresholds for the definition of effective candidate, namely 0%, 2%, 5% and 10%, respectively. Candidate characteristics for each member of the pair include: gender, age, age squared, marital status, high school and college degrees, incumbency status, the logarithm of campaign expenditures and party affiliation to PT, PMDB and PSDB. We also control for a dummy indicating whether the pair was formed by candidates from PT and PSDB. All regressions include municipality and election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Table 5: Runner-Up Effect and Electoral Advantage Effect on Campaign Attacks

	Local Linear Regression				Alternative Specifications		
	3rd place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Runner-Up Effect							
Candidacy in $t+1$	0.249	0.162	3,032	0.116***	0.120***	0.120***	0.119***
				[0.034]	[0.033]	[0.040]	[0.031]
Vote Share in $t+1$	0.088	0.178	3,264	0.042***	0.044***	0.041**	0.044***
				[0.014]	[0.015]	[0.018]	[0.014]
Winner in $t+1$	0.094	0.183	3,330	0.066***	0.068***	0.039	0.064***
				[0.024]	[0.024]	[0.034]	[0.023]
Panel B: Campaign Attacks							
Attacked in $t+1$	0.012	0.177	3,240	0.020**	0.021**	0.013	0.017**
				[0.009]	[0.010]	[0.013]	[0.008]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. Each estimate reported in columns 1−4 is obtained from a separate regression; each line corresponds to a different dependent variable. The sample includes only candidates who placed 2nd and 3rd in the election at t, focusing only on races held under single ballot plurality. All specifications use a triangular kernel and include election-year fixed effects. The 3rd place mean corresponds to the mean of the dependent variable to the left of the threshold, calculated based on the sample used in the specification reported in column 1. Additional controls include state fixed effects and the following candidate characteristics: gender, age, age squared, marital status, high school and college degrees, the logarithm of campaign expenditures and party affiliation to PT, PMDB and PSDB. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Table 6: Single vs Dual Ballot Plurality

	Local Linear Regression				Alternative Specifications		
	SB Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Full Sample							
All pairs	0.147	29,552	306	0.216***	0.247***	0.250**	0.151***
				[0.080]	[0.072]	[0.101]	[0.058]
Panel B: By Attacking Candidate							
1st place Candidate	0.158	39,705	136	0.015	0.021	0.020	−0.014
				[0.109]	[0.051]	[0.141]	[0.096]
2nd place Candidate	0.141	40,672	138	0.187	0.214*	0.240	0.224*
				[0.154]	[0.121]	[0.192]	[0.134]
3rd place Candidate	0.095	38,283	134	0.293**	0.419***	0.371**	0.243*
				[0.147]	[0.068]	[0.176]	[0.126]
Bandwidth Method				CCT	CCT	CCT	15,000
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality systems at the $200,000$ voters threshold. The dependent variable is a dummy indicating whether an attack took place between candidates of an ordered pair. Each estimate reported in columns 1−4 is obtained from a separate regression; each line corresponds to a different (sub)sample. The sample is restricted to include only observations (ordered pairs of candidates) from races with three or more effective candidates, using a 5% threshold for the definition of effective candidate. In Panel A, we report estimates for the full sample, where we consider all ordered pairs formed by the first three effective candidates. In Panel B, we report separate estimates for subsamples of pairs where the attacking candidate placed 1st, 2nd and 3rd, as indicated. All specifications use a triangular kernel and include election-year fixed effects. The SB mean corresponds to the mean of the dependent variable to the left of the threshold, calculated based on the sample used in the specification reported in column 1. Additional controls include state fixed effects and the following candidate characteristics for each member of the pair: gender, age, age squared, marital status, high school and college degrees, incumbency status, the logarithm of campaign expenditures and party affiliation to PT, PMDB and PSDB. We also control for a dummy indicating whether the pair was formed by candidates from PT and PSDB. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Online Appendix (NOT FOR PUBLICATION)

A Additional Tables and Figures

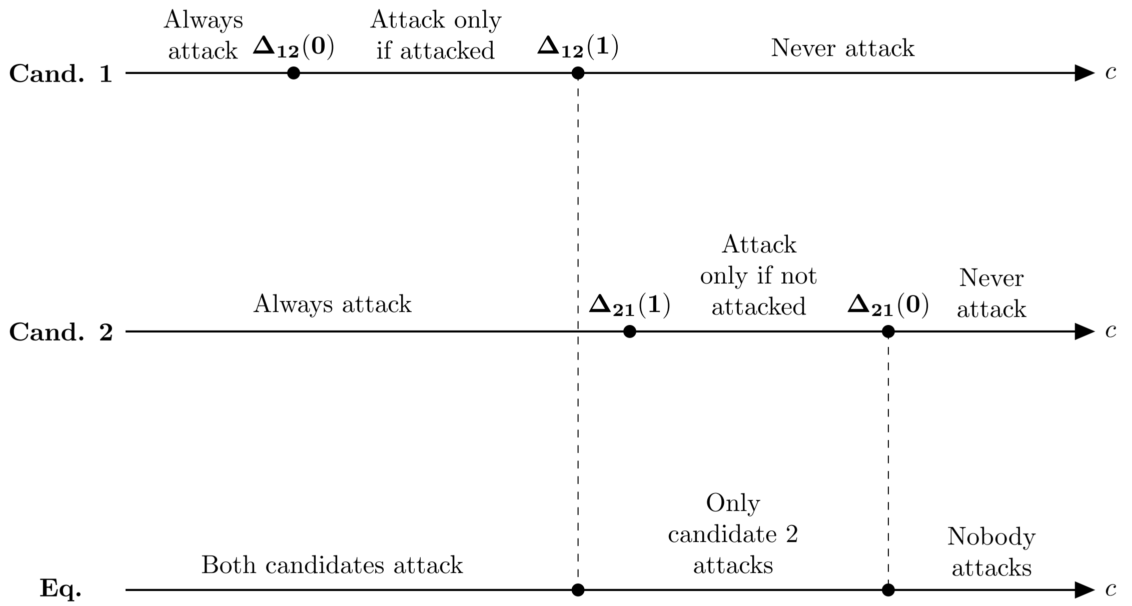


Figure A.1: Two-Candidate Races: Equilibria

Notes: This figure depicts (i) the best-response function of candidate 1, (ii) the best-response function of candidate 2 and (iii) the unique equilibrium existing in each region of the parameters.

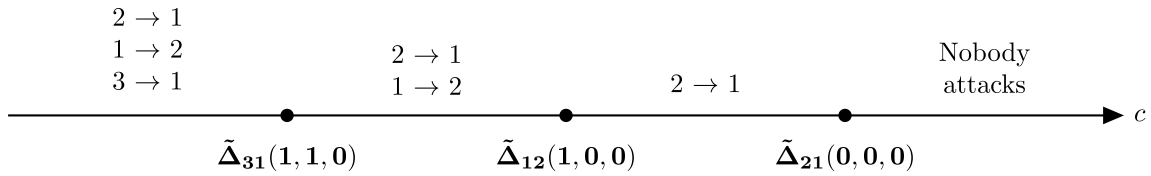


Figure A.2: Three-Candidate Races: Equilibria

Notes: This figure depicts the unique equilibrium existing in each region of the parameters for the case where $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. $i \to j$ denotes an attack from candidate i against j.

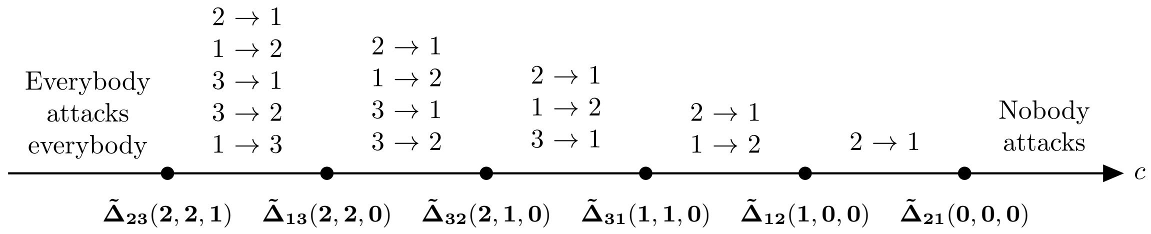


Figure A.3: Three-Candidate Races with Multiple Targets: Equilibria

Notes: This figure depicts the unique equilibrium existing in each region of the parameters for the case where candidates are allowed to target multiple opponents, and supposing that $s_1^o = 0.4$, $s_1^o = 0.3$, $s_1^o = 0.2$ and $\phi = 0.1$. $i \to j$ denotes an attack from candidate i against j.

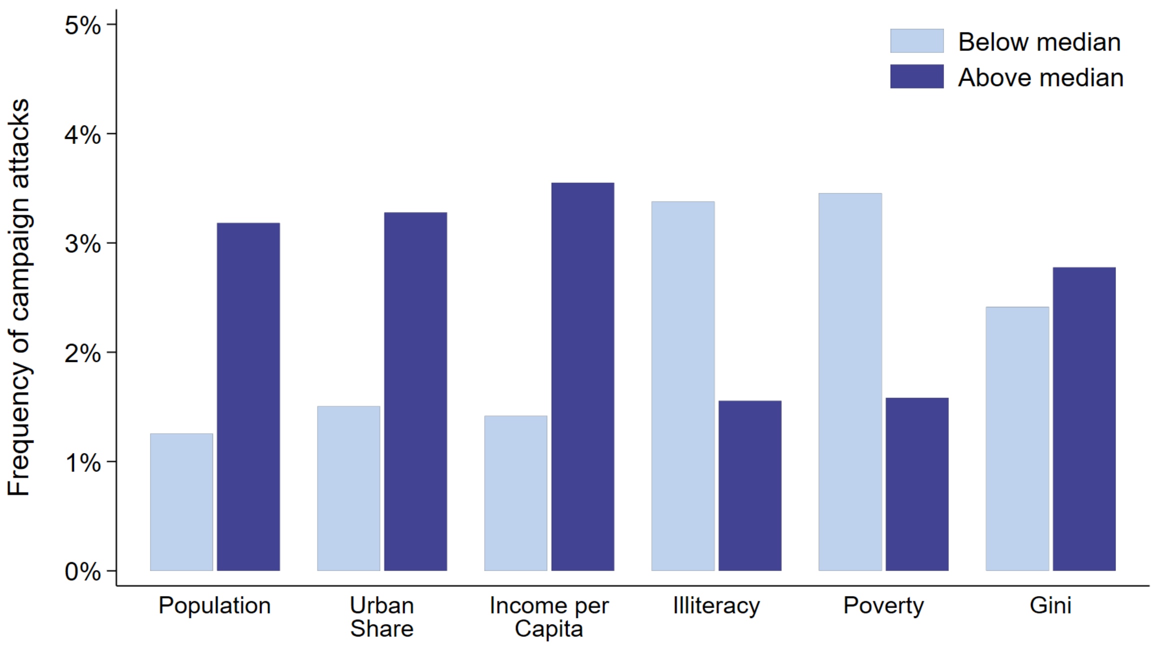


Figure A.4: Frequency of Campaign Attacks and Municipality Characteristics

Notes: This figure shows the frequency of campaign attacks in races held in municipalities below and above the median of the following socioeconomic characteristics (medians reported inside parenthesis): population (10,761), share of urban population (0.65), monthly income per capita (R$ 472.28), illiteracy rate (0.13), poverty rate (0.06) and Gini index (0.49). The reported statistics are computed based on the full sample of ordered pairs of candidates.

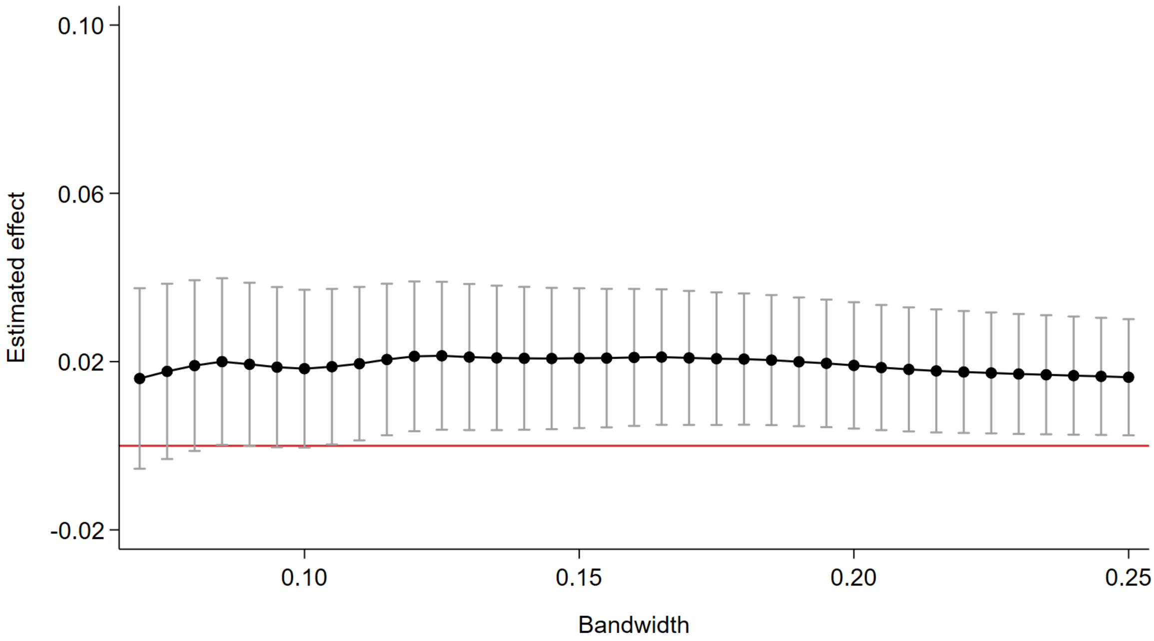


Figure A.5: Electoral Advantage Effect: Robustness to Bandwidth Choice

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidths. The specification is based on a triangular kernel and includes election-year fixed effects. 90% confidence intervals for each estimate are computed based on standard errors clustered at the municipality level. For additional details, see footnote to Table 5.

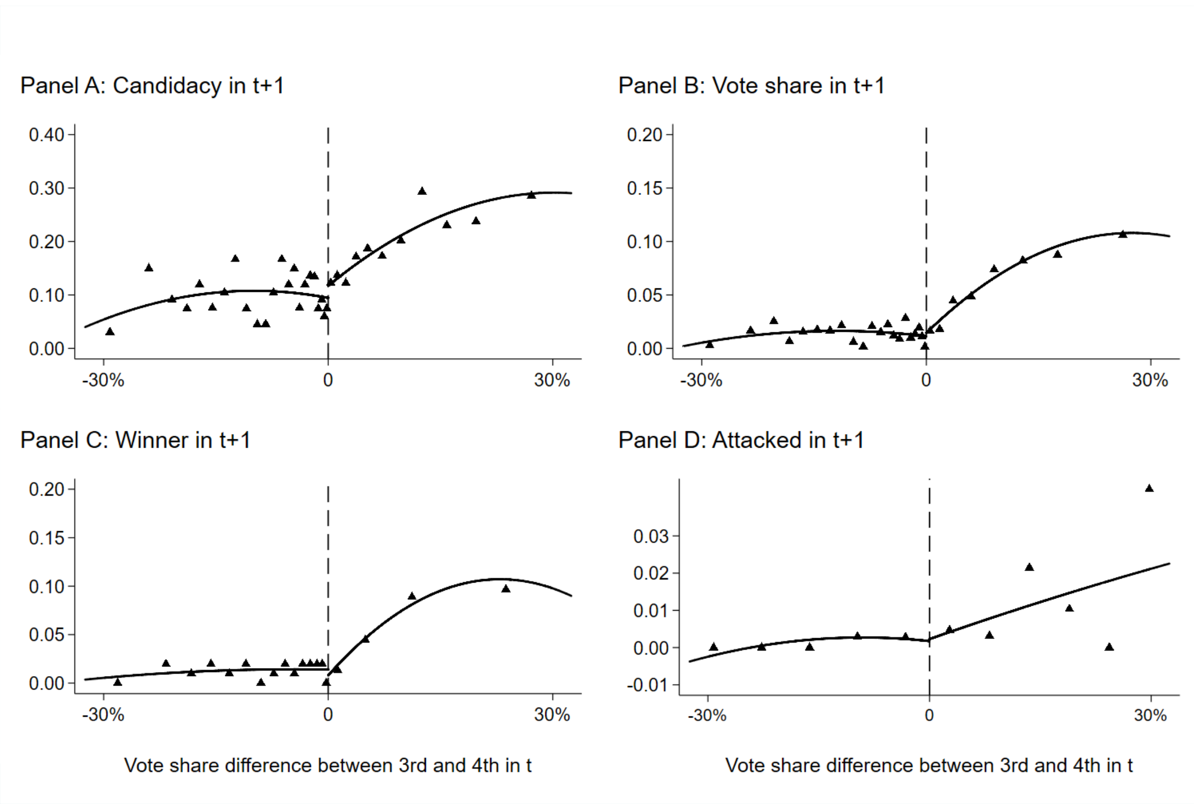


Figure A.6: Electoral Advantage Effect: Placebo Test (3rd vs 4th place)

Notes: This figure plots local averages for candidacy in t+1 (Panel A), vote share in t+1 (Panel B), winner in t+1 (Panel C) and campaign attack in t+1 (Panel D). The sample includes only candidates who placed 3rd and 4th in the election at t, focusing only on races held under single ballot plurality. For additional details, see footnote to Figure 2.

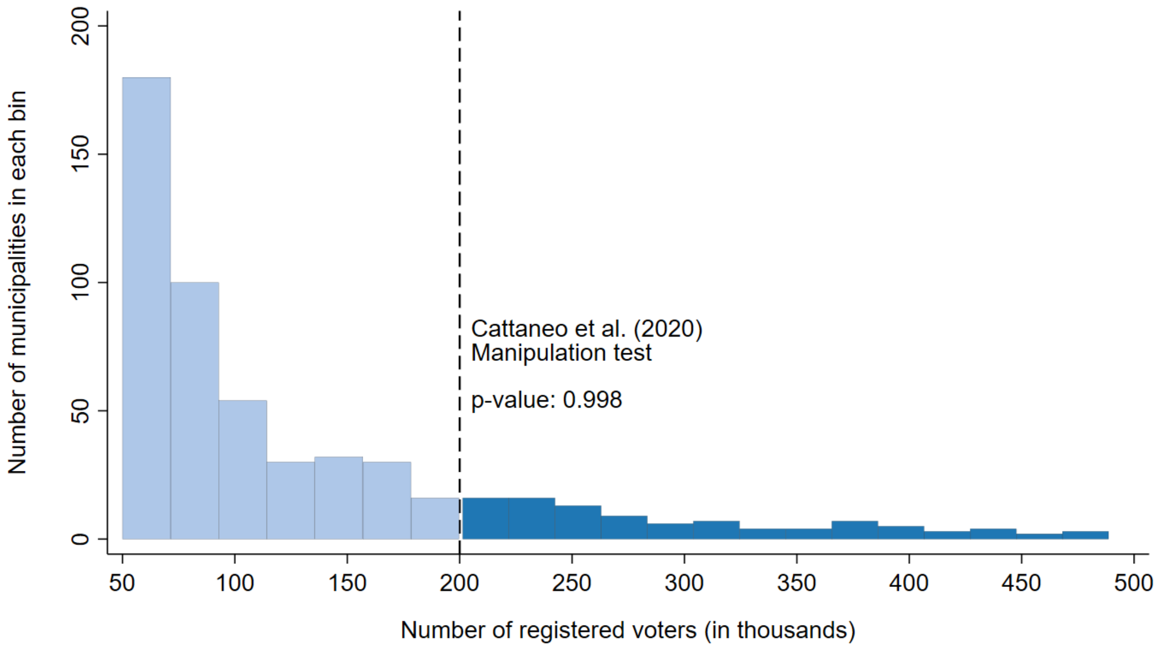


Figure A.7: Distribution of the Number of Registered Voters

Notes: This figure plots the distribution of the number of registered voters, pooling data from the 2012 and 2016 elections. The graph reports the p-value for the manipulation test proposed by Cattaneo et al. (2020).

