Optimal Consumption and Investment Decisions with Disastrous Income Risk: Revisiting Rietz's Rare Disaster Risk Hypothesis

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Abstract

We develop an analytically tractable dynamic model of optimal consumption and investment decisions with disastrous income risk in the context of Rietz's rare disaster risk hypothesis. We first empirically explore the relations among consumption changes, aggregate income, disaster shock severity, and fiscal measures in 55 countries during the Covid-19 period. We then by empirical motivation investigate an important role of insurance with a focus on the recovery of income in a disaster. We highlight how extent of disastrous income risk to which an agent is exposed and her income recovery post disaster jointly affect the agent's optimal decisions. Overall, availability of insurance can be particularly important for both the poor and the wealthy in the sense that they could even consume more, save less, and invest more post disaster as long as their future income is (partly) recovered.

Keywords: Optimal Consumption, Optimal Investment, Disastrous Income Shock, Precautionary Savings, Insurance

JEL Codes: D15, D58, G11, G12

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1 Introduction

How should disastrous income risk affect the optimal consumption and investment decisions of individuals? One of the best understood relationships between income risk and optimal decisions in the consumption/savings literature is that individuals save precautionarily when their income is threatened by a shock, which is consistent with the permanent income hypothesis.

We shed new light on dynamic models of optimal consumption and investment decisions for individuals who exhibit constant absolute risk aversion (CARA) utility preferences[1] by exploring insights into how possibility of a disastrous income shock combined with a non-negative constraint on borrowing[2] affects both the consumption/savings and (positive) wealth allocation decisions between bonds and equity. In particular, we address the interactions between optimal decisions and a change in perception where the changed probability of a future pandemic affects both probability and duration of disaster.

> In particular, by specifying Mehra and Prescott's model to include a low-probability, depression-like third state, I can explain both high equity risk premia and low risk-free returns without abandoning the Arrow-Debreu paradigm (Rietz , 1988).

[1]The CARA utility model keeps all the derived optimal strategies in analytically tractable form. In addition to an analytical convenience, the use of the CARA utility model is necessary for making the problem with disastrous income risk well defined. Specifically, with other utility preferences (e.g., constant relative risk aversion (CRRA) utility model, Epstin-Zin recursive utility model) the maximized expected discounted utility (or the value function of wealth) becomes $-\infty$ with a possibility of a full default of income as wealth approaches the borrowing limit, i.e., when the borrowing constraint binds. This technical challenges associated with disastrous income risk have been first tackled by Bensoussan *et al.* (2016) in which the mix of a logarithmic utility model and a piecewise connected utility model is proposed to prevent the value function from being $-\infty$. In case of CRRA utility model, only a partial default of income can be considered (Jang *et al.*, 2019). The CARA utility model used in this paper allows us to investigate both cases of a full and partial default of income for the consideration of disastrous income risk.

[2]In reality, it is not possible for an individual to fully pledge future income. This constraint is mainly driven by market frictions such as asymmetric information, agency conflicts, and limited enforcement. According to the Survey of Consumer Finances (2017), "In 2016, 20.8 percent of families were considered credit constrained – those who reported being denied credit in the past year, as well as those who did not apply for credit for fear of being denied in the past year."

This is precisely the direction we take on further in this paper. We develop an analytically tractable dynamic model of optimal consumption and investment decisions with disastrous income risk in the context of Rietz's rare disaster risk hypothesis. As a generalization of the rare disaster risk hypothesis to include disastrous income risk, we consider a version of the Merton (1969, 1971) model with the special feature that income can abruptly jump from a positive value to a smaller positive value or even to zero.[3] This specification is also motivated by the recent Covid-19 pandemic. We first empirically explore the relations among consumption changes, aggregate income, disaster shock severity, and fiscal measures in 55 countries during the Covid-19 period.

We then by empirical motivation investigate an important role of insurance with a focus on the recovery of income in a disaster. The currently available social securities and private insurance market are insufficient to perfectly hedge against disastrous income risk, making the financial market essentially incomplete (Cocco *et al.*, 2005; Bensoussan *et al.*, 2016; Jang *et al.*, 2019; Jang *et al.*, 2020). The individuals' ability to recover from and respond to disastrous income shocks is further impeded by their ill-preparedness for future consumption needs.[4] If there is an insurance market for (partially) hedging against disastrous income risk, the individual's income is partly wiped out when a disastrous income shock occurs.[5] In the extreme case of no access to the insurance market, the income is

[3]The disastrous income shocks to which an individual is exposed could be driven by the risks of loss of employment and/or of episode of severe ill health that prevents work and also the duration of unemployment/ill health, which all results in a permanent loss of income.

[4]Indeed, recent empirical evidence points out that many individuals are vulnerable to unexpected economic hardship. According to the 2017 Federal Reserve report, 44% of U.S. households are unable to pay for their emergency expenses of just $400. For more details, refer to "Report on the Economic Well-Being of U.S. Households in 2016" published by Board of Governors of the Federal Reserve System on May 2017. According to EU statistics on Income and Living Conditions (EU-SILC 2017), about 218 million E.U. households are struggling to prepare for their future consumption needs.

