How do Tom and Jerry Play? A Simple Application of Convex Analysis in Hide-and-Seek Games

Xinmi Li, Master of Finance[1]; Jie Zheng, Professor[2]

[1]School of Economics and Management, Tsinghua University

[2]The Center for Economic Research, Shandong University

Abstract

We propose a simultaneous-move hide-and-seek game, where one player wins by matching the other player, while the other player wins by mismatching in a continuous space X in Euclidean space.

- A complete characterization of Type I Nash Equilibrium where the seeker plays a pure strategy, showing that the center of mass of the hider's strategy coincides with the seeker's strategy at the center of the minimal cover ball.
- A characterization Type II Nash Equilibrium where the seeker plays a non-pure strategy, showing that the shape of X matters and the seeker will only allocate the probability weights along a straight line.
- Discussions of results under alternative settings.

These results can be applied to a large number of scenarios, characterizing the behavior of two players in a zero-sum game, where one player aims to maximize the distance between them, while the other aims to minimize it.

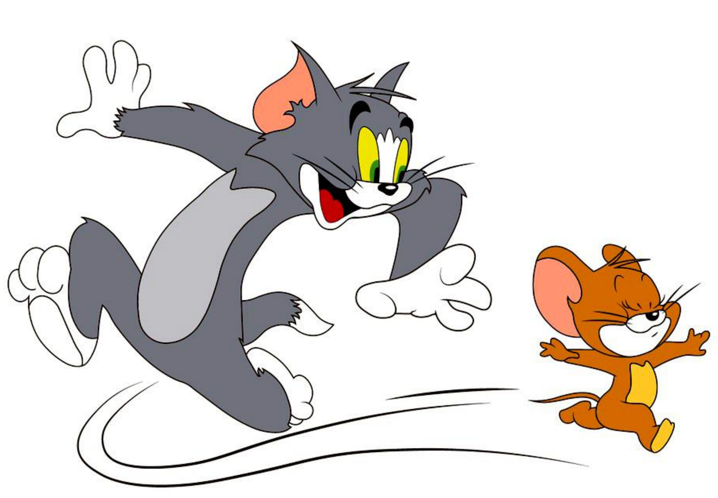
Introduction

Hide-and-seek is a well-known game, where one player aims to win by matching the other's decision, while the other aims to win by mismatching. Typical hide-and-seek games take place every day in the real world:

- **Animal society**. In a jungle, predators hope to catch their preys, while preys always struggle to move away from predators' territories.
- **City management**. Police officers hope to catch criminals, while criminals hope to stay as far away as possible from police officers.
- **Race between innovation and imitation**. Innovators hope to develop new methods, ideas or products, while imitators hope to mimic them.

Games in a space have attracted many game theorists for over a century.

- Hotelling model (Hotelling, 1929) studies how location affects duopoly competition, proposing early concepts linking games with space.
- Von Neumann (1953) studies the 2-dimension zero-sum hide-and-seek game with 2 players.
- Petrosjan (1993) discusses the hide-and-seek problem briefly based on triangles.
- Other different branches, such as 3-player matching pennies games (Jordan, 1993; McCabe et al., 2000; Cao & Yang, 2014; Cao et al., 2019; among many others) or experiments (Crawford & Iriberri, 2007; among many others).



Model

Suppose there are two players, A (seeker) and B (hider). The territory is denoted as a compact convex set $X \subseteq \mathbb{R}^n$. Player i's pure strategy is a point $x_i \in X$. Player i's mixed strategy is a probability measure $\sigma_i \in \Delta(X)$. Assume a 2-norm distance metric. The expected utility functions of the seeker A and hider B are denoted as

$$U_A(\sigma_A, \sigma_B) = \int_{X\times X} -\|x_A - x_B\|_2 \, d\sigma_A d\sigma_B$$

$$U_B(\sigma_A, \sigma_B) = \int_{X\times X} \|x_A - x_B\|_2 \, d\sigma_A d\sigma_B$$

A strategy profile (σ_A^*, σ_B^*) is a mixed strategy Nash Equilibrium, if and only if for any player i, for any deviation strategy $\sigma_i \in \Delta(X)$, we have

$$U_i(\sigma_i^*, \sigma_{-i}^*) \geq U_i(\sigma_i, \sigma_{-i}^*)$$

Definition (Minimal cover ball)

The ball $b(x^*, r^*)$ is a minimal cover ball of a compact convex set $X \subseteq \mathbb{R}^n$, if $X \subseteq b(x^*, r^*)$ and for any ball $b(x, r)$ with $X \subseteq b(x, r)$, we have $r^* \leq r$.

Results

Theorem 1 (Type I Nash Equilibrium)

Type I Nash Equilibrium always exists. A strategy profile (x_A^*, σ_B^*) is a Type I Nash Equilibrium, if and only if:

- Seeker A adopts a pure strategy $x_A^* = x^*$, where x^* is the center of $b_{mc}(X) = b(x^*, r^*)$, the minimal cover ball of X.
- Hider B adopts a mixed strategy σ_B^*, where
  - σ_B^* is supported by the intersection of the boundary of X and its minimal cover ball;
  - the center of mass of σ_B^* locates at x^*.

Figure 1. Examples when X is a compact convex set in $\mathbb{R}$ or $\mathbb{R}^2$.