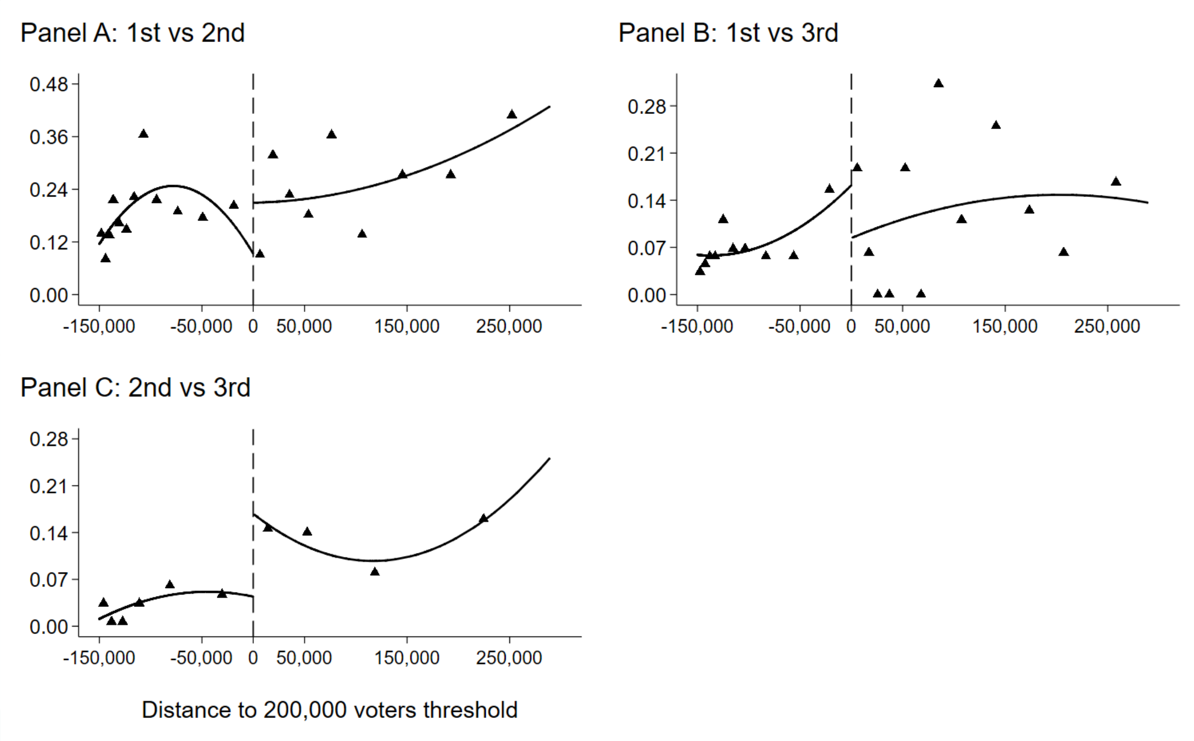


Figure A.8: Single vs Dual Ballot Plurality: Heterogeneous Effects by Pairs

Notes: This figure plots local averages of a dummy indicating whether a campaign attacked took place between candidates of an ordered pair. Panel A considers a subsample of ordered pairs formed by 1st and 2nd placed candidates; panel B focuses on a subsample of pairs formed by 1st and 3rd placed candidates; and panel C considers a subsample of pairs formed by 2nd and 3rd placed candidates. For additional details, see footnote to Figure 3.

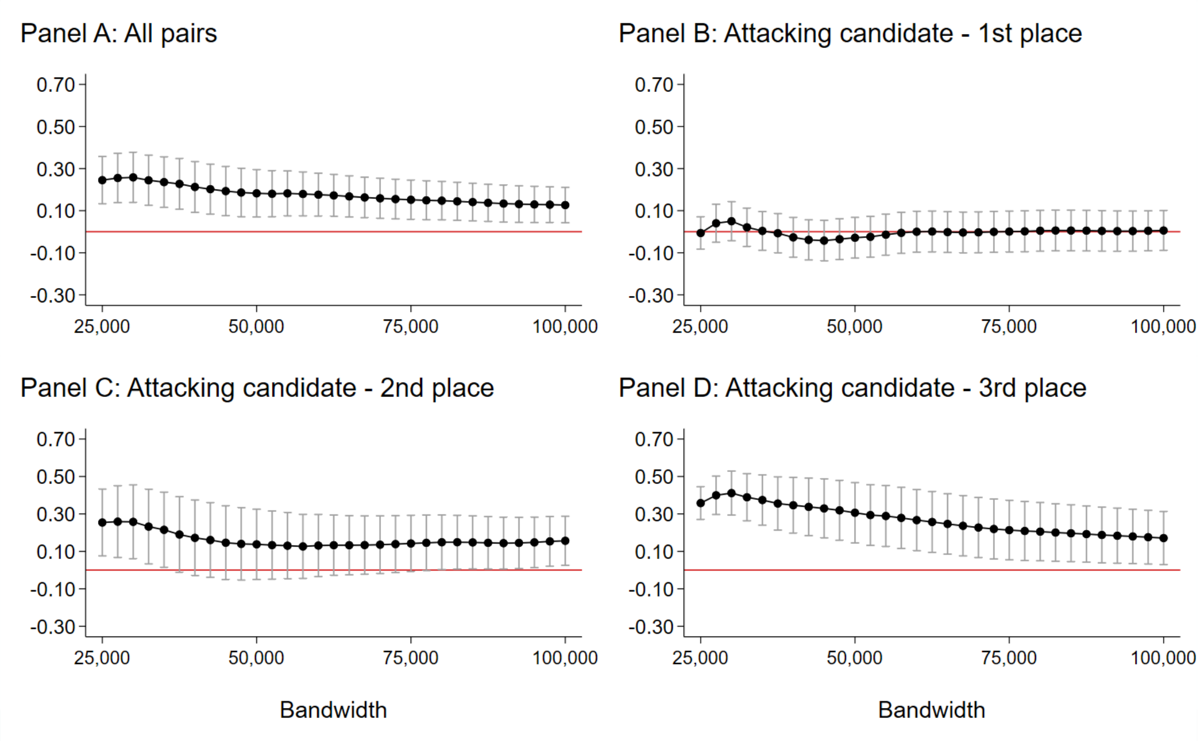


Figure A.9: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice (All Pairs and by Attacking Candidate)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidths. The specification is based on a triangular kernel and includes election-year fixed effects. 90% confidence intervals for each estimate are computed based on standard errors clustered at the municipality level. For additional details, see footnote to Table 6.

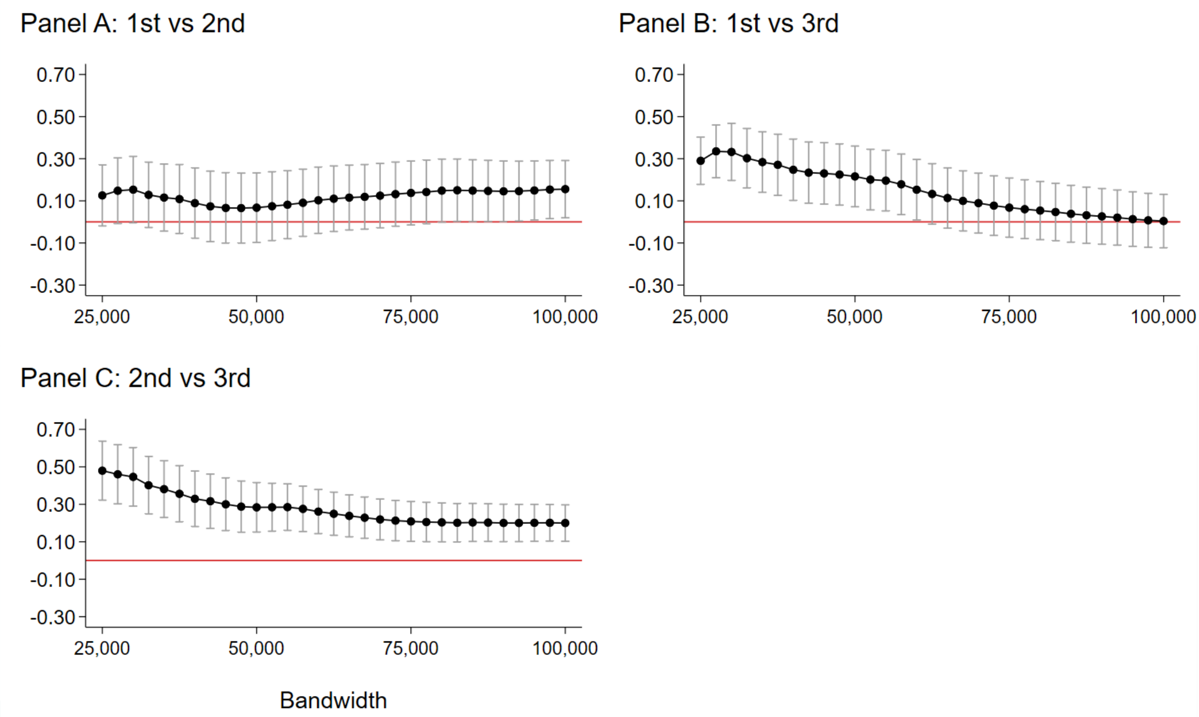


Figure A.10: Single vs Dual Ballot Plurality: Robustness to Bandwidth Choice (By Pairs)

Notes: This figure plots point estimates obtained from separate local linear regressions using various bandwidths. The specification is based on a triangular kernel and includes election-year fixed effects. 90% confidence intervals for each estimate are computed based on standard errors clustered at the municipality level. For additional details, see footnote to Table 6.

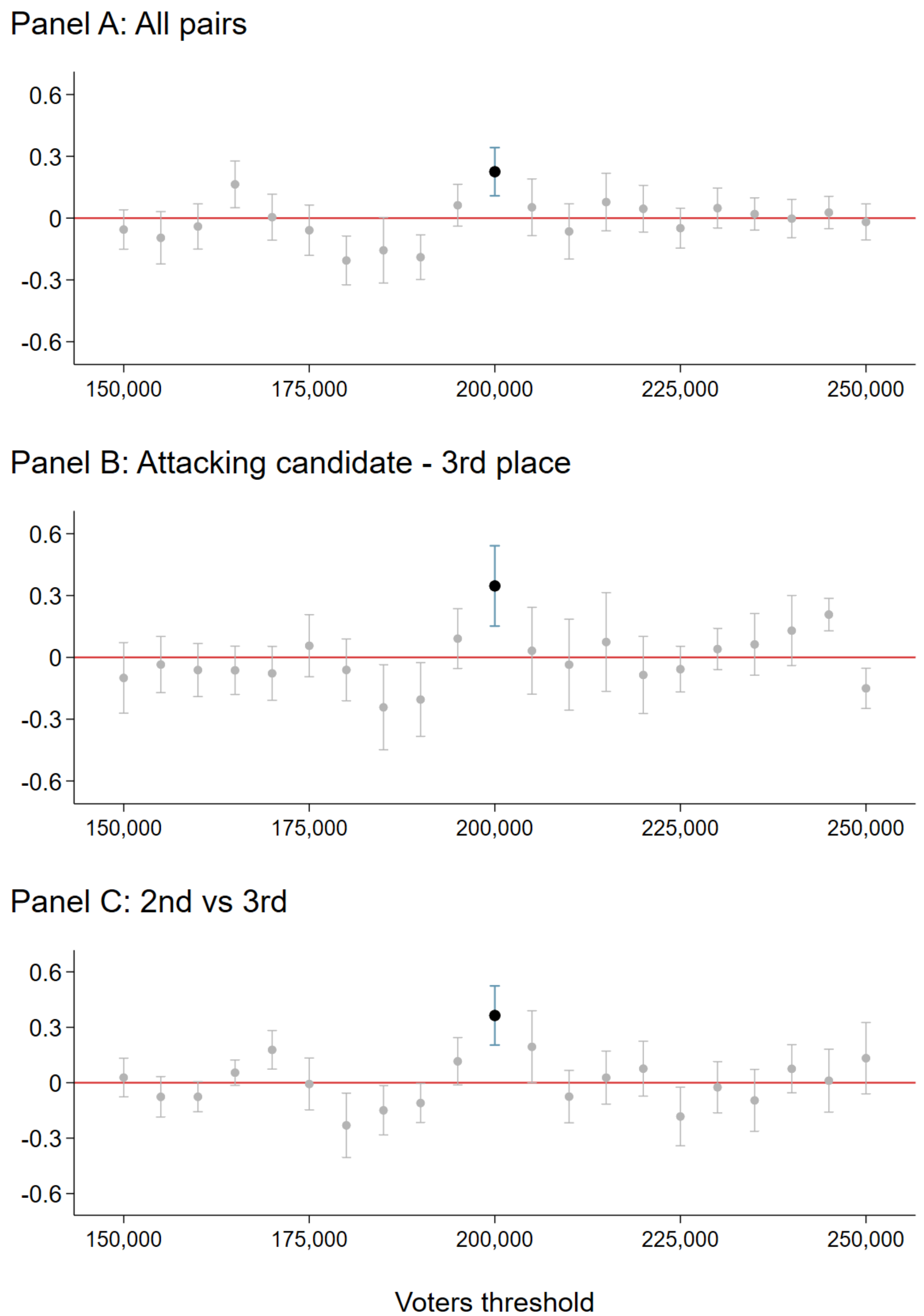


Figure A.11: Single vs Dual Ballot Plurality: Placebo Thresholds

Notes: This figure plots point estimates obtained from separate local linear regressions using false thresholds, above and below the true discontinuity at 200,000 registered voters. The specification is based on a triangular kernel with the Calonico et al. (2014) optimal bandwidth and includes election-year fixed effects. 90% confidence intervals for each estimate are computed based on standard errors clustered at the municipality level. For additional details, see footnote to Table 6.

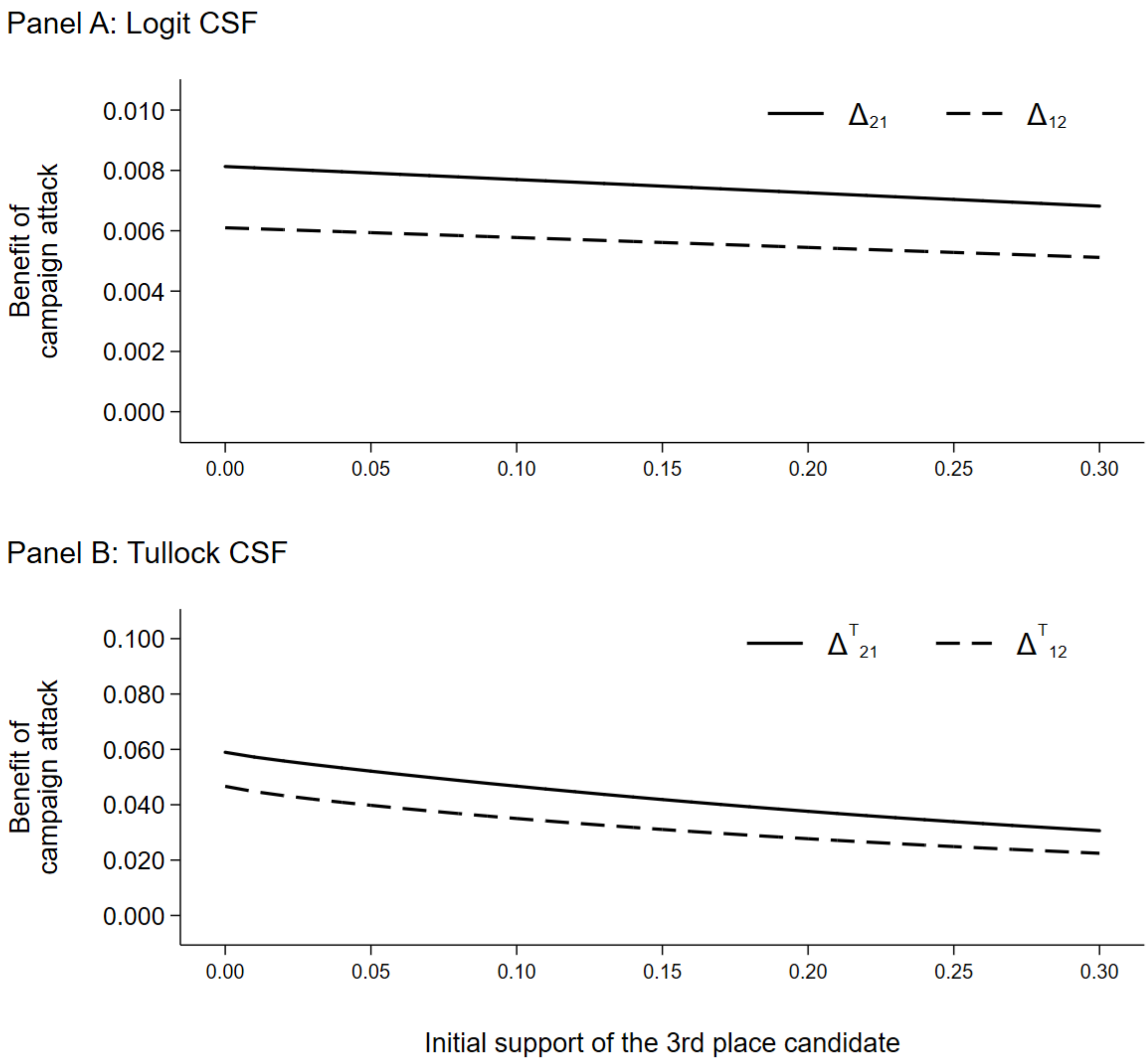


Figure A.12: Simulations: The Dillution Effect

Notes: This figure plots the benefits of campaign attacks between candidates 1 and 2 as a function of the initial support of candidate 3 for both the Logit CSF (Panel A) and the Tullock CSF with $\mu = 1.5$ (Panel B). The simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$ and $\phi = 0.1$, with s_3^o varying in the interval $[0, 0.3)$. The benefit functions are always evaluated at $n = (0, 0, 0)$.