[5]As an alternative to the insurance market, there are a wide range of financial mechanisms used in planning for and managing the financial consequences of disasters. In particular, the state (e.g., national-level budget contingencies) has much of the responsibility for managing the financial consequences of large scale disasters. The state plays a key role in covering at least some disaster losses. For instance, disaster relief funds ensure resources readily available to provide prompt assistance following a disaster.

completely defaulted or destroyed in the disastrous shock.[6] In other terms, for the income, there can be a full or partial default in a disastrous shock, and the recovery of income in the aftermath of the shock depends crucially on availability of insurance. In this paper, the income in the aftermath of a disastrous shock is the income before the shock multiplied by the recovery rate.[7]

We offer new insights helping the consumption/savings literature. First, we highlight how extent of disastrous income risk to which an agent is exposed affects low-wealth people's optimal decisions. As is fairly standard in the literature, we find a large precautionary savings motive for income-risk-exposed agents to consume less. Interestingly, the agent's optimal consumption shows a significant discontinuity and the dramatic change in the concavity of consumption when disastrous income risk is more significant than it is slim. We find that if disastrous income risk is small, there does not exist a significant precautionary savings term between agents with little wealth so that there are no significant consumption reductions in response to the income risk. The highly likelihood of a rare event could change the consumption demand of low-wealth people. Overall, how much people do cut back their consumption optimally is determined by the extent to which disastrous income risk is identified quantitatively.

Second, we clarify how the presence of disastrous income risk affects the agent's optimal portfolio choice. The standard consumption/savings literature typically assumes that the agent can invest in riskless bonds only. We allow the agent to trade a risky asset in the market. The agent's optimal investment decision with disastrous income risk can then be further complex for optimally allocating the precautionary savings in both bonds and equity. The precautionary savings terms turn out to contribute to an increase in the risky investment surprisingly. The agent in our incomplete markets has an incentive

[6] This completely default of income in a disaster can be caused by insurer default (Lopes and Michaelides, 2007; Babbel and Merrill, 2007; Jang *et al.*, 2019).

[7] When the recovery rate is zero, the income is fully wiped out (completely defaulted) in the disastrous shock. When the recovery rate is one, there is no income loss after the shock.

to invest more in the stock market. Risky assets would serve as a partial hedging tool against disastrous income risk. The positive risk premium obtained from risky investments allows the agent to promptly accumulate sufficient wealth to be in a more liquid position. The liquidity with far enough wealth buffers against disastrous income risk and helps the agent avoid binding borrowing constraints and efficiently finance future consumption needs. In light of the growing equity demand in the presence of disastrous income risk, the conventional wisdom of precautionary savings should be understood by the interactions between consumption/savings and portfolio choice.

Finally, we demonstrate that the individual's optimal consumption and investment decisions with disastrous income risk are significantly affected by the recovery rate of income after the occurrence of income disaster. The individual's optimal consumption increases, savings decreases, and investment in the stock market increases as the recovery rate increases. This particularly highlights the relation between household decisions and the income recovery by access to (private) insurance and various types of government safety nets.

This paper contributes to the risk management literature with an interesting application of rare disaster risk hypothesis to household finance with disastrous income risk. Standard risk management models may fail to take sufficient account of the likelihood of extreme risk events (e.g., major market crashes, catastrophes, pandemic outbreak). In light of such a major limitation, significance of the low-probability, high-impact aspect of disastrous income risk has been neither recognized in the literature nor taken into account in modeling of disastrous income risk. However, large and negative earnings losses are observed at job displacement (Low *et al.*, 2010), and such substantial losses can have a significant impact on household investment and consumption decisions (Guvenen *et al.*, 2015). Addressing the above limitation is to follow the rare disaster literature. We focus on the extremes of the probability distribution of income, deviating from log-normality substantially. The disastrous income risk in the paper is caused by exogenous discrete Poisson jump shocks (rather than continuous Brownian shocks).

By extending the seminal study of Rietz (1988), Barro (2006), Gabaix (2008, 2012), Wachter (2013), Pindyck and Wang (2013), Farhi and Gabaix (2016), Barro *et al.* (2022), Hong *et al.* (2023) and others develop different rare disaster models having focuses on asset pricing implications based on general equilibrium models. Our work can be viewed as complementary to these earlier studies, but having distinct features compared to those. Our disastrous income shock resembles the low-probability, depression-like third state in the rare disaster risk hypothesis of Rietz (1988). In existing rare disaster models with representative agents, the economy has complete markets because the price of any Arrow-Debrew security is such that the net holding of the security by the representative agent is zero since markets must clear. Contrary to complete markets, we have incomplete markets where the agent cannot insure against disastrous income risk, thus giving rise to the demand for incomplete-markets precautionary savings. We do not assume dynamic market completeness implying the existence of a unique state price density process that is necessarily required in the literature to characterize general equilibrium quantities with representative agents. Rather, we directly solve the partial equilibrium incomplete markets problem using the convex-duality approach of Bensoussan *et al.* (2016) without resorting to the unique existence of state price density process. Further, general equilibrium rare disaster models attempt to explain empirical regularities (e.g., the equity premium puzzle, the risk-free rate puzzle), whereas this paper is a theoretical exercise in the understanding of optimal consumption/savings and investment behaviors with disastrous income risk in a partial equilibrium incomplete-market environment.