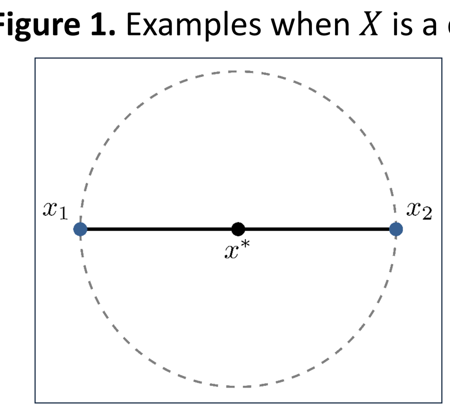


Figure 1-a. X is a closed interval.

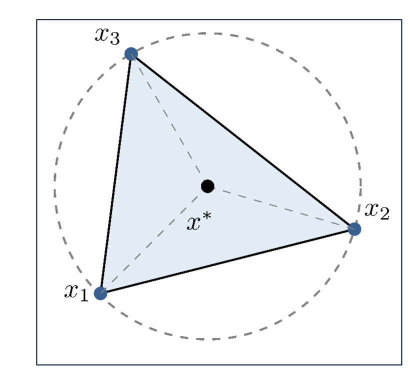


Figure 1-b. X is an acute triangle.

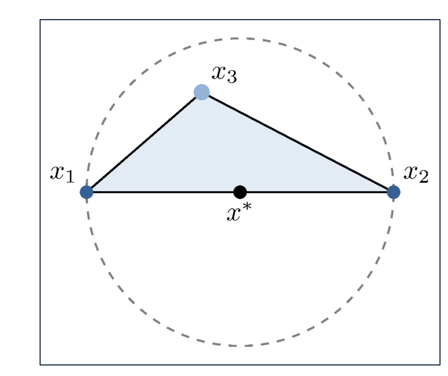


Figure 1-c. X is an obtuse triangle.

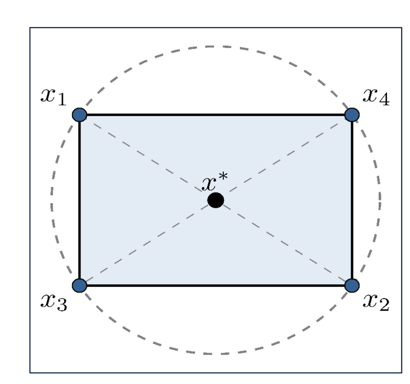


Figure 1-d. X is a box (rectangle in $\mathbb{R}^2$).

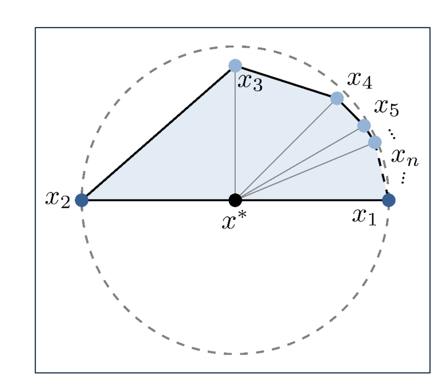


Figure 1-e. Weird X in $\mathbb{R}^2$, where there exists no Type II Nash Equilibrium.

Theorem 2 (Type II Nash Equilibrium)

The existence of Type II Nash Equilibrium depends on the shape of X:

- **Necessary condition**: $\partial X \cap \partial b_{mc}(X)$ contains exactly 1 pair of antipodal points (formally, $\partial X \cap \partial b_{mc}(X) = \{x_1, x_2\}$).
- **Sufficient condition**: There is no converging point sequence in $EP(X) - \{x_1, x_2\}$ that converges to x_1 or x_2, where $EP(X)$ is the extreme point set of X.

If a strategy profile (σ_A^*, σ_B^*) is a Type II Nash Equilibrium, then:

- Seeker A adopts a non-pure strategy σ_A^*, which is supported by the diameter $\overline{x_1 x_2}$ and has a center of mass locating at x^*.
- Hider B adopts a mixed strategy σ_B^*, which distributes equal probability weights only to the points x_1 and x_2. Formally, $\sigma_B^*(x_1) = \sigma_B^*(x_2) = \frac{1}{2}$.

Table 1. The number of Type I and Type II Nash Equilibria. ✔ means a possible combination and ✖ means an impossible one. For each possible combination, an example where $X \subseteq \mathbb{R}$ or $X \subseteq \mathbb{R}^2$ is provided in the bracket.

Type I \ Type II	0	continuum
1	✔ (acute triangle)	✔ (closed interval)
continuum	✔ (box)	✖

Discussion

- Mathematical properties of minimal cover ball
  - existence & uniqueness
  - a convex optimization problem about the minimal cover ball
- Alternative settings
  - when X is no longer a compact convex set in Euclidean space
  - when X is a ball surface with the cosine distance metric

Conclusions

Many social, economic, political and military interactions between two parties with conflict of interest share the feature of a game between a distance-maximizing hider and a distance-minimizing seeker. In this paper, we formally characterize the Nash Equilibrium of a simultaneous-move version of such a hide-and-seek game with a commonly shared compact strategy space X. Alongside this direction, many questions (e.g. the characterization of Nash Equilibrium under alternative settings) still remain open, which we leave for further exploration.

Contact

Xinmi Li
Tsinghua University
Email: lixinmi1999@gmail.com
Website: https://xinmili.weebly.com
Phone: (+86) 134 2749 5219

Jie Zheng
Shandong University
Email: jie.academic@gmail.com
Website: https://jzheng.weebly.com
Phone: (+86) 139 1198 7818

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