Table A.1: Electoral Advantage Effect: Covariate Smoothness

	Local Linear Regression				Quadratic Specification
	3rd place Mean	Optimal BW	Obs	(1)	(2)
Female	0.127	0.148	2,804	−0.016	−0.009
				[0.025]	[0.034]
Age	49.181	0.177	3,242	0.373	1.019
				[0.754]	[1.110]
Married	0.719	0.126	2,480	−0.016	−0.015
				[0.036]	[0.042]
High School	0.829	0.175	3,218	−0.022	−0.043
				[0.027]	[0.041]
College	0.495	0.151	2,844	0.029	0.018
				[0.038]	[0.050]
Campaign Expenditures (in Reais)	85,588	0.185	3,145	-3,333	-13,287
				[18,613]	[28,823]
Party Filliation: PMDB	0.146	0.181	3,304	−0.030	−0.040
				[0.025]	[0.034]
Party Filliation: PT	0.144	0.141	2,704	0.045*	0.053
				[0.026]	[0.034]
Party Filliation: PSDB	0.103	0.163	3,040	−0.019	−0.028
				[0.022]	[0.026]
Bandwidth Method				CCT	CCT
Polynomial Order				1	2
Additional Controls				No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. All specifications use a triangular kernel and include election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.2: Electoral Advantage Effect: Placebo Test (3rd vs 4th place)

	Local Linear Regression				Alternative Specifications		
	4th place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: 3rd vs 4th Place Effect							
Candidacy in $t+1$	0.105	0.062	1,408	0.041	0.033	0.053	0.037*
				[0.030]	[0.033]	[0.035]	[0.020]
Vote Share in $t+1$	0.014	0.059	1,368	0.001	0.001	0.002	0.009**
				[0.006]	[0.007]	[0.007]	[0.005]
Winner in $t+1$	0.016	0.057	1,348	−0.009	−0.011	−0.009	0.003
				[0.009]	[0.011]	[0.011]	[0.007]
Panel B: Campaign Attacks							
Attacked in $t+1$	0.002	0.043	1,076	0.000	−0.000	−0.000	0.001
				[0.002]	[0.002]	[0.003]	[0.002]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 3rd and 4th place candidates. All specifications use a triangular kernel and include election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.3: Electoral Advantage Effect: Placebo Test (Attacked in t)

	Local Linear Regression				Alternative Specifications		
	3rd place Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Attacked in t	0.050	0.156	1,514	0.001	0.010	0.005	0.011
				[0.024]	[0.021]	[0.029]	[0.021]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. The dependent variable is a dummy indicating whether the candidate received a campaign attack in the election at t (i.e. *same election*). All specifications use a triangular kernel. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.4: Electoral Advantage Effect: Heterogeneous Effects

	Local Linear Regression				Alternative Specifications		
	3rd place mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
Panel A: Strong Runner-Ups ($s_2 + s_3 \geq s_1$)							
Candidacy in $t+1$	0.299	0.116	1,614	0.137***	0.138***	0.133**	0.147***
				[0.049]	[0.047]	[0.058]	[0.038]
Winner in $t+1$	0.123	0.105	1,512	0.064	0.075*	0.058	0.085***
				[0.040]	[0.038]	[0.043]	[0.030]
Attacked in $t+1$	0.012	0.127	1,758	0.033**	0.034**	0.025	0.029***
				[0.013]	[0.013]	[0.016]	[0.011]
Panel B: Weak Runner-Ups ($s_2 + s_3 < s_1$)							
Candidacy in $t+1$	0.168	0.153	858	0.043	0.038	0.053	0.047
				[0.051]	[0.056]	[0.058]	[0.047]
Winner in $t+1$	0.056	0.139	792	0.016	0.012	0.018	0.009
				[0.033]	[0.033]	[0.039]	[0.029]
Attacked in $t+1$	0.012	0.170	954	−0.012	−0.011	−0.018	−0.011
				[0.011]	[0.013]	[0.019]	[0.011]
Bandwidth Method				CCT	CCT	CCT	0.05
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting virtual ties between 2nd and 3rd place candidates. Panel A reports estimates obtained from a subsample of candidates where the combined vote shares of 2nd and 3rd place candidates were larger than the vote share of the winner in the election at t ($s_2 + s_3 \geq s_1$). Panel B, in turn, reports estimates obtained from a subsample where the combined vote shares of 2nd and 3rd place candidates were smaller than the vote share of the winner in the election at t ($s_2 + s_3 < s_1$). All specifications use a triangular kernel and include election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 5.

Table A.5: Single vs Dual Ballot Plurality: Covariate Smoothness

	Local Linear Regression				Quadratic Specification
	SB Mean	Optimal BW	Obs	(1)	(2)
Latitude	−18.432	49,605	52	−4.375	−3.824
				[4.488]	[5.283]
Longitude	−46.763	52,301	53	0.935	−4.734
				[3.701]	[4.634]
Percentage of Urban Population	0.951	66,979	75	0.022	0.011
				[0.031]	[0.040]
Monthly Income per Capita (in Reais)	780.45	49,503	52	123.40	136.59
				[157.16]	[186.55]
Illiteracy Rate	6.380	56,977	65	0.535	−0.341
				[2.049]	[2.543]
Poverty Rate	2.672	67,880	78	−1.958	−2.486
				[1.545]	[2.000]
Gini Index	0.506	49,009	52	−0.015	−0.016
				[0.031]	[0.038]
Bandwidth Method				CCT	CCT
Polynomial Order				1	2
Additional Controls				No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality systems at the 200,000 voters threshold. The unit of observation is a municipality. The sample is restricted to include only municipalities considered in the main analysis reported in Table 6, i.e. those that had races with three or more effective candidates in the 2012 or 2016 elections. All socioeconomic characteristics were taken from the 2010 Population Census. All specifications use a triangular kernel and include election-year fixed effects. Robust standard errors are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.6: Single vs Dual Ballot Plurality: Heterogeneous Effects by Pairs

	Local Linear Regression				Alternative Specifications		
	SB Mean	Optimal BW	Obs	(1)	(2)	(3)	(4)
1st vs 2nd	0.207	34,374	118	0.143	0.140	0.198	0.090
				[0.098]	[0.091]	[0.148]	[0.075]
1st vs 3rd	0.139	37,762	132	0.135	0.273***	0.184	0.118
				[0.165]	[0.087]	[0.182]	[0.146]
2nd vs 3rd	0.052	44,889	162	0.236***	0.356***	0.320***	0.244***
				[0.086]	[0.090]	[0.111]	[0.074]
Bandwidth Method				CCT	CCT	CCT	15,000
Polynomial Order				1	1	2	0
Additional Controls				No	Yes	No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality systems at the 200,000 voters threshold. All specifications use a triangular kernel and include election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.7: Single vs Dual Ballot Plurality: Alternative Mechanisms

	Local Linear Regression				Alternative Specifications
	SB Mean	Optimal BW	Obs	(1)	(2)
Turnout	0.827	65,128	126	−0.010	−0.018
				[0.018]	[0.024]
No. of Candidates	5.045	42,298	76	0.356	−0.509
				[0.693]	[0.918]
Max. Cand. Campaign Expenditure (in Reais)	1,521,154	28,538	47	284,038	406,963
				[327,067]	[430,463]
Avg. Cand. Campaign Expenditure (in Reais)	558,000	30,991	56	45,482	108,746
				[133,529]	[168,202]
Vote Share: 1st placed Candidate	0.482	68,944	133	0.014	0.005
				[0.051]	[0.079]
Vote Share: 2nd placed Candidate	0.289	65,652	127	0.007	−0.008
				[0.025]	[0.037]
Vote Share: 3rd placed Candidate	0.145	67,173	128	−0.015	−0.015
				[0.022]	[0.030]
Vote Share: 3rd and Lower placed Candidates	0.227	76,150	143	−0.007	−0.016
				[0.043]	[0.058]
Vote Share: 4th and Lower placed Candidates	0.080	69,852	135	0.009	0.013
				[0.031]	[0.052]
Bandwidth Method				CCT	CCT
Polynomial Order				1	2
Additional Controls				No	No

Notes: This table reports RDD estimates of local polynomial regressions exploiting the discontinuous assignment of single and dual ballot plurality systems at the 200,000 voters threshold. The unit of observation is an electoral race. The sample is restricted to include only races considered in the main analysis reported in Table 6, i.e. those with three or more effective candidates. All specifications use a triangular kernel and include election-year fixed effects. Standard errors clustered at the municipality level are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively. For additional details, see footnote to Table 6.

Table A.8: Simulations: Two-Candidate Races

	Logit CSF			Tullock CSF					
				$\mu = 1.5$			$\mu = 2$		
	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$
$\Delta_{12}(0)$	0.0098	0.0197	0.0294	0.0282	0.0557	0.0825	0.0346	0.0673	0.0979
$\Delta_{12}(1)$	0.0099	0.0199	0.0299	0.0290	0.0588	0.0891	0.0370	0.0765	0.1173
$\Delta_{21}(0)$	0.0149	0.0298	0.0488	0.0433	0.0876	0.1325	0.0549	0.1128	0.1723
$\Delta_{21}(1)$	0.0148	0.0296	0.0433	0.0425	0.0845	0.1259	0.0525	0.1036	0.1529

Notes: This table reports the benefits of campaign attacks in two-candidate races under both Logit and Tullock CSFs. The simulations assume that $s_1^o = 0.6$ and $s_2^o = 0.4$, and we consider $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$.

Table A.9: Simulations: Three-Candidate Races

	Logit CSF			Tullock CSF					
				$\mu = 1.5$			$\mu = 2$		
	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$	$\phi = 0.05$	$\phi = 0.10$	$\phi = 0.15$
$\widetilde{\Delta}_{12}(0,0,0)$	0.0027	0.0054	0.0081	0.0140	0.0277	0.0409	0.0202	0.0394	0.0574
$\widetilde{\Delta}_{13}(0,0,0)$	0.0016	0.0032	0.0049	0.0071	0.0140	0.0206	0.0078	0.0150	0.0216
$\widetilde{\Delta}_{21}(0,0,0)$	0.0036	0.0072	0.0108	0.0188	0.0376	0.0563	0.0273	0.0548	0.0822
$\widetilde{\Delta}_{23}(0,0,0)$	0.0014	0.0029	0.0044	0.0066	0.0131	0.0195	0.0069	0.0136	0.0200
$\widetilde{\Delta}_{31}(0,0,0)$	0.0032	0.0065	0.0098	0.0147	0.0297	0.0449	0.0170	0.0346	0.0527
$\widetilde{\Delta}_{32}(0,0,0)$	0.0022	0.0044	0.0066	0.0099	0.0199	0.0299	0.0103	0.0207	0.0310

Notes: This table reports the benefits of campaign attacks in three-candidate races under both Logit and Tullock CSFs. The simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$ and $s_3^o = 0.2$, and we consider $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$. The benefit functions are always evaluated at $n = (0,0,0)$.

Table A.10: Simulations: Single vs Dual Ballot Plurality

	Logit CSF			Tullock CSF ($\mu = 1.5$)		
	Single Ballot	Dual Ballot	Variation (%)	Single Ballot	Dual Ballot	Variation (%)
$\widetilde{\Delta}_{12}(0,0,0)$	0.0054	0.0057	+0.05	0.0277	0.0114	−0.58
$\widetilde{\Delta}_{13}(0,0,0)$	0.0032	0.0041	+0.28	0.0140	0.0189	+0.35
$\widetilde{\Delta}_{21}(0,0,0)$	0.0072	0.0077	+0.06	0.0376	0.0139	−0.63
$\widetilde{\Delta}_{23}(0,0,0)$	0.0029	0.0044	+0.51	0.0131	0.0354	+1.70
$\widetilde{\Delta}_{31}(0,0,0)$	0.0065	0.0083	+0.27	0.0297	0.0428	+0.44
$\widetilde{\Delta}_{32}(0,0,0)$	0.0044	0.0067	+0.52	0.0199	0.0533	+1.67

Notes: This table reports the benefits of campaign attacks in three-candidate races under single and dual ballot plurality for both the Logit CSF and the Tullock CSF with $\mu = 1.5$. The simulations assume that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$. The benefit functions are always evaluated at $n = (0,0,0)$.

B Model: Extensions and Robustness Checks

B.1 Additional Results

This subsection provides a number of additional results which are intended to complement our main analysis. The following proposition describes how the relationship between $\widetilde{\Delta}_{12}(n)$ and $\widetilde{\Delta}_{31}(n)$ depend on the initial level of support of the candidates.

Proposition B.1. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ and s_2^o is sufficiently close to s_1^o then:*

$$\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

Alternatively, if $\phi < \overline{\phi}$ and s_2^o is sufficiently close to s_3^o then:

$$\widetilde{\Delta}_{31}(n_3) > \widetilde{\Delta}_{12}(n_1) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

Next, for completeness, we extend the characterization of the benefit function $\widetilde{\Delta}_{ij}(n)$ to all remaining pairs of candidates.

Proposition B.2. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then we have:*

i. The benefit of an attack of candidate 1 on 2 is larger than that of an attack of 3 on 2:

$$\widetilde{\Delta}_{32}(n_3) < \widetilde{\Delta}_{12}(n_1) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

ii. The benefit of an attack of candidate 3 on 1 is larger than that of an attack of 1 on 3:

$$\widetilde{\Delta}_{13}(n_1) < \widetilde{\Delta}_{31}(n_3) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

iii. The benefit of an attack of candidate 1 on 3 is larger than that of an attack of 2 on 3:

$$\widetilde{\Delta}_{23}(n_2) < \widetilde{\Delta}_{13}(n_1) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

iv. The benefit of an attack of candidate 3 on 2 is larger than that of an attack of 2 on 3:

$$\widetilde{\Delta}_{23}(n_2) < \widetilde{\Delta}_{32}(n_3) \qquad \textit{for any } n_i \in \mathcal{N}_i$$

The following corollary combines the results derived in Propositions 3 and B.2, summarizing our main results in a more concise way.

Corollary B.1. *From Propositions 3 and B.2, it follows that there exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then:*

$$\widetilde{\Delta}_{23}(n_2) < \{\widetilde{\Delta}_{13}(n_1), \widetilde{\Delta}_{32}(n_3)\} < \{\widetilde{\Delta}_{12}(n_1^{'}), \widetilde{\Delta}_{31}(n_3^{'})\} < \widetilde{\Delta}_{21}(n_2^{'}) \qquad \textit{for any } n_i, n_i^{'} \in \mathcal{N}_i$$

Interestingly, our analysis shows that while the most likely direction of attack is from candidate 2 against 1, the least likely direction is from 2 against 3. Moreover, note that both the relationship between $\widetilde{\Delta}_{12}(n)$ and $\widetilde{\Delta}_{31}(n)$, as discussed in the main text (see also Proposition B.1), and that between $\widetilde{\Delta}_{13}$ and $\widetilde{\Delta}_{32}$ are ambiguous. With respect to the latter, it is possible to show that if s_2^o is sufficiently close to

s_1^o, then $\widetilde{\Delta}_{13}(n) < \widetilde{\Delta}_{32}(n)$; whereas if s_2^o is sufficiently close to s_3^o, then the opposite holds. Intuitively, everything else constant, a larger initial support s_2^o increases the benefit of an attack against candidate 2. The following proposition provides a formal statement for this result.

Proposition B.3. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ and s_2^o is sufficiently close to s_1^o, then:*

$$\widetilde{\Delta}_{13}(n_1) < \widetilde{\Delta}_{32}(n_3) \qquad \text{for any } n_i \in \mathcal{N}$$

Conversely, if $\phi < \overline{\phi}$ and s_2^o is sufficiently close to s_3^o, then:

$$\widetilde{\Delta}_{13}(n_1) > \widetilde{\Delta}_{32}(n_3) \qquad \text{for any } n_i \in \mathcal{N}$$

Next, based on the properties of the function $\widetilde{\Delta}_{ij}(n)$ derived in Proposition 3, the following result provides a complete characterization of the unique equilibrium of the game with three candidates.[57,58]

Proposition B.4. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ then:*

i. First, suppose that $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. There exists a unique Nash equilibrium with the following characteristics:

- *i.a Candidates 2 and 3 attack 1 and candidate 1 attacks 2 if, and only if, $c \leq \widetilde{\Delta}_{31}(1,1,0)$.*
- *i.b Candidate 2 attacks 1 and candidate 1 attacks 2 if, and only if, $\widetilde{\Delta}_{31}(1,1,0) < c \leq \widetilde{\Delta}_{12}(1,0,0)$.*
- *i.c Candidate 2 attacks 1 if, and only if, $\widetilde{\Delta}_{12}(1,0,0) < c \leq \widetilde{\Delta}_{21}(0,0,0)$.*
- *i.d No candidate attacks if, and only if, $\widetilde{\Delta}_{21}(0,0,0) < c$.*

ii. Second, suppose that $\widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. There exists a unique Nash equilibrium with the following characteristics:

- *ii.a Candidates 2 and 3 attack 1 and candidate 1 attacks 2 if, and only if, $c \leq \widetilde{\Delta}_{12}(2,0,0)$.*
- *ii.b Candidate 2 and 3 attack 1 if, and only if, $\widetilde{\Delta}_{12}(2,0,0) < c \leq \widetilde{\Delta}_{31}(1,0,0)$.*
- *ii.c Candidate 2 attacks 1 if, and only if, $\widetilde{\Delta}_{31}(1,0,0) < c \leq \widetilde{\Delta}_{21}(0,0,0)$.*
- *ii.d No candidate attacks if, and only if, $\widetilde{\Delta}_{21}(0,0,0) < c$.*

Figure A.2 depicts the regions of parameters where each class of equilibrium exists for the case where $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ (Proposition B.4, item *i*). Note that, as the cost of attacking increases, we move through four different regions where the following equilibria exist: (*a*) an equilibrium where 2 and 3 attack 1 and 1 attacks 2, (*b*) an equilibrium where 2 attacks 1 and 1 attacks 2, (*c*) an equilibrium where 2 attacks 1, and (*d*) an equilibrium where nobody attacks.

Next, regarding the comparison between races with two and three candidates, the following proposition shows that the benefit of an attack between candidates 1 and 2 are always smaller in three-candidate races relatively to two-candidate races, as a result of the "dillution effect".

Proposition B.5. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ then:*

$$\Delta_{12}(n_1) > \widetilde{\Delta}_{12}(n_1^{'}) \qquad \text{for any } n_1 \in \{0,1\} \text{ and } n_1^{'} \in \mathcal{N}_1$$

and

$$\Delta_{21}(n_2) > \widetilde{\Delta}_{21}(n_2^{'}) \qquad \text{for any } n_2 \in \{0,1\} \text{ and } n_2^{'} \in \mathcal{N}_2$$

[57] Note that from Proposition 3, item *iii* it follows that there are two cases to consider: (*i*) $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ and (*ii*) $\widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{21}(n_2)$.

[58] As a reminder, note that $\widetilde{\Delta}_{ij}(n_1,n_2,n_3)$ represents candidate *i*'s benefit of attacking *j* when candidate 1 receives n_1 attacks, candidate 2 receives n_2 attacks and candidate 3 receives n_3 attacks. Thus, for example, the condition $c \leq \widetilde{\Delta}_{31}(1,1,0)$ implies that candidate 3 is willing to attack 1 when $n_1 = 1$, $n_2 = 1$ and $n_3 = 0$.

We next show that under dual ballot plurality candidates 2 and 3 might prefer to attack each other rather than the front-runner − a situation like that would never occur under single ballot plurality in equilibrium (see Proposition 3, items *ii* and *iii*). The following proposition characterizes the conditions under which this might happen.

Proposition B.6. *There exist a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ and s_1^o is sufficiently large, then under dual ballot we have:*

i. For candidate 2, the benefit of an attack on 3 is larger than that of an attack on 1:

$$\widetilde{\Delta}_{21}^{DB}(n) < \widetilde{\Delta}_{23}^{DB}(n^{'}) \qquad \text{for any } n, n^{'} \in \mathcal{N}_2$$

ii. For candidate 3, the benefit of an attack on 2 is larger than that of an attack on 1:

$$\widetilde{\Delta}_{31}^{DB}(n) < \widetilde{\Delta}_{32}^{DB}(n^{'}) \qquad \text{for any } n, n^{'} \in \mathcal{N}_3$$

Intuitively, if the front-runner is sufficiently ahead of the other candidates, then it pays for 2 and 3 to attack each other in an attempt to secure a place in the second round. Finally, we show that all three candidates might become more aggressive under dual ballot plurality if the race is sufficiently competitive in the sense that the initial supports of candidates are close enough.

Proposition B.7. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ and the distance $s_1^o - s_3^o > 0$ is sufficiently small, then for any i and j with $i \neq j$ we have:*

$$\widetilde{\Delta}_{ij}(n_i) < \widetilde{\Delta}_{ij}^{DB}(n_i^{'}) \quad \text{for any } n_i, n_i^{'} \in \mathcal{N}_i$$

B.2 Voter Demobilization

In this subsection, we extend our basic model to consider an alternative setting where campaign attacks lead to voter demobilization. Our main goal is to incorporate in a reduced form fashion the idea that campaign attacks and mudslinging may generate voter apathy and mistrust in politics (Ansolabehere and Iyengar (1995)). To do so, we allow voters to cast a null vote or simply choose not to vote, denoting this alternative (outside option) by $\emptyset$ with $s_{\emptyset}^o > 0$. Our proposed framework captures two sources of voter demobilization while maintaining the assumption that the overall effect of an attack is given by $\phi \in (0,1)$. First, we suppose that a fraction $\alpha\phi$ of voters of the attacked candidate become demobilized and switch to not supporting anyone, while a fraction $(1-\alpha)\phi$ move to other candidates, with $\alpha \in [0,1]$. Second, we assume that a fraction $\gamma\phi$ of voters of the attacking candidate also become disengaged and switch to not voting (i.e. "backlash effect"), with $\gamma \in [0,1]$.