Three papers that present parallel and independent works on the issues associated with disastrous income risk are Cocco *et al.* (2005), Bensoussan *et al.* (2016), and Wang *et al.* (2016). These three papers study an incomplete market consumption/savings or portfolio choice model with uninsurable labor income risk. Cocco *et al.* (2005) investigate the role of market incompleteness caused by uninsurable labor income risk in individuals' optimal policies. Bensoussan *et al.* (2016) explore the effects of the risk of forced unemployment on interdependent consumption/savings, portfolio selection, and retirement decisions. Wang *et al.* (2016) study the impact of stochastic income on optimal consumption and savings decisions with recursive utility. The distinct feature of this paper from these three studies is state-dependent and stochastically time-varying severity of disaster (or duration of disaster) in the model. We think about disastrous income shocks as recurring events that repeat over time, so disastrous income risk is modeled and interpreted as state dependent disasters that fluctuate in extreme scenarios. We therefore shed new insights, beyond existing studies, into the relations among stochastic income disasters, consumption/savings, and portfolio choice.[8]

The paper is organized as follows. In Section 2, we provide for empirical motivation exploratory analysis of the relationship between disastrous shock and consumption, using data covering the periods of Covid-19 and 2008 financial crisis. In Section 3, we develop the basic model for the optimal consumption and investment framework without disastrous income risk. In Section 4, we develop three general models to include in the basic model borrowing constraints, a disastrous income shock, and state-dependent and time-varying disastrous income risk. We solve all these three models in analytically tractable forms. In Section 5, we conduct quantitative analysis to discuss various properties of optimal strategies, having an interest in the role of insurance. In Section 6, we conclude the paper.

[8]Further, Wang *et al.* (2016) do not consider the agent's asset allocation decision that is one of the most important financial decisions over the life cycle.

2 Empirical Motivation

The ratios of aggregate consumption to aggregate income (GDP) for almost all advance and emerging market economies have been very badly affected by rare disasters for the last ten years. The last 2008 global financial crisis was disastrous to cause a large fall of consumption to income ratios and the recent Covid-19 was more disruptive causing even lower ratios in the pandemic recovery period than in the period post 2008 crisis (Figure 1).

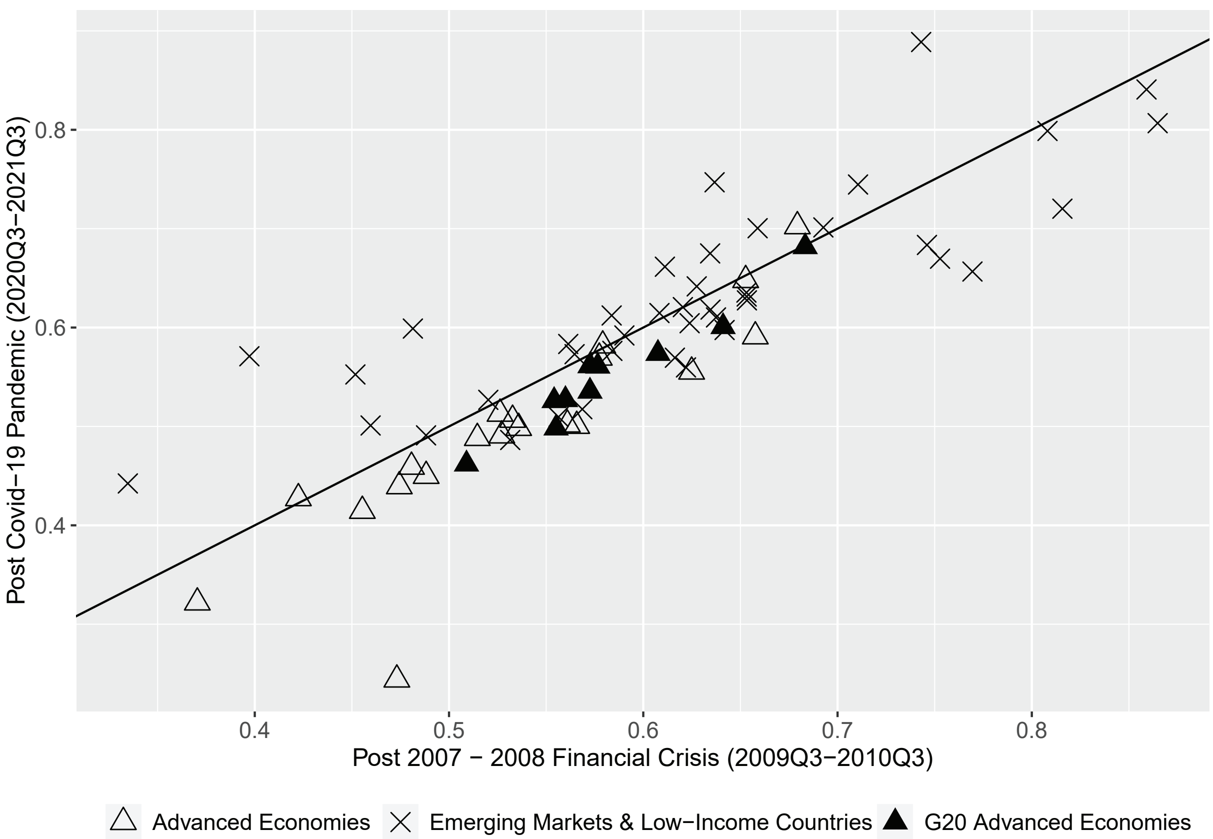

Figure 1: Consumption to GDP Ratio by Country

It is, therefore, of interest to empirically examine what underlying economic mechanisms have contributed to substantial consumption reductions at the times when rare disasters occur. Focusing on the recent pandemic, the relations among consumption (per capita growth rate) changes, aggregate income (GDP), disaster shock severity, and fiscal measures while controlling for other variables are reported in Table 1 using quarterly panel data with 55 countries over the period 2011Q1-2021Q2.[9] The strong positive estimate of $GDP\_pc_{i,t}$ as a proxy of aggregate income supports the consumption theory of Friedman (1957) that income is one of the major determinants for consumption. The negative estimate of $Str\_index_{i,t}$ and positive estimate of $Vaccine_{i,t}$ provide evidence of a negative relation between consumption and the severity of Covid-19 shock.