Let $a_{ij} \in \{0,1\}$ denote candidate i's binary decision to attack j. In two-candidate races the final support of candidate i is now given by:

$$x_i(a_{ij}, a_{ji}) = (1 - \phi a_{ji})s_i^o + a_{ij}((1-\alpha)\phi s_j^o - \gamma\phi s_i^o) + \epsilon_i$$

Thus, on the one hand, if candidate i receives an attack from j, she loses a fraction ϕ of her initial support. On the other hand, if candidate i attacks j, she is able to steal a fraction $(1-\alpha)\phi$ of j's initial support, but at the same time loses a fraction $\gamma\phi$ of her own support due to the backlash effect. The overall level of voter demobilization is thus given by:

$$x_{\emptyset}(a_{12}, a_{21}) = s_{\emptyset}^o + a_{12}\phi(\alpha s_2^o + \gamma s_1^o) + a_{21}\phi(\alpha s_1^o + \gamma s_2^o) + \epsilon_{\emptyset}$$

Finally, the probability that a candidate i wins the election is still given by expression (2) while the share of abstentions and null votes is:

$$p_{\emptyset}(a_{12}, a_{21}) = \frac{\exp(s_{\emptyset}(a_{12}, a_{21}))}{\exp(s_1(a_{12}, a_{21})) + \exp(s_2(a_{12}, a_{21}) + \exp(s_{\emptyset}(a_{12}, a_{21})))}$$

Note that this general version of the model reduces to our basic setup when $\alpha = 0$ and $\gamma = 0$.

Similarly, in three-candidate races the final level of support of candidate i is given by:

$$x_i(a) = (1 - \phi n_i)s_i^o + \textstyle\sum_{j \neq i} \frac{n_j(1-\alpha)\phi s_j^o}{2} - (\sum_{j \neq i} a_{ij})\gamma\phi s_i^o + \epsilon_i$$

where n_i denotes the number of attacks received by candidate i, while $\sum_{j \neq i} a_{ij}$ captures whether i attacked one of her opponents. In this case, the overall level of voter demobilization can be written as:

$$x_{\emptyset}(a) = s_{\emptyset}^o + \textstyle\sum_i \sum_{j \neq i} a_{ij}\phi(\alpha s_j^o + \gamma s_i^o) + \epsilon_{\emptyset}$$

Moreover, the probability that a candidate i wins the election under single ballot plurality is still given by expression (6). As before, our basic setup is obtained when $\alpha = 0$ and $\gamma = 0$.

We conduct our analysis in two parts. First, we consider a version of the model without backlash effects, $\gamma = 0$. Interestingly, in this case, we are able to show that all results go through for any $\alpha \in [0, 1]$.[59] Thus, it follows that our main conclusions remain unchanged even if an attack serves only to demobilize voters of rival candidates. In particular, none of our results depend on voters switching candidates − i.e. it is enough that they become disengaged for a campaign attack to have its desired effect.[60] Next, we consider the general version of the model with backlash effects, $\gamma > 0$. While the model in this case becomes significantly less tractable, it is still possible to show that our main findings remain unchanged for any $\alpha \in [0, 1]$ provided that γ is not too large. Specifically, for two-candidate races we show that the restriction on γ is given by $\gamma < \frac{(1-\alpha)s_2^o}{s_1^o}$, which simply guarantees that the benefit of an attack for candidate 1 is always strictly positive. Intuitively, the benefit of an attack for candidate 1 is increasing in $(1 - \alpha)s_2^o$, i.e. the amount of support that she is able to steal from 2, and decreasing in γs_1^o, i.e. the size of the backlash effect for 1. For three-candidate races, the exact condition is much harder to pin down, but we can still show that, conditional on attacking someone, each candidate prefers to target her highest-ranked opponent for any $\gamma > 0$. Intuitively, while the backlash effect may reduce the overall willingness to attack, it does not affect the choice of whom to attack. The other results can be shown to hold when $\gamma > 0$ is sufficiently close to zero.[61]

B.3 Multiple Attacks per Candidate

In this subsection, we extend our basic model to allow candidates to target multiple opponents. Observe that the set of all possible profiles of attacks is now given by $\mathcal{N} = \{0, 1, 2\}^3$. Let $\mathcal{N}_{i,\ell} \subset \mathcal{N}$ denote the set of possible 3-tuples which the vector n may assume when we impose the restriction that player i does not attack ℓ, where ℓ denotes an individual opponent or a pair of rivals.[62] Observe that, as in our main analysis, for any $n \in \mathcal{N}_{i,j}$, the benefit obtained by candidate i when she attacks a single opponent j is:

$$\widetilde{\Delta}_{i,j}(n) = p_i(n_i, n_j + 1, n_k) - p_i(n_i, n_j, n_k),$$

[59] All proofs can be adapted in a direct manner. Additional details are available upon request.

[60] In this sense, this version of the setup resembles models of negative vote buying, where voters are paid to abstain (Morgan and Várdy (2012)).

[61] Additional details are available upon request.

[62] For the sake of clarity, we alter the notation slightly by adding a comma in the subindex to separate the attacking candidate i from the attacked candidates ℓ.

whereas, for any $n \in \mathcal{N}_{i,jk}$, the benefit *per attack* obtained by candidate i when she targets both j and k is:

$$\widetilde{\Delta}_{i,jk}(n) = \frac{p_i(n_i, n_j+1, n_k+1) - p_i(n_i, n_j, n_k)}{2},$$

where we normalize the total benefit by the number of attacks in order to make the measures of cost and benefit comparable.

As before, we focus our analysis on the case where the parameter ϕ is small, so that all results derived in Proposition 3 still hold in the present setting. The following proposition complements the characterization of the benefit function $\widetilde{\Delta}_{i,\ell}(n)$ by extending it to consider multiple attacks.

Proposition B.8. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then we have:*

i. The benefit of attacking for candidate 1 is such that:

$$\widetilde{\Delta}_{1,3}(n_{1,3}) < \widetilde{\Delta}_{1,23}(n_{1,23}) < \widetilde{\Delta}_{1,2}(n_{1,2}) \qquad \textit{for any } n_{1,\ell} \in \mathcal{N}_{1,\ell}$$

ii. The benefit of attacking for candidate 2 is such that:

$$\widetilde{\Delta}_{2,3}(n_{2,3}) < \widetilde{\Delta}_{2,13}(n_{2,13}) < \widetilde{\Delta}_{2,1}(n_{2,1}) \qquad \textit{for any } n_{2,\ell} \in \mathcal{N}_{2,\ell}$$

iii. The benefit of attacking for candidate 3 is such that:

$$\widetilde{\Delta}_{3,2}(n_{3,2}) < \widetilde{\Delta}_{3,12}(n_{3,12}) < \widetilde{\Delta}_{3,1}(n_{3,1}) \qquad \textit{for any } n_{3,\ell} \in \mathcal{N}_{3,\ell}$$

Thus, similarly to Proposition 3, each candidate prefers to target her highest ranked opponent alone rather than any other rival or pair of rivals. Moreover, every candidate prefers to target both of her rivals together rather than the lowest ranked opponent alone, i.e. $\widetilde{\Delta}_{1,3}(n_{1,3}) < \widetilde{\Delta}_{1,23}(n_{1,23})$, $\widetilde{\Delta}_{2,3}(n_{2,3}) < \widetilde{\Delta}_{2,13}(n_{2,13})$ and $\widetilde{\Delta}_{3,2}(n_{3,2}) < \widetilde{\Delta}_{3,12}(n_{3,12})$. In other words, an attack against the lowest ranked opponent is more valuable when coupled with an attack against the highest ranked opponent. We also note that the relationship among $\widetilde{\Delta}_{1,23}(n)$, $\widetilde{\Delta}_{2,13}(n)$ and $\widetilde{\Delta}_{3,12}(n)$ is ambiguous in general, so that it is not possible to completely pin down the structure of the equilibrium. However, from Propositions 3 and B.8, it follows that in any equilibrium two properties must always hold: (*i*) the most likely direction of attack is from candidate 2 against 1, and (*ii*) no candidate attacks her lowest ranked opponent alone.

For specific parameter values, we are able to provide a complete characterization of the unique equilibrium of the game in pure strategies. Suppose, for instance, that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$, in which case we have the following ordering:[63]

$$\widetilde{\Delta}_{1,23}(n_{1,\ell}) < \widetilde{\Delta}_{2,13}(n_{2,\ell}) < \widetilde{\Delta}_{1,2}(n'_{1,\ell}) < \widetilde{\Delta}_{3,12}(n_{3,\ell}) < \widetilde{\Delta}_{3,1}(n'_{3,\ell}) < \widetilde{\Delta}_{2,1}(n_{2,\ell}),$$

for any $n_{i,\ell}, n'_{i,\ell} \in \mathcal{N}_{i,\ell}$. In this case, the equilibrium is such that:

i. Every candidate attacks every other candidate if, and only if, $c \leq \widetilde{\Delta}_{1,23}(2,1,1)$.

ii. Candidate 2 attacks 1 and 3, candidate 3 attacks 1 and 2 and candidate 1 attacks 2 if, and only if, $\widetilde{\Delta}_{1,23}(2,1,1) < c \leq \widetilde{\Delta}_{2,13}(1,2,0)$.

[63] For $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$, we have: $\widetilde{\Delta}_{1,23}(2,1,1) = 0.0043$, $\widetilde{\Delta}_{2,13}(1,2,0) = 0.0050$, $\widetilde{\Delta}_{1,2}(2,1,0) = 0.0051$, $\widetilde{\Delta}_{3,12}(1,0,0) = 0.0056$, $\widetilde{\Delta}_{3,1}(1,0,0) = 0.0064$ and $\widetilde{\Delta}_{2,1}(0,0,0) = 0.0072$. Additional results using alternative parameter values are available upon request.

iii. Candidate 2 attacks 1, candidate 3 attacks 1 and 2 and candidate 1 attacks 2 if, and only if, $\widetilde{\Delta}_{2,13}(1,2,0) < c \leq \widetilde{\Delta}_{1,2}(2,1,0)$.

iv. Candidate 2 attacks 1 and candidate 3 attacks 1 and 2 if, and only if, $\widetilde{\Delta}_{1,2}(2,1,0) < c \leq \widetilde{\Delta}_{3,12}(1,0,0)$.

v. Candidate 2 attacks 1 and candidate 3 attacks 1 if, and only if, $\widetilde{\Delta}_{3,12}(1,0,0) < c \leq \widetilde{\Delta}_{3,1}(1,0,0)$.

vi. Candidate 2 attacks 1 if, and only if, $\widetilde{\Delta}_{3,1}(1,0,0) < c \leq \widetilde{\Delta}_{2,1}(0,0,0)$.

vii. No candidate attacks if, and only if, $c > \widetilde{\Delta}_{2,1}(0,0,0)$.

Figure A.3 depicts the region of parameters where each class of equilibrium exists in this case. Note that, as the cost of attacking increases, we move through seven distinct regions corresponding to the different classes of equilibria describe above. Note that the equilibrium structure is similar to the one characterized in Proposition B.4, although the fact that candidates are now allowed to target multiple opponents adds significant complexity to it.

B.4 Tullock Contest Sucess Function

In this subsection, we consider an alternative specification for the probability of winning based on the widely used contest sucess function proposed by Tullock (1980). Specifically, suppose that in a two-candidate race the final level of support of a candidate i is given by:

$$x_i(n_i, n_j) = \log\left(\left((1-\phi n_i)\, s_i^o + n_j \phi s_j^o\right)^\mu\right) + \epsilon_i,$$

where $\mu > 0$ and ϵ_i is an iid shock with Type I Extreme Value distribution. As before, let $s_i(n_i, n_j) := (1-\phi n_i)\, s_i^o + n_j \phi s_j^o$. In this case, the probability that candidate i wins is given by:

$$p_i^T(n_i, n_j) = \frac{s_i(n_i, n_j)^\mu}{s_i(n_i, n_j)^\mu + s_j(n_i, n_j)^\mu} \tag{17}$$

This particular functional form is known as Tullock CSF. Note that the parameter μ captures the sensitivity of a candidate's winning chances with respect to changes in her electoral support.

Similarly, in a three-candidate race, we assume that the final level of support of a candidate i is:

$$x_i(n) = \log\left(\left((1-\phi n_i)\, s_i^o + \textstyle\sum_{j\neq i} \frac{n_j \phi s_j^o}{2}\right)^\mu\right) + \epsilon_i,$$

where $\mu > 0$ and ϵ_i is an iid shock with Type I EV distribution. Let $s_i(n) := (1-\phi n_i)\, s_i^o + \sum_{j\neq i} \frac{n_j \phi s_j^o}{2}$. In this case, the probability that candidate i wins is given by:

$$\widetilde{p}_i^T(n) = \frac{s_i(n)^\mu}{\sum_{k=1}^3 s_k(n)^\mu} \tag{18}$$

The benefit functions for two and three-candidate races are defined as in the main text and denoted by $\Delta_{ij}^T(n)$ and $\widetilde{\Delta}_{ij}^T(n)$, respectively. In what follows, we investigate the robustness of our findings under this alternative setting. Overall, the Tullock CSF is considerably less tractable, although we are still able to show that our main qualitative results hold in this case.

Two-candidate Races. We begin our analysis by studying races with two candidates. We show that a version of Proposition 1 holds in this case, provided that ϕ is sufficiently small.

Proposition B.9. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$, then candidate 2 is more aggressive than candidate 1 in the sense that:*

$$\Delta_{12}^{T}(n_1) < \Delta_{21}^{T}(n_2) \qquad \text{for any } n_1, n_2 \in \{0,1\}$$

Therefore, it follows that our previous result that candidate 2 is always the more aggressive candidate holds also for a Tullock CSF. Thus, the equilibrium structure remains very similar to that characterized in Proposition 2. In particular, it is possible to show that, as the cost of attacking increases, we move through three different parameter regions where the following equilibria exist: (*i*) an equilibrium where both candidates attack, (*ii*) an equilibrium where only candidate 2 attacks, and (*iii*) an equilibrium where nobody attacks.

Three-candidate Races. Next, for races with three candidates, we show that a version of Proposition 3 holds for the Tullock CSF, provided that $\mu > 1$.

Proposition B.10. *There exists a threshold $\overline{\phi} > 0$ such that if $\phi < \overline{\phi}$ and $\mu > 1$, then we have:*

i. For candidate 1, the benefit of an attack on 2 is larger than that of an attack on 3:

$$\widetilde{\Delta}_{13}^{T}(n) < \widetilde{\Delta}_{12}^{T}(n^{'}) \qquad \text{for any } n, n^{'} \in \mathcal{N}_1$$

ii. For candidate 2, the benefit of an attack on 1 is larger than that of an attack on 3:

$$\widetilde{\Delta}_{23}^{T}(n) < \widetilde{\Delta}_{21}^{T}(n^{'}) \qquad \text{for any } n, n^{'} \in \mathcal{N}_2$$

iii. For candidate 3, the benefit of an attack on 1 is larger than that of an attack on 2:

$$\widetilde{\Delta}_{32}^{T}(n) < \widetilde{\Delta}_{31}^{T}(n^{'}) \qquad \text{for any } n, n^{'} \in \mathcal{N}_3$$

iv. Candidate 2 is the most aggressive candidate in the sense that:

$$\max\{\widetilde{\Delta}_{12}^{T}(n_1), \widetilde{\Delta}_{31}^{T}(n_3)\} < \widetilde{\Delta}_{21}^{T}(n_2) \qquad \text{for any } n_i \in \mathcal{N}_i$$

Therefore, as in our main analysis, all candidates prefer to target their highest ranked opponent and candidate 2 is always the most aggressive candidate, provided that $\mu > 1$. Finally, the model becomes too intractable to allow us to investigate the robustness of the comparative static results regarding the dilution effect and the comparison between single and dual ballot plurality systems, derived in Propositions 4 and 5 respectively. In the next subsection, we present simulation results suggesting that those findings also hold under a Tullock CSF.

B.5 Simulations

Most of our theoretical results, particularly those obtained for three-candidate races, were derived under the assumption that the impact of a campaign attack, as captured by the parameter ϕ, was sufficiently small. In this subsection, we present simulation results showing that our main findings hold for "reasonable", i.e. not vanishingly small, values of ϕ. We begin our analysis by examining the case of two-candidate races, assuming that $s_1^o = 0.6$ and $s_2^o = 0.4$.[64] We compute the benefits of attacking for

[64]Our simulation results are robust to the choice of initial levels of support, provided that $s_1^o > s_2^o$. Additional results are available upon request.

candidates 1 and 2 under both Logit and Tullock CSFs, for $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$.[65] The simulation results are presented in Table A.8. Note that, consistently with Propositions 1 and B.9, we always have $\Delta_{12}(n_1) < \Delta_{21}(n_2)$ for any $n_1, n_2 \in \{0, 1\}$ for both Logit and Tullock CSFs, so that candidate 2 is always the most aggressive. Moreover, in all cases, we have $\Delta_{12}(0) < \Delta_{12}(1) < \Delta_{21}(1) < \Delta_{21}(0)$.

Next, we study the case of three-candidate races, assuming that $s_1^o = 0.4$, $s_2^o = 0.3$ and $s_3^o = 0.2$, and considering $\phi \in \{0.05, 0.1, 0.15\}$ and $\mu \in \{1.5, 2\}$. For this exercise, and the ones to follow, we evaluate the benefit function $\widetilde{\Delta}_{ij}(n)$ at $n = (0, 0, 0)$, i.e. assuming that no candidate is receiving an attack.[66] The results presented in Table A.9 show that in all cases we have $\widetilde{\Delta}_{13} < \widetilde{\Delta}_{12}$, $\widetilde{\Delta}_{23} < \widetilde{\Delta}_{21}$ and $\widetilde{\Delta}_{32} < \widetilde{\Delta}_{31}$, with $\max\{\widetilde{\Delta}_{12}, \widetilde{\Delta}_{31}\} < \widetilde{\Delta}_{21}$, which is consistent with Propositions 3 and B.10. That is, all candidates prefer to target their highest-ranked opponent and candidate 2 is always the most aggressive. Furthermore, in line with Proposition B.2 and Corollary B.1, we always have $\widetilde{\Delta}_{23} < \{\widetilde{\Delta}_{13}, \widetilde{\Delta}_{32}\} < \{\widetilde{\Delta}_{12}, \widetilde{\Delta}_{31}\} < \widetilde{\Delta}_{21}$ for both Logit and Tullock CSFs.

We next investigate how the incentives for the first two candidates to attack each other vary when we increase the strength of the 3rd place candidate. Under the assumption that $s_1^o = 0.4$, $s_2^o = 0.3$ and $\phi = 0.1$, Figure A.12 plots the benefits of campaign attacks between candidates 1 and 2 as a function of the initial support of candidate 3, for both the Logit CSF (Panel A) and the Tullock CSF with $\mu = 1.5$ (Panel B). In accordance with the "dillution effect" derived in Proposition 4, we find that $\widetilde{\Delta}_{12}$ and $\widetilde{\Delta}_{21}$ are strictly decreasing in s_3^o, implying that both candidates become less aggressive towards each other when the 3rd place candidate becomes stronger. Interestingly, although Proposition 4 applies only to the case of a Logit CSF, our simulation results suggest that the "dillution effect" also holds under a Tullock CSF.

Finally, we compare the benefits of campaign attacks under single and dual ballot plurality systems for both the Logit CSF and the Tullock CSF with $\mu = 1.5$, assuming that $s_1^o = 0.4$, $s_2^o = 0.3$, $s_3^o = 0.2$ and $\phi = 0.1$. Consistently with Proposition 5, the simulation results reported in Table A.10 show that $\max\{\widetilde{\Delta}_{12}, \widetilde{\Delta}_{13}\} < \max\{\widetilde{\Delta}_{21}, \widetilde{\Delta}_{23}\} < \max\{\widetilde{\Delta}_{31}, \widetilde{\Delta}_{32}\}$ under dual ballot plurality for both CSFs, i.e. candidate 3 is always the most aggressive candidate under dual ballot followed by candidates 2 and 1. Moreover, a change from single to dual ballot plurality leads to a particularly large increase in $\widetilde{\Delta}_{23}$ and $\widetilde{\Delta}_{32}$. This result is in line with the gist of Proposition B.6, which shows that candidates 2 and 3 may become more aggressive towards each other under dual ballot plurality. In particular, note that for the Tullock CSF we actually have $\widetilde{\Delta}_{23} > \widetilde{\Delta}_{21}$ and $\widetilde{\Delta}_{32} > \widetilde{\Delta}_{31}$ under dual ballot. [67]

[65] We remind the reader that μ is a parameter of the Tullock CSF. Additional results using alternative values for ϕ and μ are available upon request.

[66] We obtain similar results for alternative values of $n = (n_1, n_2, n_2)$. Additional results are available upon request.

[67] Interestingly, the results for the Tullock CSF also show that the benefits of campaign attacks between candidates 1 and 2 actually decrease when we move from single to dual ballot plurality.

C Proofs

C.1 Proposition 1

Item i. We want to show that $\Delta_{12}(0) < \Delta_{12}(1)$, which can be expressed as:

$$\frac{\exp(s_1^o + \phi s_2^o)}{\exp(s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o)} - \frac{\exp(s_1^o)}{\exp(s_1^o) + \exp(s_2^o)} < \frac{\exp((1-\phi)s_1^o + \phi s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o + \phi s_1^o)} - \frac{\exp((1-\phi)s_1^o)}{\exp((1-\phi)s_1^o) + \exp(s_2^o + \phi s_1^o)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp(s_1^o + s_2^o)(\exp(2\phi s_2^o) - 1)}{(\exp(s_1^o) + \exp(s_2^o))(\exp(s_2^o) + \exp(s_1^o + 2\phi s_2^o))} < \frac{\exp((1+2\phi)s_1^o + s_2^o)(\exp(2\phi s_2^o) - 1)}{(\exp(s_1^o) + \exp(2\phi s_1^o + s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))}$$

Rearranging and simplifying, we get:

$$\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)(\exp(2\phi s_2^o) - 1)(\exp(2(s_1^o + \phi s_2^o)) - \exp(2(\phi s_1^o + s_2^o))) > 0$$

which always holds, since $\exp(2\phi s_1^o) > 1$, $\exp(2\phi s_2^o) > 1$ and $\exp(2(s_1^o + \phi s_2^o)) > \exp(2(\phi s_1^o + s_2^o))$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$.

Item ii. We want to show that $\Delta_{21}(1) < \Delta_{21}(0)$, which can be expressed as:

$$\frac{\exp(\phi s_1^o + (1-\phi)s_2^o)}{\exp((1-\phi)s_1^o + \phi s_2^o) + \exp(\phi s_1^o + (1-\phi)s_2^o)} - \frac{\exp((1-\phi)s_2^o)}{\exp(s_1^o + \phi s_2^o) + \exp((1-\phi)s_2^o)} < \frac{\exp(\phi s_1^o + s_2^o)}{\exp((1-\phi)s_1^o) + \exp(\phi s_1^o + s_2^o)} - \frac{\exp(s_2^o)}{\exp(s_1^o) + \exp(s_2^o)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp(s_1^o + (1+2\phi)s_2^o)(\exp(2\phi s_1^o) - 1)}{(\exp(s_2^o) + \exp(s_1^o + 2\phi s_2^o))(\exp(2\phi s_1^o + s_2^o) + \exp(s_1^o + 2\phi s_2^o))} < \frac{\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)}{(\exp(s_1^o) + \exp(s_2^o))(\exp(s_1^o) + \exp(2\phi s_1^o + s_2^o))}$$

Rearranging and simplifying, we get:

$$\exp(s_1^o + s_2^o)(\exp(2\phi s_1^o) - 1)(\exp(2\phi s_2^o) - 1)(\exp(2(s_1^o + \phi s_2^o)) - \exp(2(\phi s_1^o + s_2^o))) > 0$$

which always holds, since $\exp(2\phi s_1^o) > 1$, $\exp(2\phi s_2^o) > 1$ and $\exp(2(s_1^o + \phi s_2^o)) > \exp(2(\phi s_1^o + s_2^o))$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$.

Item iii. We want to show that $\Delta_{12}(1) < \Delta_{21}(1)$, which can be expressed as:

$$\frac{\exp\left((1-\phi)s_1^o + \phi s_2^o\right)}{\exp\left((1-\phi)s_1^o + \phi s_2^o\right) + \exp\left((1-\phi)s_2^o + \phi s_1^o\right)} - \frac{\exp\left((1-\phi)s_1^o\right)}{\exp\left((1-\phi)s_1^o\right) + \exp\left(s_2^o + \phi s_1^o\right)} < \frac{\exp\left(\phi s_1^o + (1-\phi)s_2^o\right)}{\exp\left((1-\phi)s_1^o + \phi s_2^o\right) + \exp\left(\phi s_1^o + (1-\phi)s_2^o\right)} - \frac{\exp\left((1-\phi)s_2^o\right)}{\exp\left(s_1^o + \phi s_2^o\right) + \exp\left((1-\phi)s_2^o\right)}$$

After some algebra, we can rewrite the above expression as:

$$\frac{\exp\left((1+2\phi)s_1^o + s_2^o\right)\left(\exp\left(2\phi s_2^o\right) - 1\right)}{\left(\exp\left(s_1^o\right) + \exp\left(2\phi s_1^o + s_2^o\right)\right)\left(\exp\left(2\phi s_1^o + s_2^o\right) + \exp\left(s_1^o + 2\phi s_2^o\right)\right)} < \frac{\exp\left(s_1^o + (1+2\phi)s_2^o\right)\left(\exp\left(2\phi s_1^o\right) - 1\right)}{\left(\exp\left(s_2^o\right) + \exp\left(s_1^o + 2\phi s_2^o\right)\right)\left(\exp\left(2\phi s_1^o + s_2^o\right) + \exp\left(s_1^o + 2\phi s_2^o\right)\right)}$$

Rearranging and simplifying, we get:

$$\begin{aligned}&\exp\left(2\phi s_1^o + s_2^o\right) + 2\exp\left(s_1^o + 2\phi s_1^o + 2\phi s_2^o\right) + \exp\left(4\phi s_1^o + s_2^o + 2\phi s_2^o\right) \\ &\quad - \exp\left(s_1^o + 2\phi s_2^o\right) - \exp\left(s_1^o + 2\phi s_1^o + 4\phi s_2^o\right) - 2\exp\left(2\phi s_1^o + s_2^o + 2\phi s_2^o\right) \ > \ 0\end{aligned}$$

which in turn can be re-expressed as:

$$\exp\left((1+2\phi)\left(s_1^o + s_2^o\right)\right)\left(\exp\left(2\phi s_1^o\right) - \exp\left(2\phi s_2^o\right)\right) > 0$$

which always holds since $\exp\left(2\phi s_1^o\right) - \exp\left(2\phi s_2^o\right) > 0$ for any $0 < \phi < 1$ and $s_1^o > s_2^o$. ■

C.2 Proposition 2

Note that from Proposition 1, we have $\Delta_{12}(0) < \Delta_{12}(1) < \Delta_{21}(1) < \Delta_{21}(0)$. Thus, we need to consider five cases for the cost parameter $c \in \mathbb{R}_+$:

i. Case 1: Suppose that $c \leq \Delta_{12}(0)$. In this case, both candidates are always willing to attack their opponent. Thus, in equilibrium, both candidates attack.

ii. Case 2: Suppose that $\Delta_{12}(0) < c \leq \Delta_{12}(1)$. In this case, candidate 2 is always willing to attack, whereas candidate 1 is willing to attack only if attacked. Thus, in equilibrium, both candidates attack.

iii. Case 3. Suppose that $\Delta_{12}(1) < c \leq \Delta_{21}(1)$. In this case, candidate 2 is always willing to attack, whereas candidate 1 is never willing to attack. Thus, in equilibrium, only candidate 2 attacks.

iv. Case 4. Suppose that $\Delta_{21}(1) < c \leq \Delta_{21}(0)$. In this case, candidate 2 is willing to attack only if not attacked, whereas candidate 1 is never willing to attack. Thus, in equilibrium, only candidate 2 attacks.

v. Case 5. Suppose that $\Delta_{21}(0) < c$. In this case, both candidates are never willing to attack. Thus, in equilibrium, no candidate attacks.

Therefore, in equilibrium, we have: (*i*) both candidates attack if and only if $c \leq \Delta_{12}(1)$; (*ii*) only candidate 2 attacks if and only if $\Delta_{12}(1) < c \leq \Delta_{21}(0)$; and (*iii*) no candidate attacks if and only if $\Delta_{21}(0) < c$.■

C.3 Proposition 3

Item i. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{12}(n) > \widetilde{\Delta}_{13}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_1$. To do so, let $f_1(\phi, n, n^{'}) := \widetilde{\Delta}_{12}(n) - \widetilde{\Delta}_{13}(n^{'})$. We will show that $f_1(0, n, n^{'}) = 0$ and $\partial f_1(0, n, n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_1$, so that by taking a first-order Taylor expansion of the function $f_1(\phi, n, n^{'})$ around $\phi = 0$ it follows that $f_1(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f_1(0, n, n^{'}) = 0$, given that $\widetilde{\Delta}_{12}(n) = \widetilde{\Delta}_{13}(n^{'}) = 0$ when $\phi = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(\phi, n, n^{'})}{\partial\phi} = \frac{3\exp(s_1^o)\left(s_2^o \exp(s_2^o) - s_3^o \exp(s_3^o)\right)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o \exp(s_2^o) > s_3^o \exp(s_3^o)$.

Item ii. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{21}(n) > \widetilde{\Delta}_{23}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n^{'}) := \widetilde{\Delta}_{21}(n) - \widetilde{\Delta}_{23}(n^{'})$. We will show that $f_2(0, n, n^{'}) = 0$ and $\partial f_2(0, n, n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_2$. It is immediate to see that $f_2(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_2^o)\left(s_1^o \exp(s_1^o) - s_3^o \exp(s_3^o)\right)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_3^o \exp(s_3^o)$.

Item iii. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{31}(n) > \widetilde{\Delta}_{32}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n^{'}) := \widetilde{\Delta}_{31}(n) - \widetilde{\Delta}_{32}(n^{'})$. We will show that $f_3(0, n, n^{'}) = 0$ and $\partial f_3(0, n, n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_3(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_3^o)\left(s_1^o \exp(s_1^o) - s_2 \exp(s_2^o)\right)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o \exp(s_1^o) > s_2^o \exp(s_2^o)$.

Item iv. First, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{21}(n^{'}) > \widetilde{\Delta}_{12}(n)$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. To do so, let $h_{12}(\phi, n, n^{'}) := \widetilde{\Delta}_{21}(n^{'}) - \widetilde{\Delta}_{12}(n)$. We will show that $h_{12}(0, n, n^{'}) = 0$ and $\partial h_{12}(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $h_{12}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{12}(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_1^o + s_2^o)(s_1^o - s_2)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o > s_2^o$. Next, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{21}(n^{'}) > \widetilde{\Delta}_{31}(n^{''})$ for any $n^{'} \in \mathcal{N}_2$ and $n^{''} \in \mathcal{N}_3$. To do so, let $h_{23}(\phi, n^{'}, n^{''}) := \widetilde{\Delta}_{21}(n^{'}) - \widetilde{\Delta}_{31}(n^{''})$. We will show that $h_{23}(0, n^{'}, n^{''}) = 0$ and $\partial h_{23}(0, n^{'}, n^{''})/\partial\phi > 0$ for any $n^{'} \in \mathcal{N}_2$ and $n^{''} \in \mathcal{N}_3$. It is immediate to see that $h_{23}(0, n^{'}, n^{''}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{23}(0, n^{'}, n^{''})}{\partial\phi} = \frac{3\exp(s_1^o)\left(\exp(s_2^o) - \exp(s_3^o)\right)s_1^o}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_2^o) > \exp(s_3^o)$. Therefore, we conclude that $\widetilde{\Delta}_{21}(n^{'}) > \max\{\widetilde{\Delta}_{12}(n), \widetilde{\Delta}_{31}(n^{''})\}$ for any $n \in \mathcal{N}_1$, $n^{'} \in \mathcal{N}_2$ and $n^{''} \in \mathcal{N}_3$.$\blacksquare$

C.4 Proposition 4

First, we want to show that if ϕ is small enough then $\partial\widetilde{\Delta}_{12}(n)/\partial s_3^o < 0$ for any $n = (n_1, n_2, n_3) \in \mathcal{N}_1$. Taking the derivative of $\widetilde{\Delta}_{12}(n)$ w.r.t. s_3^o, re-arranging and simplifying, we get:

$$\frac{\partial\widetilde{\Delta}_{12}(n)}{\partial s_3^o} = \frac{\Gamma_1\Gamma_2}{2\left(\Gamma_3\right)^2\left(\Gamma_4\right)^2}\left(3n_3\phi - 2\right)$$

where

$$\Gamma_1 = \exp\left(\left(1 + n_1\phi\right)s_1^o + 3n_2\phi s_2^o + \left(1 + n_3\phi\right)s_3^o\right)\left(\exp\left(\frac{3}{2}\phi s_2^o\right) - 1\right) > 0$$

$$\begin{aligned}\Gamma_2 \;=\;& \exp\left(\frac{3}{2}\phi\left(n_1 s_1^o + n_3 s_3^o\right)\right) + \exp\left(\frac{3}{2}\phi\left(n_1 s_1^o + s_2^o + n_3 s_3^o\right)\right)\\ &+ 2\exp\left(\frac{1}{2}\left(3n_1\phi s_1^o - \left(2 - 3\left(1 + n_2\right)\phi\right)s_2^o + 2s_3^o\right)\right)\\ &+ 2\exp\left(\frac{1}{2}\left(2s_1^o - \left(2 - 3\left(1 + n_2\right)\phi\right)s_2^o + 3n_3\phi s_3^o\right)\right) \;>\; 0\end{aligned}$$

$$\begin{aligned}\Gamma_3 \;=\;& \exp\left(\frac{1}{2}\phi\left(n_1 s_1^o + n_3 s_3^o\right)\right) + \exp\left(\frac{1}{2}\left(2\left(1 - n_1\phi\right)s_1^o - \left(2 - 3\left(1 + n_2\right)\phi\right)s_2^o + n_3\phi s_3^o\right)\right)\\ &+ \exp\left(\frac{1}{2}\left(n_1\phi s_1^o - \left(2 - 3\left(1 + n_2\right)\phi\right)s_2^o + 2\left(1 - n_3\phi\right)\right)s_3^o\right) \;>\; 0\end{aligned}$$

$$\begin{aligned}\Gamma_4 \;=\;& \exp\left(\frac{1}{2}\left(3n_1\phi s_1^o + 3n_2\phi s_2^o + 2s_3^o\right)\right) + \exp\left(\frac{1}{2}\left(3n_1\phi s_1^o + 2s_2^o + 3n_3\phi s_3^o\right)\right)\\ &+ \exp\left(\frac{1}{2}\left(2s_1^o + 3n_2\phi s_2^o + 3n_3\phi s_3^o\right)\right) \;>\; 0\end{aligned}$$

Note that the sign of the derivative is completely determined by the term $3n_3\phi - 2$, which is guaranteed to be negative as long as ϕ is sufficiently small. More specifically, since $n_3 \leq 2$, the sufficient condition is $\phi < \frac{1}{3}$.

Next, we want to show that if ϕ is small enough then $\partial\widetilde{\Delta}_{21}(n)/\partial s_3^o < 0$ for any $n = (n_1, n_2, n_3) \in \mathcal{N}_2$. Taking the derivative of $\widetilde{\Delta}_{21}(n)$ w.r.t. s_3^o, re-arranging and simplifying, we get:

$$\frac{\partial\widetilde{\Delta}_{21}(n)}{\partial s_3^o} = \frac{\Psi_1\Psi_2}{2\left(\Psi_3\right)^2\left(\Psi_4\right)^2}\left(3n_3\phi - 2\right)$$

where

$$\Psi_1 = \exp\left(3n_1\phi s_1^o + \left(1 + 3n_2\phi\right)s_2^o + \left(1 + n_3\phi\right)s_3^o\right)\left(\exp\left(\frac{3}{2}\phi s_1^o\right) - 1\right) > 0$$

$$\Psi_2 = \exp\left(\frac{3}{2}\phi\left(n_2 s_2^o + n_3 s_3^o\right)\right) + \exp\left(\frac{3}{2}\phi\left(s_1^o + n_2 s_2^o + n_3 s_3^o\right)\right) + 2\exp\left(\frac{1}{2}\left(-\left(2-3\left(1+n_1\right)\phi\right)s_1^o - 3n_2\phi s_2^o + 2s_3^o\right)\right) + 2\exp\left(\frac{1}{2}\left(-\left(2-3\left(1+n_1\right)\phi\right)s_1^o + s_2^o + 3n_3\phi s_3^o\right)\right) > 0$$

$$\Psi_3 = \exp\left(-\frac{1}{2}\left(-\left(2-3\left(1+n_1\right)\phi\right)s_1^o + 3n_2\phi s_2^o + 2s_3^o\right)\right) + \exp\left(\frac{3}{2}\left(n_2\phi s_2^o + n_3\phi s_3^o\right)\right) + \exp\left(-\frac{1}{2}\left(-\left(2-3\left(1+n_1\right)\phi\right)s_1^o + 2s_2^o + 3n_3\phi s_3^o\right)\right) > 0$$

$$\Psi_4 = \exp\left(\frac{1}{2}\left(3n_1\phi s_1^o + 3n_2\phi s_2^o + 2s_3^o\right)\right) + \exp\left(\frac{1}{2}\left(3n_1\phi s_1^o + 2s_2^o + 3n_3\phi s_3^o\right)\right) + \exp\left(\frac{1}{2}\left(2s_1^o + 3n_2\phi s_2^o + 3n_3\phi s_3^o\right)\right) > 0$$

Note that, as before, the sign of the derivative is completely determined by the term $3n_3\phi - 2$, which is guaranteed to be negative as long as ϕ is sufficiently small. More specifically, since $n_3 \leq 2$, the sufficient condition is $\phi < \frac{1}{3}$.■

C.5 Proposition 5

First, we want to show that if ϕ is small enough then $\max\{\widetilde{\Delta}_{31}^{DB}(n_3), \widetilde{\Delta}_{32}^{DB}(n_3^{'})\} > \max\{\widetilde{\Delta}_{21}^{DB}(n_2), \widetilde{\Delta}_{23}^{DB}(n_2^{'})\}$ for any $n_i, n_i^{'} \in \mathcal{N}_i$. To do so, we will show that: (i) $\widetilde{\Delta}_{31}^{DB}(n_3) > \widetilde{\Delta}_{21}^{DB}(n_2)$ for any $n_i \in \mathcal{N}_i$ and (ii) $\widetilde{\Delta}_{32}^{DB}(n_3) > \widetilde{\Delta}_{23}^{DB}(n_2)$ for any $n_i \in \mathcal{N}_i$. First, let $f_{23}(\phi, n, n^{'}) := \widetilde{\Delta}_{31}^{DB}(n^{'}) - \widetilde{\Delta}_{21}^{DB}(n)$. We will show that $f_{23}(0, n, n^{'}) = 0$ and $\partial f_{23}(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_2$ and $n^{'} \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f_{23}(\phi, n, n^{'})$ around $\phi = 0$ if follows that $f_{23}(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f_{23}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{23}(0, n, n^{'})}{\partial\phi} = \frac{3s_1^o \exp(s_1^o + s_2^o + s_3^o)\Gamma\left(\exp(s_2^o) - \exp(s_3^o)\right)}{2},$$

where

$$\Gamma = \exp(2s_2^o) + \exp(2s_3^o) + 3\exp(2s_1^o) + 3\exp(s_2^o + s_3^o) + 4\exp(s_1^o + s_2^o) + 4\exp(s_1^o + s_3^o) > 0$$

Note that the derivative above is always strictly positive, since $\exp\left(s_2^o\right) > \exp\left(s_3^o\right)$. Next, let $g_{23}(\phi, n, n^{'}) := \widetilde{\Delta}_{32}^{DB}(n^{'}) - \widetilde{\Delta}_{23}^{DB}(n)$. We will show that $g_{23}(0, n, n^{'}) = 0$ and $\partial g_{23}(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_2$ and $n^{'} \in \mathcal{N}_3$. It is immediate to see that $g_{23}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_{23}(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_1^o + s_2^o + s_3^o)(\exp(s_1^o) + 2\exp(s_2^o) + 2\exp(s_3^o))(s_2^o - s_3^o)}{2(\exp(s_2^o) + \exp(s_3^o))^2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$. Therefore, from the arguments above, it follows that $\max\{\widetilde{\Delta}_{31}^{DB}(n_3), \widetilde{\Delta}_{32}^{DB}(n_3^{'})\} > \max\{\widetilde{\Delta}_{21}^{DB}(n_2), \widetilde{\Delta}_{23}^{DB}(n_2^{'})\}$.