Many governments have provided in two ways financial supports to economic entities as a response to the recent pandemic. On one hand, they have offered credit and liquidity supports (hereafter liquidity measures) through loans at a lower interest rate than before the pandemic. On the other hand, they have approved the injection of the so-called 'helicopter money' (hereafter stimulus measures) to the economy through massive tax cuts and other direct money provisions.[10] The scale of both stimulus and liquidity measures in response to the pandemic has turned out to be unprecedented. Using 185 countries' data, the distribution of the size of stimulus and liquidity measures as of 2021Q3 is shown as a percentage of each economy's GDP (Figure 2). The stimulus measures take up more than 5% GDP in majority advanced economies and 2%$\sim$5% GDP in emerging and low-income developing economies. The liquidity measures account for 2%$\sim$5% GDP for advanced economies, 1%$\sim$4% GDP for emerging market economies, and 0.2%$\sim$1% for low-income developing countries. Notice that advanced economies are likely to provide in size stimulus measures more aggressively than liquidity measures.

With a focus on consumption, both the stimulus measures and the liquidity measures are effective in restoring consumption with positive estimates for $Stimulus_{i,t-1}$ and

[9] The shock severity is measured by three factors: new Covid-19 confirmed cases, changes in the stringency index measuring Covid-19 disruption to the social economic life, increment in vaccine coverage. As a convention of consumption function studies, proxies of stock return, interest rate and wealth level are included.

[10] Examples include US's relief program that pays stimulus check to households and UK Coronavirus Job Retention Scheme that pays furloughed employees.

Table 1: Regressions for per capita consumption change

	Dependent variable:	
	$Cons\_pc_{i,t}$	
	Fixed effects model	Random effects model
$Covid_{i,t}$	-0.056^{***}	-0.056^{***}
	(0.008)	(0.008)
$Str\_index_{i,t}$	-0.090^{***}	-0.091^{***}
	(0.006)	(0.006)
$Vaccine_{i,t}$	0.049^{***}	0.049^{***}
	(0.008)	(0.008)
$Stimulus_{i,t-1}$	0.164^{***}	0.162^{***}
	(0.048)	(0.048)
$Liquidity_{i,t-1}$	0.119^{***}	0.115^{***}
	(0.038)	(0.038)
$GDP\_pc_{i,t}$	0.775^{***}	0.773^{***}
	(0.024)	(0.023)
$Stock\_rtn_{i,t}$	-0.010	-0.011
	(0.007)	(0.007)
$Bank\_rate_{i,t}$	-0.087^{*}	-0.086^{*}
	(0.047)	(0.047)
$GDP\_pc_{i,2019}$		-0.008^{***}
		(0.003)
Constant		0.769^{***}
		(0.202)
Observations	2,332	2,332
R^2	0.643	0.641
Adjusted R^2	0.633	0.640
F Statistic	509.796^{***} (df = 8; 2268)	$4{,}145.739^{***}$

*p<0.1; **p<0.05; ***p<0.01

Notes. Taking $Cons\_pc_{i,t}$ as the dependent variable, we lag stimulus and liquidity variables by one quarter in reduced form regressions to capture the short-term but not long-term impact. Table 1 reports estimates from fixed effects models and random effects models, assuming that the average value of $Cons\_pc_{i,t}$ varies across country and remains constant over time. These two models capturing short-run effects produce similar estimates, which are significant for all variables other than $Stock_rtn_{i,t}$, thus implying no significant relation between stock return and consumption in the short-run. The data contain seven dimensions. (1) Consumption per capita: $Cons\_pc_{i,t}$, percentage change of seasonally- and inflation-adjusted consumption per capita from previous quarter. (2) Disaster risk intensity: $Covid_{i,t}$, quarterly new confirmed cases of Covid-19 per 1,000 people; $Str\_index_{i,t}$, quarterly change in Covid-19 Stringency Index, which ranges from 0 to 100 (100 = strictest) and is based on nine metrics of social response (e.g. school closures); $Vaccine_{i,t}$, quarterly increment in people who received at least one vaccine dose per 100 people in the total population. (3) Fiscal measures in response to the Covid-19 pandemic: $Stimulus_{i,t-1}$, quarterly change in the value of stimulus measures as a percentage of GDP, lagged by one quarter; $Liquidity_{i,t-1}$, quarterly change in the value of liquidity measures as a percentage of GDP, lagged by one quarter. Stimulus measures are government additional spending and forgone revenue; liquidity measures are equity, loans and guarantees offered by governments. (4) GDP per capita as a proxy of aggregate income: $GDP\_pc_{i,t}$, quarterly change (in percentage) of seasonally- and inflation-adjusted GDP per capita. (5) Stock return, $Stock\_rtn_{i,t}$, the difference in quarterly stock return (in percentage), calculated from stock market index from each country, between current and previous quarter. (6) Bank rate as a measure of interest rate, $Bank\_rate_{i,t}$, quarterly difference in central bank policy rate (in percentage). (7) Wealth proxy: $GDP\_pc_{i,2019}$, percentile of 2019 GDP per capita. Data to describe dimensions 1 and 4-6 are from Passport database; data for dimensions 2, 3 and 7 are respectively from Our World in Data database, IMF and World Bank.