Second, we want to show that if ϕ is small enough then $\max\{\widetilde{\Delta}_{21}^{DB}(n_2), \widetilde{\Delta}_{23}^{DB}(n_2^{'})\} > \max\{\widetilde{\Delta}_{12}^{DB}(n_1), \widetilde{\Delta}_{13}^{DB}(n_1^{'})\}$

for any $n_i, n_i^{'} \in \mathcal{N}_i$. To do so, we will show that: (i) $\widetilde{\Delta}_{21}^{DB}(n_2) > \widetilde{\Delta}_{12}^{DB}(n_1)$ for any $n_i \in \mathcal{N}_i$ and (ii) $\widetilde{\Delta}_{23}^{DB}(n_2) > \widetilde{\Delta}_{13}^{DB}(n_1)$ for any $n_i \in \mathcal{N}_i$. First, let $g_{12}(\phi, n, n^{'}) := \widetilde{\Delta}_{21}^{DB}(n^{'}) - \widetilde{\Delta}_{12}^{DB}(n)$. We will show that $g_{12}(0, n, n^{'}) = 0$ and $\partial g_{12}(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $g_{12}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_{12}(0, n, n^{'})}{\partial \phi} = \frac{3\exp(s_1^o + s_2^o + s_3^o)(2\exp(s_1^o) + 2\exp(s_2^o) + \exp(s_3^o))(s_2^o - s_3^o)}{2(\exp(s_2^o) + \exp(s_3^o))^2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$. Similarly, let $f_{12}(\phi, n, n^{'}) := \widetilde{\Delta}_{23}^{DB}(n^{'}) - \widetilde{\Delta}_{13}^{DB}(n)$. We will show that $f_{12}(0, n, n^{'}) = 0$ and $\partial f_{12}(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $f_{12}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{12}(0, n, n^{'})}{\partial \phi} = \frac{3s_3^o \exp(s_1^o + s_2^o + s_3^o)\Psi\left(\exp(s_1^o) - \exp(s_2^o)\right)}{2},$$

where

$$\Psi = \exp(2s_1^o) + \exp(2s_2^o) + 3\exp(s_1^o + s_2^o) + 3\exp(2s_3^o) + 4\exp(s_1^o + s_3^o) + 4\exp(s_2^o + s_3^o) > 0$$

Note that the derivative above is always strictly positive, since $\exp\left(s_1^o\right) > \exp\left(s_2^o\right)$. Therefore, from the arguments above, it follows that $\max\{\widetilde{\Delta}_{21}^{DB}(n_2), \widetilde{\Delta}_{23}^{DB}(n_2^{'})\} > \max\{\widetilde{\Delta}_{12}^{DB}(n_1), \widetilde{\Delta}_{13}^{DB}(n_1^{'})\}$.$\blacksquare$

D Additional Proofs

D.1 Proposition B.1

First, we want to show that if ϕ is small enough and s_2^o is sufficiently close to s_1^o, then $\widetilde{\Delta}_{12}(n) > \widetilde{\Delta}_{31}(n^{'})$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$. To do so, let $f(\phi, n, n^{'}) := \widetilde{\Delta}_{12}(n) - \widetilde{\Delta}_{31}(n^{'})$. We will show that $f(0, n, n^{'}) = 0$ and $\lim_{s_2^o \to s_1^o} \partial f(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi, n, n^{'})$ around $\phi = 0$ it follows that $f(\phi, n, n^{'}) > 0$ for ϕ sufficiently small and s_2^o close enough to s_1^o. It is immediate to see that $f(0, n_1, n_2) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f(0, n, n^{'})}{\partial \phi} = \frac{3\exp(s_1^o)\left(s_2^o \exp(s_2^o) - s_1^o \exp(s_3^o)\right)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is positive provided that s_2^o is close enough to s_1^o. Specifically, observe that $\lim_{s_2^o \to s_1^o}(s_2^o \exp(s_2^o) - s_1^o \exp(s_3^o)) > 0$. Therefore, $\lim_{s_2^o \to s_1^o} \partial f(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$. Next, following a similar argument, we can show that if ϕ is small enough and s_2^o is sufficiently close to s_3^o, then $f(\phi, n, n^{'}) < 0$. In fact, note that $\lim_{s_2^o \to s_3^o} s_2 \exp(s_2) - s_1 \exp(s_3)) < 0$, so that $\lim_{s_2^o \to s_3^o} \partial f_1(0, n, n^{'})/\partial\phi < 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$.$\blacksquare$

D.2 Proposition B.2

Item i. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{12}(n) > \widetilde{\Delta}_{32}(n^{'})$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$. To do so, let $f(\phi, n, n^{'}) := \widetilde{\Delta}_{12}(n) - \widetilde{\Delta}_{32}(n^{'})$. We will show that $f(0, n, n^{'}) = 0$ and $\partial f(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi, n, n^{'})$ around $\phi = 0$ it follows that $f(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f(0, n, n^{'})}{\partial \phi} = \frac{3\exp(s_2^o)s_2^o(\exp(s_1^o) - \exp(s_3^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_1^o) > \exp(s_3^o)$.

Item ii. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{13}(n) < \widetilde{\Delta}_{31}(n^{'})$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$. To do so, let $g(\phi, n, n^{'}) := \widetilde{\Delta}_{31}(n^{'}) - \widetilde{\Delta}_{13}(n)$. We will show that $g(0, n, n^{'}) = 0$ and $\partial g(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_3$. It is immediate to see that $g(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g(0, n, n^{'})}{\partial \phi} = \frac{3\exp(s_1^o + s_3^o)(s_1^o - s_3^o)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o > s_3^o$.

Item iii. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{13}(n) > \widetilde{\Delta}_{23}(n^{'})$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. To do so, let $h(\phi, n, n_2) := \widetilde{\Delta}_{13}(n) - \widetilde{\Delta}_{23}(n^{'})$. We will show that $h(0, n, n^{'}) = 0$ and $\partial h(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $h(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h(0, n, n^{'})}{\partial \phi} = \frac{3\exp(s_3^o)s_3^o(\exp(s_1^o) - \exp(s_2^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $\exp(s_1^o) > \exp(s_2^o)$.

Item iv. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{23}(n) < \widetilde{\Delta}_{32}(n^{'})$ for any $n \in \mathcal{N}_2$ and $n^{'} \in \mathcal{N}_3$. To do so, let $t(\phi, n, n^{'}) := \widetilde{\Delta}_{32}(n^{'}) - \widetilde{\Delta}_{23}(n)$. We will show that $t(0, n, n^{'}) = 0$ and $\partial t(0, n, n^{'})/\partial\phi > 0$

for any $n \in \mathcal{N}_2$ and $n' \in \mathcal{N}_3$. It is immediate to see that $t(0,n,n') = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h_{23}(0,n,n')}{\partial \phi} = \frac{3\exp(s_2^o + s_3^o)(s_2^o - s_3^o)}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o > s_3^o$.■

D.3 Proposition B.3

First, we want to show that if ϕ is small enough and s_2^o is sufficiently close to s_1^o then $\widetilde{\Delta}_{13}(n) < \widetilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. To do so, let $f(\phi,n,n') := \widetilde{\Delta}_{32}(n') - \widetilde{\Delta}_{13}(n)$. We will show that $f(0,n,n') = 0$ and $\lim_{s_2^o \to s_1^o} \partial f(0,n,n')/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$, so that by taking a first-order Taylor expansion of the function $f(\phi,n,n')$ around $\phi = 0$ it follows that $f(\phi,n,n') > 0$ for ϕ and $s_2^o - s_1^o$ sufficiently small. It is straightforward to show that $f(0,n,n') = 0$. Furthermore, after some algebra, we obtain:

$$\frac{\partial f(0,n,n')}{\partial \phi} = \frac{3\exp(s_3^o)(s_2^o\exp(s_2^o) - s_3^o\exp(s_1^o))}{2(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2}$$

which is positive provided that s_2^o is close enough to s_1^o. Specifically, observe that $\lim_{s_2^o \to s_1^o} (s_2^o\exp(s_2^o) - s_3^o\exp(s_1^o)) = (s_1^o - s_3^o)\exp(s_2^o) > 0$, so that $\lim_{s_2^o \to s_1^o} \partial f(0,n,n')/\partial\phi > 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. On the other hand, we can show that if s_2^o is sufficiently close to s_3^o, then $\widetilde{\Delta}_{13}(n) < \widetilde{\Delta}_{32}(n')$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$. In fact, note that $\lim_{s_2^o \to s_3^o}(s_2^o\exp(s_2^o) - s_3^o\exp(s_1^o)) = -(\exp(s_1^o) - \exp(s_3^o))s_3^o < 0$, so that $\lim_{s_2^o \to s_3^o} \partial f(0,n,n')/\partial\phi < 0$ for any $n \in \mathcal{N}_1$ and $n' \in \mathcal{N}_3$.■

D.4 Proposition B.4

Let us denote an attack from candidate i against j by $i \to j$. From Proposition 3, we know that $\widetilde{\Delta}_{21}(n_2) > \max\{\widetilde{\Delta}_{12}(n_1), \widetilde{\Delta}_{31}(n_3)\}$ for any $n_i \in \mathcal{N}_i$. Thus, there are two cases to consider, namely: (i) $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ and (ii) $\widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{21}(n_2)$.

Item i. Suppose, first, that $\widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. Here, we need to consider four cases for the cost parameter $c \in \mathbb{R}_+$:

a. Case 1: Suppose that $c \leq \widetilde{\Delta}_{31}(1,1,0)$. In this case, candidates 2 and 3 are willing to attack 1 and candidate 1 is willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \to 1$, $1 \to 2$ and $3 \to 1$.

b. Case 2: Suppose that $\widetilde{\Delta}_{31}(1,1,0) < c \leq \widetilde{\Delta}_{12}(1,0,0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attacks occur: $2 \to 1$ and $1 \to 2$.

c. Case 3: Suppose that $\widetilde{\Delta}_{12}(1,0,0) < c \leq \widetilde{\Delta}_{21}(0,0,0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is not willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attack occur: $2 \to 1$.

d. Case 4: Suppose that $\widetilde{\Delta}_{21}(0,0,0) < c$. In this case, no candidate is willing to attack an opponent. Thus, in equilibrium, no candidate attacks.

Item ii. Next, suppose that $\widetilde{\Delta}_{12}(n_1) < \widetilde{\Delta}_{31}(n_3) < \widetilde{\Delta}_{21}(n_2)$ for any $n_i \in \mathcal{N}_i$. As before, we need to consider the following four cases:

a. Case 1: Suppose that $c \leq \widetilde{\Delta}_{12}(2,0,0)$. In this case, candidates 2 and 3 are willing to attack 1 and candidate 1 is willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \to 1$, $3 \to 1$ and $1 \to 2$.

b. Case 2: Suppose that $\widetilde{\Delta}_{12}(2,0,0) < c \leq \widetilde{\Delta}_{31}(1,0,0)$. In this case, candidate 2 is willing to attack 1, candidate 3 is willing to attack 1 and candidate 1 is not willing to attack 2. Thus, in equilibrium, the following attacks occur: $2 \to 1$ and $3 \to 1$.

c. Case 3: Suppose that $\widetilde{\Delta}_{31}(1,0,0) < c \leq \widetilde{\Delta}_{21}(0,0,0)$. In this case, candidate 2 is willing to attack 1, candidate 1 is not willing to attack 2 and candidate 3 is not willing to attack 1. Thus, in equilibrium, the following attack occurs: $2 \to 1$.

d. Case 4: Suppose that $\widetilde{\Delta}_{21}(0,0,0) < c$. In this case, no candidate is willing to attack an opponent. Thus, in equilibrium, no candidate attacks.■

D.5 Proposition B.5

First, we want to show that if ϕ is small enough, then $\Delta_{12}(n) > \widetilde{\Delta}_{12}(n^{'})$ for any $n \in \{0,1\}$ and $n^{'} \in \mathcal{N}_1$. To do so, let $f_1(\phi,n,n^{'}) := \Delta_{12}(n) - \widetilde{\Delta}_{12}(n^{'})$. We will show that $f_1(0,n,n^{'}) = 0$ and $\partial f_1(0,n,n^{'})/\partial\phi > 0$ for any $n \in \{0,1\}$ and $n^{'} \in \mathcal{N}_1$. It is immediate to see that $f_1(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(0,n,n^{'})}{\partial\phi} = \frac{1}{2}\exp\left(s_1^o + s_2^o\right)\left(\frac{4}{(\exp(s_1^o)+\exp(s_2^o))^2} - \frac{3}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_2^o,$$

which is always positive, given that $(\exp(s_1^o)+\exp(s_2^o))^2 < (\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2$, so that the term inside parenthesis is positive.

Next, we want to show that if ϕ is small enough, then $\Delta_{21}(n) > \widetilde{\Delta}_{21}(n^{'})$ for any $n \in \{0,1\}$ and $n^{'} \in \mathcal{N}_2$. To do so, let $f_2(\phi,n,n^{'}) := \Delta_{21}(n) - \widetilde{\Delta}_{21}(n^{'})$. We will show that $f_2(0,n,n^{'}) = 0$ and $\partial f_2(0,n,n^{'})/\partial\phi > 0$ for any $n \in \{0,1\}$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $f_2(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0,n,n^{'})}{\partial\phi} = \frac{1}{2}\exp\left(s_1^o + s_2^o\right)\left(\frac{4}{(\exp(s_1^o)+\exp(s_2^o))^2} - \frac{3}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_1^o,$$

which, using the same argument as above, is always positive.■

D.6 Proposition B.6

Item i. We want to show that if ϕ is small enough and s_1^o is sufficiently large, then $\widetilde{\Delta}_{23}^{DB}(n) > \widetilde{\Delta}_{21}^{DB}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_2$. To do so, let $f_2(\phi,n,n^{'}) := \widetilde{\Delta}_{23}^{DB}(n) - \widetilde{\Delta}_{21}^{DB}(n^{'})$. We will show that $f_2(0,n,n^{'}) = 0$ and $\lim_{s_1^o\to\infty}\partial f_2(0,n,n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_2$, so that by taking a first-order Taylor expansion of the function $f_2(\phi,n,n^{'})$ around $\phi = 0$ it follows that $f_2(\phi,n,n^{'}) > 0$ for ϕ sufficiently small and s_1^o large enough. It is immediate to see that $f_2(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial f_2(0,n,n^{'})}{\partial\phi} = {} & \Gamma\{\exp(3s_1^o)s_3^o + 4\exp(2s_1^o+s_2^o)s_3^o + 2\exp(2s_1^o+s_3^o)s_3^o \\ & -\exp(3s_3^o)\,s_1^o - 2\exp(s_1^o+2s_3^o)s_1^o - 4\exp(s_2^o+2s_3^o)\,s_1^o - \exp(s_1^o+2s_2^o)\,(2s_1^o-5s_3^o) \\ & -\exp(2s_2^o+s_3^o)(5s_1^o-2s_3^o) - 2\exp(3s_2^o)\,(s_1^o-s_3^o) - 4\exp(s_1^o+s_2^o+s_3^o)(s_1^o-s_3^o)\}\end{aligned}$$

where

$$\Gamma = \frac{3\exp\left(s_1^o + s_2^o + s_3^o\right)}{2\left(\exp\left(s_1^o\right) + \exp\left(s_2^o\right)\right)^2\left(\exp\left(s_2^o\right) + \exp\left(s_3^o\right)\right)^2\left(\exp\left(s_1^o\right) + \exp\left(s_2^o\right) + \exp\left(s_3^o\right)\right)^2} > 0$$

Note that since we are interested in the sign of the derivative when s_1^o is arbitrarily large, it is enough to consider only the terms inside the curly brackets where s_1^o appears as an exponent, given that these terms will dominate the expression when $s_1^o \to \infty$. Therefore, we are left with:

$$\begin{aligned}&\exp(3s_1^o)s_3^o + 4\exp(2s_1^o + s_2^o)s_3^o + 2\exp(2s_1^o + s_3^o)s_3^o - 2\exp(s_1^o + 2s_3^o)s_1^o\\ &- \ \exp\left(s_1^o + 2s_2^o\right)\left(2s_1^o - 5s_3^o\right) - 4\exp(s_1^o + s_2^o + s_3^o)(s_1^o - s_3^o)\end{aligned}$$

Next, observe that the exponential terms become dominant when $s_1^o \to \infty$, so that we can also ignore the terms multiplying them. Thus, we get:

$$\exp(3s_1^o) + \exp(2s_1^o + s_2^o) + \exp(2s_1^o + s_3^o) - \exp(s_1^o + 2s_3^o) - \exp\left(s_1^o + 2s_2^o\right) - \exp(s_1^o + s_2^o + s_3^o),$$

which is always positive when s_1^o is sufficiently large, since $\lim_{s_1^o\to\infty}\exp(3s_1^o)$, $\lim_{s_1^o\to\infty}\exp(2s_1^o + s_2^o)$ and $\lim_{s_1^o\to\infty}\exp(2s_1^o + s_3^o)$ are all greater than $\lim_{s_1^o\to\infty}\exp(s_1^o + 2s_3^o)$, $\lim_{s_1^o\to\infty}\exp\left(s_1^o + 2s_2^o\right)$ and $\lim_{s_1^o\to\infty}\exp(s_1^o + s_2^o + s_3^o)$. Therefore, $\lim_{s_1^o\to\infty}\partial f_2(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_2$.

Item ii. We want to show that if ϕ is small enough and s_1^o is sufficiently large, then $\widetilde{\Delta}_{32}^{DB}(n) > \widetilde{\Delta}_{31}^{DB}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n^{'}) := \widetilde{\Delta}_{32}^{DB}(n) - \widetilde{\Delta}_{31}^{DB}(n^{'})$. We will show that $f_3(0, n, n^{'}) = 0$ and $\lim_{s_1^o\to\infty}\partial f_3(0, n, n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial f_3(0, n, n^{'})}{\partial\phi} \quad = \quad & \Psi\{\exp(3s_1^o)s_2^o + 4\exp(2s_1^o + s_3^o)s_2^o + 2\exp(2s_1^o + s_2^o)s_2^o\\ &- \exp\left(3s_2^o\right)s_1^o - 2\exp(s_1^o + 2s_2^o)s_1^o - 4\exp\left(2s_2^o + s_3^o\right)s_1^o - \exp\left(s_1^o + 2s_3^o\right)\left(2s_1^o - 5s_2^o\right)\\ &- \exp(s_2^o + 2s_3^o)(5s_1^o - 2s_2^o) - 2\exp\left(3s_3^o\right)\left(s_1^o - s_2^o\right) - 4\exp(s_1^o + s_2^o + s_3^o)(s_1^o - s_2^o)\}\end{aligned}$$

where

$$\Psi = \frac{3\exp\left(s_1^o + s_2^o + s_3^o\right)}{2\left(\exp\left(s_1^o\right) + \exp\left(s_3^o\right)\right)^2\left(\exp\left(s_2^o\right) + \exp\left(s_3^o\right)\right)^2\left(\exp\left(s_1^o\right) + \exp\left(s_2^o\right) + \exp\left(s_3^o\right)\right)^2} > 0$$

Proceeding as above, the sign of the derivative when $s_1^o \to \infty$ is determined by the following expression:

$$\exp(3s_1^o) + \exp(2s_1^o + s_3^o) + \exp(2s_1^o + s_2^o) - \exp(s_1^o + 2s_2^o) - \exp\left(s_1^o + 2s_3^o\right) - \exp(s_1^o + s_2^o + s_3^o),$$

which is always positive when s_1^o is sufficiently large. Therefore, $\lim_{s_1^o\to\infty}\partial f_3(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_3$.$\blacksquare$

D.7 Proposition B.7

We want to show that if ϕ is small enough and the distance $s_1^o - s_3^o$ is sufficiently small, then $\widetilde{\Delta}_{ij}^{DB}(n) > \widetilde{\Delta}_{ij}(n^{'})$ for any i and j with $i \neq j$ and $n, n^{'} \in \mathcal{N}_i$. To do so, let $f_{ij}(\phi, n, n^{'}) := \widetilde{\Delta}_{ij}^{DB}(n) - \widetilde{\Delta}_{ij}(n^{'})$. We will show that $f_{ij}(0, n, n^{'}) = 0$ and $\lim_{(s_1^o, s_2^o, s_3^o)\to(s^o, s^o, s^o)}\partial f_{ij}(0, n, n^{'})/\partial\phi > 0$ for any $n, n^{'} \in \mathcal{N}_i$, so that by taking a first-order Taylor expansion of the function $f_{ij}(\phi, n, n^{'})$ around $\phi = 0$ it follows that

$f_{ij}(\phi, n, n^{'}) > 0$ for ϕ and $s_1^o - s_3^o$ sufficiently small. In what follows, we consider all possible combinations for i and j.

i. Case 1: Let $i = 1$ and $j = 2$. It is immediate to see that $f_{12}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{12}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_1^o + s_2^o\right)\left(\frac{1}{(\exp(s_1^o)+\exp(s_2^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_2^o$$

Then, note that $\lim_{(s_1^o,s_2^o,s_3^o)\to(s^o,s^o,s^o)} \frac{\partial f_{12}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(2s^o\right)\left(\frac{1}{4\exp(s^o)} - \frac{2}{9\exp(s^o)}\right)s^o > 0.$

ii. Case 2: Let $i = 1$ and $j = 3$. It is immediate to see that $f_{13}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{13}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_1^o + s_3^o\right)\left(\frac{1}{(\exp(s_1^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_3^o$$

Then, note that $\lim_{(s_1^o,s_2^o,s_3^o)\to(s^o,s^o,s^o)} \frac{\partial f_{13}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(2s^o\right)\left(\frac{1}{4\exp(s^o)} - \frac{2}{9\exp(s^o)}\right)s^o > 0.$

iii. Case 3: Let $i = 2$ and $j = 1$. It is immediate to see that $f_{21}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{21}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_1^o + s_2^o\right)\left(\frac{1}{(\exp(s_1^o)+\exp(s_2^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_1^o$$

Then, note that $\lim_{(s_1^o,s_2^o,s_3^o)\to(s^o,s^o,s^o)} \frac{\partial f_{21}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(2s^o\right)\left(\frac{1}{4\exp(s^o)} - \frac{2}{9\exp(s^o)}\right)s^o > 0.$

iv. Case 4: Let $i = 2$ and $j = 3$. It is immediate to see that $f_{23}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{23}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_2^o + s_3^o\right)\left(\frac{1}{(\exp(s_2^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_3^o$$

Note that this expression is always positive since $\frac{1}{(\exp(s_2^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2} > 0$ for any $s_1^o > s_2^o > s_3^o$. Thus, the result that $\widetilde{\Delta}_{23}^{DB}(n) > \widetilde{\Delta}_{23}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_2$ does not require that s_1^o be close to s_3^o.

v. Case 5: Let $i = 3$ and $j = 1$. It is immediate to see that $f_{31}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{31}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_1^o + s_3^o\right)\left(\frac{1}{(\exp(s_1^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_1^o$$

Then, note that $\lim_{(s_1^o,s_2^o,s_3^o)\to(s^o,s^o,s^o)} \frac{\partial f_{31}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(2s^o\right)\left(\frac{1}{4\exp(s^o)} - \frac{2}{9\exp(s^o)}\right)s^o > 0.$

vi. Case 6: Let $i = 3$ and $j = 2$. It is immediate to see that $f_{32}(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_{32}(0,n,n^{'})}{\partial \phi} = \frac{3}{2}\exp\left(s_2^o + s_3^o\right)\left(\frac{1}{(\exp(s_2^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2}\right)s_2^o$$

Note that this expression is always positive since $\frac{1}{(\exp(s_2^o)+\exp(s_3^o))^2} - \frac{2}{(\exp(s_1^o)+\exp(s_2^o)+\exp(s_3^o))^2} > 0$ for any $s_1^o > s_2^o > s_3^o$. Thus, the result that $\widetilde{\Delta}_{32}^{DB}(n) > \widetilde{\Delta}_{32}(n^{'})$ for any $n, n^{'} \in \mathcal{N}_3$ for any does not require that s_1^o be close to s_3^o. ■

D.8 Proposition B.8

Item i. First, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{1,2}(n) > \widetilde{\Delta}_{1,23}(n^{'})$ for any $n \in \mathcal{N}_{1,2}$ and $n^{'} \in \mathcal{N}_{1,23}$. To do so, let $f_1(\phi, n, n^{'}) := \widetilde{\Delta}_{1,2}(n) - \widetilde{\Delta}_{1,23}(n^{'})$. We will show that $f_1(0, n, n^{'}) = 0$ and $\partial f_1(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{1,2}$ and $n^{'} \in \mathcal{N}_{1,23}$, so that by taking a first-order Taylor expansion of the function $f_1(\phi, n, n^{'})$ around $\phi = 0$ it follows that $f_1(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f_1(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_1(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_1^o)\left(s_2^o\exp(s_2^o) - s_3^o\exp(s_3^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o\exp(s_2^o) > s_3^o\exp(s_3^o)$.

Next, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{1,23}(n) > \widetilde{\Delta}_{1,3}(n^{'})$ for any $n \in \mathcal{N}_{1,23}$ and $n^{'} \in \mathcal{N}_{1,3}$. To do so, let $g_1(\phi, n, n^{'}) := \widetilde{\Delta}_{1,23}(n) - \widetilde{\Delta}_{1,3}(n^{'})$. We will show that $g_1(0, n, n^{'}) = 0$ and $\partial g_1(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{1,23}$ and $n^{'} \in \mathcal{N}_{1,3}$, so that by taking a first-order Taylor expansion of the function $g_1(\phi, n, n^{'})$ around $\phi = 0$ it follows that $g_1(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $g_1(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_1(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_1^o)\left(s_2^o\exp(s_2^o) - s_3^o\exp(s_3^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_2^o\exp(s_2^o) > s_3^o\exp(s_3^o)$.

Item ii. Second, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{2,1}(n) > \widetilde{\Delta}_{2,13}(n^{'})$ for any $n \in \mathcal{N}_{2,1}$ and $n^{'} \in \mathcal{N}_{2,13}$. To do so, let $f_2(\phi, n, n^{'}) := \widetilde{\Delta}_{2,1}(n) - \widetilde{\Delta}_{2,13}(n^{'})$. We will show that $f_2(0, n, n^{'}) = 0$ and $\partial f_2(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{2,1}$ and $n^{'} \in \mathcal{N}_{2,13}$, so that by taking a first-order Taylor expansion of the function $f_2(\phi, n, n^{'})$ around $\phi = 0$ it follows that $f_2(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f_2(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_2(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_2^o)\left(s_1^o\exp(s_1^o) - s_3^o\exp(s_3^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o\exp(s_1^o) > s_3^o\exp(s_3^o)$.

Next, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{2,13}(n) > \widetilde{\Delta}_{2,3}(n^{'})$ for any $n \in \mathcal{N}_{2,13}$ and $n^{'} \in \mathcal{N}_{2,3}$. To do so, let $g_2(\phi, n, n^{'}) := \widetilde{\Delta}_{2,13}(n) - \widetilde{\Delta}_{2,3}(n^{'})$. We will show that $g_2(0, n, n^{'}) = 0$ and $\partial g_2(0, n, n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{2,13}$ and $n^{'} \in \mathcal{N}_{2,3}$, so that by taking a first-order Taylor expansion of the function $g_2(\phi, n, n^{'})$ around $\phi = 0$ it follows that $g_2(\phi, n, n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $g_2(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_2(0, n, n^{'})}{\partial\phi} = \frac{3\exp(s_2^o)\left(s_1^o\exp(s_1^o) - s_3^o\exp(s_3^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o\exp(s_1^o) > s_3^o\exp(s_3^o)$.

Item iii. Third, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{3,1}(n) > \widetilde{\Delta}_{3,12}(n^{'})$ for any $n \in \mathcal{N}_{3,1}$ and $n^{'} \in \mathcal{N}_{3,12}$. To do so, let $f_3(\phi, n, n^{'}) := \widetilde{\Delta}_{3,1}(n) - \widetilde{\Delta}_{3,12}(n^{'})$. We will show that $f_3(0, n, n^{'}) = 0$ and

$\partial f_3(0,n,n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{3,1}$ and $n^{'} \in \mathcal{N}_{3,12}$, so that by taking a first-order Taylor expansion of the function $f_3(\phi,n,n^{'})$ around $\phi = 0$ it follows that $f_3(\phi,n,n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $f_3(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial f_3(0,n,n^{'})}{\partial\phi} = \frac{3\exp(s_3^o)\left(s_1^o\exp(s_1^o) - s_2^o\exp(s_2^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o\exp(s_1^o) > s_2^o\exp(s_2^o)$.

Next, we want to show that if ϕ is small enough then $\widetilde{\Delta}_{3,12}(n) > \widetilde{\Delta}_{3,2}(n^{'})$ for any $n \in \mathcal{N}_{3,12}$ and $n^{'} \in \mathcal{N}_{3,2}$. To do so, let $g_3(\phi,n,n^{'}) := \widetilde{\Delta}_{3,12}(n) - \widetilde{\Delta}_{3,2}(n^{'})$. We will show that $g_3(0,n,n^{'}) = 0$ and $\partial g_3(0,n,n^{'})/\partial\phi > 0$ for any $n \in \mathcal{N}_{3,12}$ and $n^{'} \in \mathcal{N}_{3,2}$, so that by taking a first-order Taylor expansion of the function $g_3(\phi,n,n^{'})$ around $\phi = 0$ it follows that $g_3(\phi,n,n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $g_3(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial g_3(0,n,n^{'})}{\partial\phi} = \frac{3\exp(s_3^o)\left(s_1^o\exp(s_1^o) - s_2^o\exp(s_2^o)\right)}{4(\exp(s_1^o) + \exp(s_2^o) + \exp(s_3^o))^2},$$

which is always strictly positive, since $s_1^o\exp(s_1^o) > s_2^o\exp(s_2^o)$. ■

D.9 Proposition B.9

We want to show that if ϕ is small enough then $\Delta_{12}^T(n) < \Delta_{21}^T(n^{'})$ for any $\mu > 0$ and $n,n^{'} \in \{0,1\}$. To do so, let $h(\phi,n,n^{'}) := \Delta_{21}^T(n^{'}) - \Delta_{12}^T(n)$. We will show that $h(0,n,n^{'}) = 0$ and $\partial h(0,n,n^{'})/\partial\phi > 0$ for any $n,n^{'} \in \{0,1\}$, so that by taking a first-order Taylor expansion of the function $h(\phi,n,n^{'})$ around $\phi = 0$ it follows that $h(\phi,n,n^{'}) > 0$ for ϕ sufficiently small. It is immediate to see that $h(0,n,n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\frac{\partial h(0,n,n^{'})}{\partial\phi} = \frac{\mu(s_1^o)^{\mu-1}(s_2^o)^{\mu-1}\left(s_1^o + s_2^o\right)\left(s_1^o - s_2^o\right)}{((s_1^o)^\mu + (s_2^o)^\mu)^2},$$

which is always positive, since $s_1^o > s_2^o$. ■

D.10 Proposition B.10

Item i. We want to show that if ϕ is small enough then $\widetilde{\Delta}_{12}^T(n) > \widetilde{\Delta}_{13}^T(n^{'})$ for any $\mu > 1$ and $n,n^{'} \in \mathcal{N}_1$. To do so, let $f_1(\phi,n,n^{'}) := \widetilde{\Delta}_{12}^T(n) - \widetilde{\Delta}_{13}^T(n^{'})$. We will show that $f_1(0,n,n^{'}) = 0$ and $\partial f_1(0,n,n^{'})/\partial\phi > 0$ for any $\mu > 1$ and $n,n^{'} \in \mathcal{N}_1$, so that by taking a first-order Taylor expansion of function $f_1(\phi,n,n^{'})$ around $\phi = 0$ it follows that $f_1(\phi,n,n^{'}) > 0$ for ϕ sufficiently small and $\mu > 1$. It is immediate to see that $f_1(0,n,n^{'}) = 0$. Furthermore, after some algebra, we obtain:

$$\begin{aligned}\frac{\partial f_1(0,n,n^{'})}{\partial\phi} \quad = \quad & \frac{\mu(s_1^o)^{\mu-1}}{2s_2^o s_3^o((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2}\big[\big(2s_1^o(s_2^o)^{\mu+1}s_3^o + s_1^o(s_2^o)^\mu(s_3^o)^2 + (s_2^o)^{\mu+2}s_3^o + (s_2^o)^2(s_3^o)^{\mu+1}\big) \\ & - \big(2s_1^o s_2^o(s_3^o)^{\mu+1} + s_1^o(s_2^o)^2(s_3^o)^\mu + s_2^o(s_3^o)^{\mu+2} + (s_2^o)^{\mu+1}(s_3^o)^2\big)\big]\end{aligned}$$

Note that the term inside brackets determines the sign of the derivative and can be expressed as:

$$s_1^o(s_2^o)^\mu s_3^o(2s_2^o + s_3^o) - s_1^o s_2^o(s_3^o)^\mu(s_2^o + 2s_3^o) + ((s_2^o)^2 s_3^o - s_2^o(s_3^o)^2)((s_2^o)^\mu + (s_3^o)^\mu),$$

which is guaranteed to be positive provided that $\mu > 1$.

Item ii. We want to show that if ϕ is small enough then $\widetilde{\Delta}^T_{21}(n) > \widetilde{\Delta}^T_{23}(n^{'})$ for any $\mu > 1$ and $n, n^{'} \in \mathcal{N}_2$. To do so, let $f_2(\phi, n, n^{'}) := \widetilde{\Delta}^T_{21}(n) - \widetilde{\Delta}^T_{23}(n^{'})$. We will show that $f_2(0, n, n^{'}) = 0$ and $\partial f_2(0, n, n^{'})/\partial\phi > 0$ for any $\mu > 1$ and $n, n^{'} \in \mathcal{N}_2$. It is immediate to see that $f_2(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial f_2(0,n,n^{'})}{\partial\phi} &= \frac{\mu(s_2^o)^{\mu-1}}{2s_1^o s_3^o((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2}[\big(2(s_1^o)^{\mu+1}s_2^o s_3^o + (s_1^o)^\mu s_2^o (s_3^o)^2 + (s_1^o)^{\mu+2}s_3^o + (s_1^o)^2(s_3^o)^{\mu+1}\big) \\ &\quad - \big(2s_1^o s_2^o (s_3^o)^{\mu+1} + (s_1^o)^2 s_2^o (s_3^o)^\mu + s_1^o (s_3^o)^{\mu+2} + (s_1^o)^{\mu+1}(s_3^o)^2\big)]\end{aligned}$$

Note that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu s_2^o s_3^o(2s_1^o + s_3^o) - s_1^o s_2^o (s_3^o)^\mu (s_1^o + 2s_3^o) + ((s_1^o)^2 s_3^o - s_1^o(s_3^o)^2)((s_1^o)^\mu + (s_3^o)^\mu),$$

which is guaranteed to be positive provided that $\mu > 1$.

Item iii. We want to show that if ϕ is small enough then $\widetilde{\Delta}^T_{31}(n) > \widetilde{\Delta}^T_{32}(n^{'})$ for any $\mu > 1$ and $n, n^{'} \in \mathcal{N}_3$. To do so, let $f_3(\phi, n, n^{'}) := \widetilde{\Delta}^T_{31}(n) - \widetilde{\Delta}^T_{32}(n^{'})$. We will show that $f_3(0, n, n^{'}) = 0$ and $\partial f_3(0, n, n^{'})/\partial\phi > 0$ for any $\mu > 1$ and $n, n^{'} \in \mathcal{N}_3$. It is immediate to see that $f_3(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial f_3(0,n,n^{'})}{\partial\phi} &= \frac{\mu(s_3^o)^{\mu-1}}{2s_1^o s_2^o((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2}[\big(2(s_1^o)^{\mu+1}s_2^o s_3^o + (s_1^o)^\mu (s_2^o)^2 s_3^o + (s_1^o)^{\mu+2}s_2^o + (s_1^o)^2(s_2^o)^{\mu+1}\big) \\ &\quad - \big(2s_1^o (s_2^o)^{\mu+1} s_3^o + (s_1^o)^2 (s_2^o)^\mu s_3^o + s_1^o (s_2^o)^{\mu+2} + (s_1^o)^{\mu+1}(s_2^o)^2\big)]\end{aligned}$$

Note that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu s_2^o s_3^o\,(2s_1^o + s_2^o) - s_1^o (s_2^o)^\mu s_3^o\,(s_1^o + 2s_2^o) + \big((s_1^o)^2 s_2^o - s_1^o(s_2^o)^2\big)\,((s_1^o)^\mu + (s_2^o)^\mu)\,,$$

which is guaranteed to be positive provided that $\mu > 1$.

Item iv. Finally, we want to show that if ϕ is small enough then $\widetilde{\Delta}^T_{21}(n^{'}) > \max\{\widetilde{\Delta}^T_{12}(n), \widetilde{\Delta}^T_{31}(n^{''})\}$ for any $\mu > 1$ and $n \in \mathcal{N}_1$, $n^{'} \in \mathcal{N}_2$ and $n^{''} \in \mathcal{N}_3$. First, let $h_{12}(\phi, n, n^{'}) := \widetilde{\Delta}^T_{21}(n^{'}) - \widetilde{\Delta}^T_{12}(n)$. We will show that $h_{12}(0, n, n^{'}) = 0$ and $\partial h_{12}(0, n, n^{'})/\partial\phi > 0$ for any $\mu > 1$, $n \in \mathcal{N}_1$ and $n^{'} \in \mathcal{N}_2$. It is immediate to see that $f_3(0, n, n^{'}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial h_{12}(0,n,n^{'})}{\partial\phi} &= \frac{\mu}{2s_1^o s_2^o((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2}[\big((s_1^o)^{\mu+2}(s_2^o)^\mu s_3^o + (s_1^o)^{\mu+1}(s_2^o)^2(s_3^o)^\mu + (s_1^o)^2(s_2^o)^\mu(s_3^o)^{\mu+1}\big) \\ &\quad - \big((s_1^o)^\mu (s_2^o)^{\mu+2} s_3^o + (s_1^o)^\mu (s_2^o)^2 (s_3^o)^{\mu+1} + (s_1^o)^2(s_2^o)^{\mu+1}(s_3^o)^\mu\big)]\end{aligned}$$

Observe that the term inside brackets can be rewritten as:

$$(s_1^o)^\mu (s_2^o)^\mu s_3^o\,\big((s_1^o)^2 - (s_2^o)^2\big) + (s_1^o)^\mu s_2^o (s_3^o)^\mu\,(s_1^o - s_3^o)\,s_2^o - s_1^o (s_2^o)^\mu (s_3^o)^\mu\,(s_2^o - s_3^o)\,s_1^o,$$

But note that since $(s_1^o)^\mu (s_2^o)^\mu s_3^o > (s_1^o)^\mu s_2^o (s_3^o)^\mu > s_1^o (s_2^o)^\mu (s_3^o)^\mu$ for any $\mu > 1$, then a sufficient condition for this expression to be positive is:

$$\big((s_1^o)^2 - (s_2^o)^2\big) + (s_1^o - s_3^o)\,s_2^o - (s_2^o - s_3^o)\,s_1^o > 0,$$

which always holds.