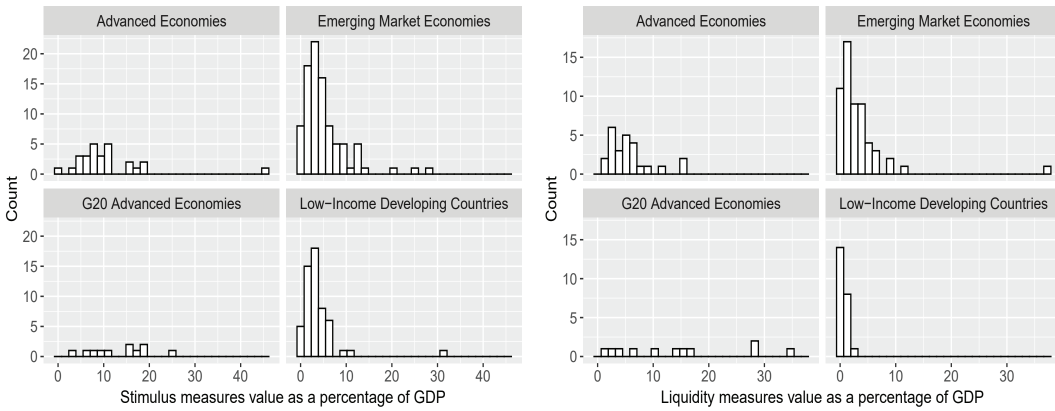

Figure 2: **Distribution of value of fiscal measures in response to Covid-19**

$Liquidity_{i,t-1}$ (Table 1).

Our empirical results would now provide a useful motivation for developing a model of consumption/savings and investment decisions considering the joint impact of disastrous income risk and income recovery. We conjecture that availability of insurance that smooths out income fluctuations would be capable of pulling consumption back to its normal level especially in a severe income shock. The model to be developed in the paper rather simplifies the role of insurance with an emphasis on income recovery only in a disaster. Having developed the model, we will investigate the effects of disastrous income risk on the agent's optimal decisions and how access to insurance for income recovery helps the agent to manage the adverse effects of disastrous income risk by optimally adjusting amounts of consumption/savings and investments.

3 The Basic Model

For the basic model allowing for analytically tractable closed-form solutions, we consider the optimal consumption/savings and investment decisions of a representative economic agent who exhibits CARA utility preference as follows:

$$U = E\Big[\int_0^\infty e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big],$$

where E is the expectation taken at time 0, c_t is per-period consumption, $\beta > 0$ is the agent's subjective discount rate, and $\gamma > 0$ is the agent's constant coefficient of absolute risk aversion. We consider constant hazard rate of death. Following Blanchard (1985), the effective discount rate can then be the sum of subjective discount rate and mortality rate. In this sense, we assume that the subjective discount rate considered in this paper accounts for both discounting and a constant probability of death.

The agent can trade securities. The securities market consists of two assets: a riskless bond (or a risk-free asset) and a risky stock (or a risky asset). The bond price B_t follows

$$dB_t = rB_t dt,$$

where $r > 0$ is the risk-free interest rate, and the stock price S_t evolves according to the following geometric Brownian motion (GBM):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1}$$

where $\mu > r$ and $\sigma > 0$ are the mean and standard deviation of the returns on the stock, i.e., they represent the expected return and risk in the financial market, and W_t is a standard one-dimensional Brownian motion defined on an appropriate probability space. We assume that r, μ, σ are constant, i.e., the investment opportunity set is constant.

The labor income is assumed to be risk free, but time varying in a deterministic way in this basic model, so its dynamics are given by: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_t dt,$$

where μ^ϵ is the income growth parameter that is nonnegative.

The agent's basic model of optimal consumption/savings and investment decisions is to maximize her life-time CARA utility of consumption by optimally managing per-period consumption c and investment π. That is, the agent's objective function is to find the following value function:

$$V(x,\epsilon) \equiv \max_{(c,\pi)} E\Big[\int_0^\infty e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt\Big], \tag{2}$$

which is subject to the following dynamic budget constraint: $X_0 = x > -\epsilon/r^\epsilon$,

$$dX_t = \big(rX_t - c_t + \epsilon_t\big)dt + \pi_t\sigma(dW_t + \theta dt), \quad \theta = \frac{\mu - r}{\sigma}, \tag{3}$$

where

$$r^\epsilon = r - \mu^\epsilon,$$

which serves as the effective interest rate for discounting future income. Here, the agent is allowed to borrow against the present value, ϵ/r^ϵ, of future income, i.e., the agent is not borrowing constrained at all.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the value function (2) is given by

$$\begin{aligned}\max_{(c,\pi)}\Big[&-\beta V(x,\epsilon) + (rx - c + \epsilon)V_x(x,\epsilon) + \frac{1}{2}\pi^2\sigma^2 V_{xx}(x,\epsilon)\\ &+ \pi\sigma\theta V_x(x,\epsilon) + \mu^\epsilon \epsilon V_\epsilon(x,\epsilon) - \frac{1}{\gamma}e^{-\gamma c}\Big] = 0.\end{aligned} \tag{4}$$

Applying the first-order-conditions (FOCs) for consumption c and investment π results in the following candidates for optimal policies:

$$\begin{aligned} c &= -\frac{1}{\gamma}\ln V_x(x,\epsilon), \\ \pi &= -\frac{\theta}{\sigma}\frac{V_x(x,\epsilon)}{V_{xx}(x,\epsilon)}. \end{aligned} \tag{5}$$

Substituting the above FOCs in the HJB equation (4), we obtain that

$$\begin{aligned} -\beta V(x,\epsilon) + (rx+\epsilon)V_x(x,\epsilon) + \frac{1}{\gamma}V_x(x,\epsilon)\Big\{\ln V_x(x,\epsilon) - 1\Big\} \\ -\frac{1}{2}\theta^2\frac{V_x(x,\epsilon)^2}{V_{xx}(x,\epsilon)} + \mu^\epsilon \epsilon V_\epsilon(x,\epsilon) = 0. \end{aligned} \tag{6}$$