Next, let $h_{23}(\phi, n^{'}, n^{''}) := \widetilde{\Delta}^T_{21}(n^{'}) - \widetilde{\Delta}^T_{31}(n^{''})$. We will show that $h_{23}(0, n, n^{''}) = 0$ and $\partial h_{23}(0, n, n^{''})/\partial\phi >$

0 for any $\mu > 1$, $n^{'} \in \mathcal{N}_2$ and $n^{''} \in \mathcal{N}_3$. It is immediate to see that $h_{23}(0, n^{'}, n^{''}) = 0$. Furthermore, after some algebra, we get:

$$\begin{aligned}\frac{\partial h_{23}(0, n^{'}, n^{''})}{\partial \phi} \quad = \quad & \frac{\mu}{2s_1^o s_2^o((s_1^o)^\mu + (s_2^o)^\mu + (s_3^o)^\mu)^2}[\left((s_1^o)^{\mu+2}(s_2^o)^\mu s_3^o + 2(s_1^o)^{\mu+1}(s_2^o)^{\mu+1}s_3^o + 2(s_1^o)^2(s_2^o)^\mu(s_3^o)^{\mu+1}\right) \\ & - \left((s_1^o)^{\mu+2}s_2^o(s_3^o)^\mu - 2(s_1^o)^{\mu+1}s_2^o(s_3^o)^{\mu+1} - 2(s_1^o)^2(s_2^o)^{\mu+1}(s_3^o)^\mu\right)]\end{aligned}$$

Observe that the term inside brackets can be written as:

$$s_1^o s_2^o s_3^o \left\{ s_1^o \left((s_1^o)^\mu(s_2^o)^{\mu-1} - (s_1^o)^\mu(s_3^o)^{\mu-1}\right) + 2(s_1^o)^\mu \left((s_2^o)^\mu - (s_3^o)^\mu\right) - 2s_1^o \left((s_2^o)^\mu(s_3^o)^{\mu-1} - (s_2^o)^{\mu-1}(s_3^o)^\mu\right)\right\},$$

which is always positive since $2(s_1^o)^\mu \left((s_2^o)^\mu - (s_3^o)^\mu\right) > 2s_1^o \left((s_2^o)^\mu(s_3^o)^{\mu-1} - (s_2^o)^{\mu-1}(s_3^o)^\mu\right)$ for any $\mu > 1$.$\blacksquare$

E Right of Reply Lawsuits in Brazil

In this appendix, we discuss the details of two right of reply lawsuits, focusing on three main aspects: (*i*) the campaign attack, (*ii*) the electoral judge's decision and (*iii*) the reply. Our accounts are based on a number of court documents, which we were able to access for these two particular cases and provide unique insight into how these lawsuits work. Both cases occurred during the 2020 municipal election in the city of Sao Paulo, the largest one in the country, and involved the same two candidates, Celso Russomanno and Guilherme Boulos. Russomanno is a Brazilian congressman who became notorious for a TV show in which he confronted businesses owners and shop workers in defense of consumer rights. Boulos, on the other hand, is the former leader of the Homeless Workers' Movement (*Movimento dos Trabalhadores sem Teto - MTST*), a social movement which acts against housing shortages in metropolitan areas, often invading unoccupied plots and abandoned buildings. Both cases occurred during the first round of the elections, where Boulos and Russomanno were in direct competition for a place in the runoff against the incumbent, Bruno Covas.

E.1 *Russomanno vs. Boulos*

Campaign Attack. On October 15, Boulos posted on his Facebook and Instagram accounts a message saying, "*Behind the cameras, Russomanno hates the poor*" (see Figure E.1). He also shared in the same post an excerpt of an interview where Russomanno stated: "*homeless people and residents of Cracolandia are possibly more resistant to Covid-19 because they don't bathe*".[68] On the following day, October 16, Russomanno filed a lawsuit against Boulos demanding that the post in question be taken down and that a right of reply be granted in his favor. In his complaint, Russomanno argued that his statement about the homeless being "*more resistant to Covid-19*" was taken out of context and that the claim that "*Russomanno hates the poor*" was defamatory and untrue. The complaint was short and concise, containing two screenshots of the offending posts and brief references to the relevant legislation and legal precedents.

Decision. On October 20, after receiving Boulos' defense statement, the electoral judge ruled in favor of Russomanno. First, with respect to the accusation that "*Russomanno hates the poor*", the magistrate understood that "*while harsh criticism is an integral part of campaigns, in the present case there was excess from the part of the defendant insofar as he made a frivolous assertion, impossible of being verified, without any foundation in reality*". However, regarding the excerpt from Russomanno's interview, the judge saw no excess from Boulos' part, since "*he just reproduced what the plaintiff actually said*". The decision granted Russomanno the right to publish a reply on Boulos' social media which was to remain visible for a period twice as long as the attack. The judge also ordered that the offending posts be excluded immediately and imposed a daily fine of R\$ $10,000$ (US\$ $1,777$) in case the defendant did not comply with the decision.

Reply. Figure E.1 also shows Russomanno's reply, as posted on Boulos's Instagram and Facebook accounts. The translation reads as follows:

> *Right of Reply granted by the Electoral Justice. Guilherme Boulos offended Celso Russomanno and was punished by the Electoral Justice. Celso Russomanno does not hate the poor, so much so that he has always fought for the most disadvantaged class in these 30 years of defending consumer rights and will continue to fight if elected mayor of Sao Paulo. Real politics are made with proposals and not with offenses!*

[68] "Cracolandia" is a region of the city of Sao Paulo well-known for its high incidence of drug trafficking and drug use in public. The story is available at Globo's G1 Portal, https://glo.bo/3HMA1X7.

E.2 *Boulos vs. Russomanno*

Campaign Attack. About two weeks after the events described above, Russomanno posted a 30 second video on his Facebook and Instagram accounts accusing Boulos of (*i*) leading the invasion of a building in the central zone of Sao Paulo and collecting rent from its occupants[69], (*ii*) omitting personal assets from the electoral justice, and (*iii*) using a supermarket cashier in a campaign ad against Russomanno. Figure E.2 provides a screenshot of the original Instagram post.[70] In his complaint, filed on October 31, Boulos argued that the accusation of breaking into the building and collecting rent was false and defamatory. In fact, the invasion of the building in question, and other unlawful acts, were carried out by a group called "*Movimento Moradia para Todos e Sem Tetos do Centro*", which had all its leaders arrested. He also argued that the omission of personal assets from the electoral justice was unintentional and very minor − something worth approximately US$ 100 − and that, in fact, he had already rectified it. Finally, regarding the supermarket cashier, he argued that she had been humiliated by Russomanno some years earlier on his TV show and that she had voluntarily agreed to participate in the campaign ad against him.

Decision. On November 3, after receiving Russomanno's defense statement, the electoral judge issued a ruling in favor of Boulos. The judge recognized that the accusation that Boulos led the invasion of the building and collected rent was false and defamatory, constituting clear offense to his honor. However, he denied the existence of any illicit act related to the other two charges, namely that Boulos omitted assets from the justice and that he used the supermarket cashier in his campaing, because both of them were factually true and neither offended his integrity. Interestingly, the magistrate stated that "*it is up to the candidate, through regular campaign channels, to clarify his voters*" about both incidents. The decision granted Boulos the right to publish a reply on Russomanno's social media which was to remain visible for a period twice as long as the attack. As in the previous case, the judge also ordered that the offending video be excluded immediately and imposed a R$ $20,000$ (US$ $3,555$) fine in case the defendant did not comply with the decision within 48 hours.

Reply. Boulos' reply was in the form of a 30 second video in which Cleide Cruz, the supermarket cashier, talks about how she felt humiliated by Russomanno in his TV show.[71] Note that in this particular case, the content of the reply is not directly related to the part of the attack which was considered defamatory by the justice. The Brazilian legislation does not impose any restriction on the content of the reply, and the attacked candidate is allowed to use the allotted space as he wishes. The English transcript of what Cleide says in the video is as follows:

> Narrator: *Right of reply granted the Electoral Justice against Russomanno for lies he told against Boulos.*
>
> Cleide Cruz, the supermarket cashier: *Not here, Celso Russomanno. You are lying. You humiliated me that day. You left me feeling terrible, like rubbish. You caused me pain, you caused me problems, Celso Russomanno. You don't have the dignity to apologize for what you did. Serious people, as you say in the video, do their job without humiliating, offending, or vilifying people.*

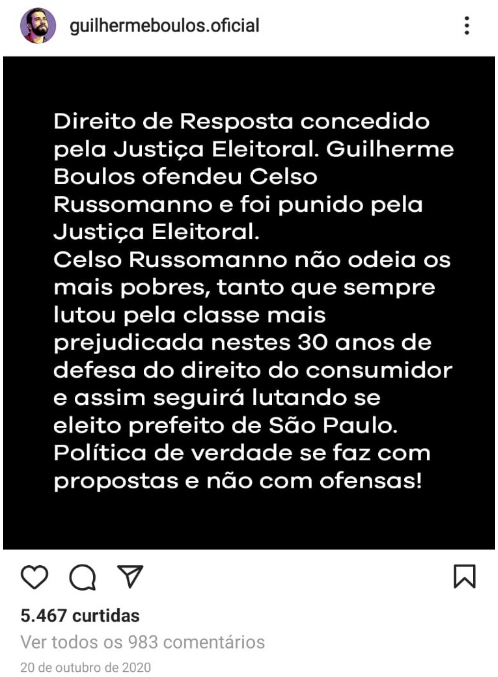
E.3 Figures

[69] The building in question, "Edifício Wilton Paes de Almeida", became notorious for having been destroyed by fire and collapsing a few years earlier. For more information, see https://bit.ly/32G4CFs.

[70] The original video is not available anymore.

[71] The video is available online at https://bit.ly/3xVpK6q

(a) Original Publication



(b) Reply

Figure E.1: *Russomanno vs. Boulos*

Figure E.2: *Boulos vs. Russomanno*: Original Publication

F Survey Data: Attacks and Voting Behavior

In this appendix, we provide further details about the data used in our discussion in Section 6 and we complement our analysis by presenting additional regression results. Our data set consists of a 3,010-person nationally representative survey conducted by the Ibope Institute between the two rounds of the 2018 elections (October 21-23, 2018).[72] In Panel A of Table F.1 we show that the sample means of several key demographic and socio-economic characteristics of surveyed individuals are, indeed, very close to the corresponding population means taken from the Tribunal Superior Eleitoral's (TSE) database of registered voters − although survey respondents do tend to have more high school education. Moreover, in Panel B we report the percentage of respondents who claimed to have voted for Bolsonaro, Haddad, and other candidates in the first round of elections, as well as the distribution of voting intentions for the second round of elections, alongside the actual electoral results. As we discussed in the paper, the 2018 presidential election was sharply polarized between Jair Bolsonaro, the far-right populist, and Fernando Haddad, the leftist candidate and former president Lula's protégé. Interestingly, as shown in Panel B, the poll results are remarkably close to the actual electoral results.

As mentioned in the main text, the Ibope survey was unique in that, in addition to the usual voting intention questions, it also included a specific query about whether individuals received a campaign attack via WhatsApp.[73] As shown in Panel C of Table F.1, 25.2% of respondents received at least one message attacking one of the candidates in the week preceding the first round of elections.[74] Not surprisingly, given the degree of polarization of the 2018 election, among those who received one or more of these messages, 97.3% said that the attacks were directed at either Bolsonaro or Haddad (not reported in Panel C), while 23.7% declared that the content of these messages actually influenced their choice of whom to vote for.[75] As discussed in the paper, these numbers imply that at least 36.8 million voters were exposed to campaign attacks in the week before the first round of elections and approximately 8.5 million of them (or about 5.7% of the electorate) were influenced to some extent by those messages.

We further examine the effect of being exposed to a campaign attack on individual electoral behavior by estimating the following regression:

$$V_i = \beta_0 + \beta_1 Attack_i + X_i\phi + \epsilon_i$$

where V_i is a dummy representing declared vote for a certain candidate (Bolsonaro, Haddad or others) in the first round of elections or intention to cast a null vote in the second round of elections, and $Attack_i$ is a dummy indicating whether the respondent received an attack via WhatsApp. Our analysis controls for a detailed set of demographic characteristics, including gender, age, education level, income level, region of residence, race, and religion. Moreover, we also control for the respondent's current level of life satisfaction, her view on the future of the country (optimistic or pessimistic), the political party (if any) of her preference, and the political party (if any) she would never vote for.

The results reported in Table F.2 suggest that being exposed to a campaign attack induces an individual to become less likely to vote for both Bolsonaro and Haddad (columns 1 and 2), although only the effect on Bolsonaro is statistically significant. These results are consistent with the idea that attacks and aggressive campaigning lead voters to demobilize and are in line with experimental evidence provided by Chong et al. (2015). Indeed, as we show in column 3, receiving an attack − which was most likely directed at either Haddad or Bolsonaro − leads to a 5.9 pp increase in the likelihood of voting for a third candidate in the first round of elections.[76] Furthermore, in column 4, we show that individuals

[72] The Ibope Institute is a reputable Brazilian company specialized in public opinion polls. See https://bit.ly/3IeTmje.

[73] WhatsApp is the most popular messaging app in Brazil. According to a recent survey, approximately 98% of smartphone owners in Brazil said they had WhatsApp installed on their mobile devices (see https://bit.ly/3IvW845). Moreover, a survey conducted by the Brazilian Senate in November 2019 found that 79% of respondents used WhatsApp as their main source of information.

[74] The exact phrasing of this question was: "*Did you receive a message through WhatsApp containing criticisms or attacks against any candidate in the week before the first round of elections?*"

[75] The exact phrasing of this question was: "*Did the content you received helped or not to decide your vote?*"

[76] This finding is consistent with experimental evidence obtained by Galasso et al. (2020) who show that negative

who were exposed to an attack are also 3.3 pp more likely to declare an intention to cast a null vote in the second round of elections.

Next, we investigate whether the effect of receiving an attack on voting behavior varies with the characteristics of the individual exposed to it. In particular, we test whether the impact of an attack is stronger among the more disillusioned groups of voters, i.e. those most likely to feel abandoned by traditional politics and electoral institutions. In Table F.3 we split the sample into two groups of voters according to whether respondents reported feeling "pessimistic" or "optimistic" about the future of the country.[77] Interestingly, we find that the effects are all concentrated in the subsample of "pessimistic" voters. In particular, in Panel A we show that among "pessimistic" voters an exposure to a campaign attack is associated with a 8.0 pp reduction in the likelihood of voting for Haddad, accompanied by a 10.3 pp increase in the probability of voting for a third candidate in the first round of elections and a 6.5 pp increase in the likelihood of declaring an intention to cast a null vote in the second round. In Panel B, on the other hand, we find that among "optimistic" voters the point estimates are all much smaller in magnitude and none of them is statistically significant. Finally, in an additional exercise (available upon request) we show that similar results are obtained when dividing the sample by income level, with all the effects concentrated on the poorer voters.

campaigning generates a positive spillover effect on third candidates (neither the target nor the attacker).

[77] For this analysis, we exclude the individuals who did not answer or claimed not to have an opinion about the future of the country.

F.1 Tables

Table F.1: Summary Statistics: Ibope Survey

Panel A: Demographic Characteristics	Ibope Survey	TSE Database
Region of Residence		
- North/Central-West	0.153	0.152
- Northeast	0.260	0.267
- Southeast	0.437	0.435
- South	0.149	0.146
Demographic and Socio-Economic Characteristics		
- Female	0.527	0.525
- 16-17 years old	0.009	0.010
- 18-24 years old	0.166	0.142
- 25-34 years old	0.234	0.212
- 35-44 years old	0.208	0.206
- 45-54 years old	0.176	0.170
- 55-64 years old	0.127	0.133
- 65+ years old	0.081	0.128
- Illiterate	0.041	0.045
- Primary Education or More	0.750	0.700
- High School Education or More	0.553	0.426
- College Education	0.129	0.105
Panel B: 2018 Presidential Elections	Ibope Survey	Election Results
1st Round of Elections		
- Voted for Bolsonaro	0.414	0.420
- Voted for Haddad	0.281	0.267
- Voted for Another Candidate	0.211	0.225
- Null Vote	0.094	0.088
2nd Round of Elections		
- Intention to Vote for Bolsonaro	0.512	0.499
- Intention to Vote for Haddad	0.380	0.406
- Intention to Cast a Null Vote	0.108	0.096
Panel C: Attacks via WhatsApp	Ibope Survey	
- Received an Attack	0.252	
- Attack Influenced Voting Decision	0.237	

Notes: This table reports summary statistics for selected variables available in the Ibope survey, which was conducted between the two rounds of the 2018 elections (October 21-23, 2018). In Panel A, we report sample means for several demographic and socio-economic characteristics of survey respondents, alongside with the corresponding population means taken from TSE's database of registered voters. In Panel B we report the percentage of respondents who claimed to have voted for each candidate in the 1st round of elections as well as their voting intentions for the 2nd round of elections, alongside with the actual election results. In Panel C, we report sample means for answers to the questions related to attacks via WhatsApp. The variable "Received an Attack" indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. "Attack Influenced Voting Decision" indicates whether the attack influenced the choice of whom to vote for conditional upon receiving an attack.

Table F.2: The Effect of Campaign Attacks on Voting Behavior

	1st Round of Elections			2nd Round of Elections
	Voted for Bolsonaro	Voted for Haddad	Voted for Another Candidate	Null Vote
	(1)	(2)	(3)	(4)
Received an Attack	−0.037**	−0.018	0.059***	0.033**
	[0.017]	[0.015]	[0.018]	[0.014]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	3,010	3,010	3,010	3,010
Adj. R-squared	0.336	0.362	0.092	0.089

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether the respondent voted for a certain candidate (Bolsonaro, Haddad or others) in the first round of elections (columns 1−3) or intends to cast a null vote in the second round of elections (column 4). The variable "Received an Attack" indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. Individual characteristics include gender, age, education level, income level, region of residence, race, and religion. We also control for the respondent's current level of life satisfaction, her view on the future of the country (optimistic or pessimistic), the political party (if any) of her preference, and the political party (if any) she would never vote for. Robust standard errors are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

Table F.3: The Effect of Campaign Attacks on Voting Behavior: Heterogeneous Effects

	1st Round of Elections			2nd Round of Elections
	Voted for Bolsonaro	Voted for Haddad	Voted for Another Candidate	Null Vote
	(1)	(2)	(3)	(4)
Panel A: Pessimistic Voters				
Received an Attack	−0.042	−0.080***	0.103***	0.065***
	[0.029]	[0.028]	[0.031]	[0.026]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	1,167	1,167	1,167	1,167
Adj. R-squared	0.256	0.338	0.147	0.084
Panel B: Optimistic Voters				
Received an Attack	−0.034	0.023	0.021	0.017
	[0.023]	[0.018]	[0.022]	[0.015]
Individual Characteristics	Yes	Yes	Yes	Yes
Observations	1,652	1,652	1,652	1,652
Adj. R-squared	0.343	0.394	0.086	0.091

Notes: This table reports OLS estimates of regressions where the dependent variable is a dummy indicating whether the respondent voted for a certain candidate (Bolsonaro, Haddad or others) in the first round of elections (columns 1−3) or intends to cast a null vote in the second round of elections (column 4). The variable "Received an Attack" indicates whether the respondent received a message via WhatsApp attacking one of the candidates in the week preceding the first round of elections. Panel A reports estimates obtained from a subsample of respondents who declared to be pessimistic about the future of the country. Panel B, in turn, reports estimates obtained from a subsample of respondents who declared to be optimistic about the future of the country. The analysis excludes the individuals who did not answer or claimed not to have an opinion about the future of the country. Individual characteristics include gender, age, education level, income level, region of residence, race, and religion. We also control for the respondent's current level of life satisfaction, the political party (if any) of her preference, and the political party (if any) she would never vote for. Robust standard errors are reported in brackets. $*$, $**$ and $***$ denote statistical significance at 10%, 5% and 1%, respectively.

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