We conjecture that the value function $V(x,\epsilon)$ has the following form:

$$V(x,\epsilon) = -\frac{A}{\gamma r}e^{-\gamma r(x+a\epsilon)}, \tag{7}$$

where A and a are the constants to be determined. Putting the conjectured value function into the HJB equation (6) determines the constants as follows

$$A = e^{-\frac{1}{r}\left(\frac{\theta^2}{2}+\beta-r\right)}, \quad a = \frac{1}{r^\epsilon}.$$

Therefore, the value function is now obtained in closed-form as follows

$$V(x,\epsilon) = -\frac{A}{\gamma r}e^{-\gamma r(x+\epsilon/r^\epsilon)}.$$

We then obtain the optimal consumption and investment strategies in closed-form by putting the derived value function into the FOCs given in (5):

$$
\begin{aligned}
c &= r\Big[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta - r)\Big)\Big],\\
\pi &= \frac{\theta}{\gamma\sigma}\frac{1}{r}.
\end{aligned}
$$

The obtained optimal strategies in this basic model suggest that the optimal consumption is affine in total wealth that is the sum of financial wealth x and human wealth ϵ/r^{ϵ}. The optimal investment policy is to maintain a constant dollar amount in the risky asset, which is independent of wealth and thus, does not capture the so-called *wealth effect*. Notice that the adoption of CARA utility has a major benefit of greatly simplifying the analysis at the expense of wealth effect property. The limitation of CARA utility will be overcome in part in the next section by considering a borrowing-constrained situation in the agent's budget constraint so that the agent's liquid wealth should have effects on investment policy.

4 General Models

Having understood some basic interactions among labor income, consumption/savings, and investment, we then establish general models to provide an intuitive insight into how state-dependent and time-varying disastrous income risk affects the agent's optimal choices. We proceed pedagogically with our analysis and develop insights by solving three models, which are categorized by borrowing constraints, a disastrous income shock, and stochastic nature of disastrous income risk.

Model 1. The basic model with borrowing constraints.

Model 2. Model 1 with a one-time-only disastrous income shock.

Model 3. Model 2 with state-dependent and time-varying disastrous income risk.

Model 1 is a generalization of Merton (1969) considering the borrowing-constrained agent who is not allowed to borrow against future income. Moving to Model 2 isolates the effects of a disastrous income shock on optimal strategies. Finally, moving to Model 3 very closely analyzes new issues introduced by state dependency and time-varying severity of disastrous income risk on consumption/savings and investment decisions.

4.1 A consumption/savings and investment model with borrowing constraints (Model 1)

The agent is borrowing constrained due to market frictions (e.g., informational asymmetry, agency conflicts, limited enforcement). In the presence of borrowing constraints against income, the agent's wealth should be nonnegative and hence,

$$X_t \geq 0 \quad \text{for all } t \geq 0. \tag{8}$$

So, Model 1 is to find the value function given in (2) subject to the dynamic budget constraint (3) and the borrowing constraints in (8).

Incorporating the borrowing constraints into the basic model causes a considerable challenge in the primal HJB equation (6) because the two-dimensional HJB equation is no longer separable in financial wealth x and income ϵ due to the wealth effect that has been ignored in the CARA utility function. That is, the wealth effect is caused with CARA utility preference by borrowing constraints and the conjecture (7) using the separable principle cannot be applied to solve the equation (6) any more. To overcome this technical difficulty, we convert the primal HJB equation (6) into the dual HJB equation by developing a modified version of the convex-duality approach of Bensoussan *et al.* (2016) to deal with two dimensions of the equation (6) instead of one dimension in Bensoussan *et al.* (2016).

We first introduce the dual variable $\lambda(x, \epsilon)$ as the first derivative of the value function

$V(x,\epsilon)$:

$$\lambda(x,\epsilon) \equiv V_x(x,\epsilon). \tag{9}$$

We next introduce the convex-dual function $G(\lambda(x,\epsilon))$ as the total wealth that is the sum of financial wealth x and the present value ϵ/r^{ϵ} of future income:

$$G\big(\lambda(x,\epsilon)\big) \equiv x + \frac{\epsilon}{r^{\epsilon}}. \tag{10}$$

We then obtain the following relations:

$$\begin{aligned} &G'\big(\lambda(x,\epsilon)\big)\lambda_x(x,\epsilon) = 1, \quad G'\big(\lambda(x,\epsilon)\big)\lambda_{\epsilon}(x,\epsilon) = \frac{1}{r^{\epsilon}}, \\ &G''\big(\lambda(x,\epsilon)\big)\lambda_x(x,\epsilon)^2 + G'\big(\lambda(x,\epsilon)\big)\lambda_{xx}(x,\epsilon) = 0. \end{aligned} \tag{11}$$

For notational simplicity, we write $G(\lambda(x,\epsilon))$ and $\lambda(x,\epsilon)$ as just $G(\lambda)$ and λ, respectively, unless there is any confusion.

Taking the first derivative with respect to financial wealth x on the both sides of the HJB equation (6), we obtain that

$$\begin{aligned} &-\beta V_x(x,\epsilon) + rV_x(x,\epsilon) + (rx+\epsilon)V_{xx}(x,\epsilon) + \frac{1}{\gamma}V_{xx}(x,\epsilon)\{\ln V_x(x,\epsilon) - 1\} \\ &+ \frac{1}{\gamma}V_x(x,\epsilon)\frac{V_{xx}(x,\epsilon)}{V_x(x,\epsilon)} - \frac{1}{2}\theta^2\frac{2V_x(x,\epsilon)V_{xx}(x,\epsilon)^2 - V_x(x,\epsilon)^2 V_{xxx}(x,\epsilon)}{V_{xx}(x,\epsilon)^2} + \mu^{\epsilon}\epsilon V_{\epsilon x}(x,\epsilon) = 0. \end{aligned}$$

Using the dual variable (9), the convex-dual function (10), and the their relations given in (11), we obtain the following dual HJB equation: for $0 < \lambda < \overline{\lambda}$,

$$rG(\lambda) = \frac{1}{2}\theta^2\lambda^2 G''(\lambda) + (\beta + \theta^2 - r)\lambda G'(\lambda) - \frac{1}{\gamma}\ln\lambda, \tag{12}$$

where $\overline{\lambda}$ is a constant to be determined according to the borrowing constraints (8). We will address proper boundary conditions of $G(\lambda)$ at $\overline{\lambda}$ after understanding the economics first behind the dual HJB equation (12).

The left-hand side of the equation (12) demonstrates that the agent's total wealth represented by the convex-dual function $G(\lambda)$ earns the risk-free rate of return by dynamically consuming and trading both the risk-free bond and the risky stock in the market. The right-hand side of the equation shows instantaneous expected changes in the agent's total wealth with respect to the dual variable λ that eventually determines the agent's marginal consumption from the FOC (5). More specifically, the convexity effect of the Sharpe ratio θ on total wealth is revealed in the first term on the right-hand side of the equation (12) involving the second derivative $G''(\lambda)$ of total wealth. The agent's expected risk premium relative to the risk-free interest rate r is captured by the drift term $\beta + \theta^2 - r$ of total wealth that is the second term on the right-hand side of the equation involving the first derivative $G'(\lambda)$ of total wealth. The very last term of the right-hand side of the equation results from the agent's instantaneous utility of consumption.

We now address the boundary conditions associated with the borrowing constraints (8). First,

$$G(\overline{\lambda}) = \frac{\epsilon}{r^{\epsilon}} \tag{13}$$

is naturally obtained by the relation (10) between financial wealth x and convex-dual function $G(\lambda)$. The convex-dual function $G(\lambda)$ gets closer to human wealth ϵ/r^{ϵ} only as financial wealth decumulates to zero, so that there exists a threshold level $\overline{\lambda}$ of λ such that $G(\overline{\lambda}) = \epsilon/r^{\epsilon}$. Second,

$$G'(\overline{\lambda}) = 0 \tag{14}$$

is a consequence of a principle in optimal stopping that is known as smooth pasting, which implies that when marginal value λ of wealth rises to the upper limit $\overline{\lambda}$, the borrowing constraints bind, thus resulting in a 100% investment in riskless bonds. From the FOC (5), the second boundary condition is thus obtained. Technically, the borrowing constraints are reminiscent of an absorbing boundary.

The converted dual HJB equation (12) with the boundary conditions (13) and (14) admits the following closed-form solution: for $0 < \lambda < \overline{\lambda}$,

$$G(\lambda) = -\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta - r)\Big) + B\lambda^{-\alpha^*}, \tag{15}$$

where B is a constant to be determined with $\overline{\lambda}$ according to the boundary conditions (13) and (14) and $-1 < \alpha^* < 0$ is the negative root of the following characteristic equation:

$$F(\alpha) \equiv -\frac{1}{2}\theta^2\alpha(\alpha - 1) + \alpha(\beta - r) + r = 0.$$

Using the boundary conditions (13) and (14), we obtain the following two relations:

$$\frac{\epsilon}{r} = -\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta - r)\Big) + B\overline{\lambda}^{-\alpha^*},$$

and

$$0 = \frac{1}{\gamma r} + \alpha^* B\overline{\lambda}^{-\alpha^*}.$$

We therefore determine B and $\overline{\lambda}$ as follows:

$$B = -\frac{\overline{\lambda}^{\alpha^*}}{\gamma r\alpha^*} > 0, \tag{16}$$

and

$$\overline{\lambda} = \exp\Big\{ -\frac{\theta^2}{2r}\Big(1 + \frac{2}{\theta^2}(\beta - r)\Big) - \frac{1}{\alpha^*} - \gamma\epsilon\Big\} > 0, \tag{17}$$

respectively.

We now derive the agent's optimal policies. The FOCs for consumption c and investment π given in (5) can be rewritten with the dual variable λ and the convex-dual function

$G(\lambda)$ as follows:

$$\begin{aligned} c &= -\frac{1}{\gamma}\ln\lambda, \\ \pi &= -\frac{\theta}{\sigma}\lambda G'(\lambda). \end{aligned} \tag{18}$$

We thus obtain the following optimal consumption and investment strategies by substituting the determined convex-dual function $G(\lambda)$ in (15) with (16) and (17) in the rewritten FOCs stated above:

$$\begin{aligned} c &= r\Big[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta - r)\Big) - B\lambda^{-\alpha^*}\Big], \\ \pi &= \frac{\theta}{\gamma\sigma}\Big(\frac{1}{r} + \alpha^* B\lambda^{-\alpha^*}\Big). \end{aligned} \tag{19}$$

Contrary to the basic model, the optimal consumption is no longer affine in total wealth. The consumption is complicated by the presence of borrowing constraints and now nonlinear in (financial) wealth by the dual variable λ having the relation (10) with wealth via the convex-dual function $G(\lambda)$. Further, keeping a constant dollar amount in stockholdings is suboptimal with borrowing constraints. Stock investments are also affected by levels of wealth through the dual variable λ. In light of such a wealth-dependent investment strategy, the issue of wealth effect that has been ignored by adopting CARA utility is partially addressed with the help of borrowing constraints.

We center our Model 1 analysis on one possible economic mechanism underlying borrowing constraints. Agents' wealth position crucially determines how much they diminish both consumption and stockholdings. With $B > 0$ and $-1 < \alpha^* < 0$, we theoretically predict from the optimal choices in (19) that the presence of borrowing constraints causes agents to decrease both their consumption and investments in risky assets. The effects of borrowing constraints are wealth dependent. We know the inverse relation between wealth x and dual variable λ implied from (11) with $G'(\lambda) < 0$ that is straightforward to be verified using the closed-form solution of $G(\lambda)$ in (15). Low levels of wealth are associated with high levels of dual variable. As a result, the presence of borrowing constraints leads to a quantitatively substantial reduction in amounts of consumption and stock investment especially for agents with low liquid wealth. Intuitively, when agents are in a low wealth position, borrowing constraints are more likely to bind. Therefore, agents with low wealth seek to avoid binding borrowing constraints by reserving sufficiently large amount of savings, thus suggesting such optimal consumption and investment policies with borrowing constraints.

4.2 A consumption/savings and investment model with a disastrous income shock (Model 2)

Model 2's focus is on the effects of a disastrous income shock on optimal policies. For simplicity and comparison purposes with Model 3 having general settings for disastrous income risk, we consider in Model 2 a one-time-only large negative Poisson shock to labor income. In the presence of a one-time Poisson shock, the labor income dynamics ϵ_t is evolved according to the following stochastic processes: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_{t-} dt - (1-k)\epsilon_{t-} dN_t, \tag{20}$$

where $k \in [0, 1)$ is the income recovery parameter and N_t is the one-time Poisson shock with intensity $\delta > 0$. The agent's income plummets to $k\epsilon_{t-}$ from ϵ_{t-} at the time when the disastrous Poisson shock occurs. Our income dynamics with the positive income growth rate μ^ϵ would reflect a realistic feature of the recent pandemic that income is substantially lost for a period of months, but taking a long time after the disaster, the income is highly likely to revert to its pre-disaster state.

Without any consideration of a potential role of insurance in the income recovery in the aftermath of the income disaster, the agent's income would be completely wiped out reducing to nothing, which corresponds to the limiting case when the income recovery parameter k equals to zero. In this case, the agent has no income source in the disaster.

If we consider in a very reduced form the role of insurance for hedging the disastrous income shock, the story post income disaster would be quite different. With access to an insurance market to hedge against the income shock, the agent's income can be partly recovered at the rate of $0 < k < 1$, so that she receives $k\epsilon$ post disaster.

The role of insurance is particularly important for the agent's human capital (present value of future labor income) which is a major staple of total wealth for managing overall risk exposure. The human capital is the following present value of future income discounted by the risk-free interest rate r (Friedman, 1957):

$$\frac{\epsilon}{r^\epsilon + \delta}\Big(1 + \frac{\delta k}{r^\epsilon}\Big).$$

For the limiting case of $k = 0$, i.e., without any access to the insurance market, the human capital decreases to

$$\frac{\epsilon}{r^\epsilon + \delta}.$$

With access to the insurance market, total available financial resources for hedging purposes could be further expanded by the larger human capital than without access to the insurance market.

Model 2 is to find the value function given in (2) subject to the dynamic budget constraint (3), the borrowing constraints in (8), and the disaster-exposed income dynamics (20).

The HJB equation associated with Model 2 is given by

$$\begin{aligned} \max_{(c,\pi)} \Big[&- (\beta + \delta)V(x,\epsilon) + (rx - c + \epsilon)V_x(x,\epsilon) + \frac{1}{2}\pi^2\sigma^2 V_{xx}(x,\epsilon) \\ &+ \pi\sigma\theta V_x(x,\epsilon) + \mu^\epsilon \epsilon V_\epsilon(x,\epsilon) - \frac{1}{\gamma}e^{-\gamma c} - \delta\frac{A}{\gamma r}e^{-\gamma r(x + k\epsilon/r^\epsilon)}\Big] = 0. \end{aligned} \tag{21}$$

Notice that the very last term on the left-hand side of the HJB equation (21) is the value function in the basic model replacing ϵ with $k\epsilon$ and it is therefore the maximal utility value post the disastrous income shock. Here, the post-disaster value function directly affects the pre-disaster value function, thus influencing optimal decisions pre disaster, which is the main departure of the HJB equation (21) from those in the basic model and Model 1.

We apply the same convex-duality approach that has been adopted in Model 1 by introducing the dual variable λ in (9) and the convex-dual function $G(\lambda)$ in (10). The HJB equation (21) by taking its first derivative with respect to x then reduces to the following dual HJB equation: for $0 < \lambda < \overline{\lambda}$,

$$rG(\lambda) = \frac{1}{2}\theta^2\lambda^2 G''(\lambda) + \Big\{\beta + \delta\Big(1 - \frac{A}{\lambda}e^{-\gamma r\left(G(\lambda)-\epsilon/r^{\epsilon}+k\epsilon/r^{\epsilon}\right)}\Big) + \theta^2 - r\Big\}\lambda G'(\lambda) - \frac{1}{\gamma}\ln\lambda, \quad (22)$$

subject to the boundary conditions in (13) and (14). The additional risk in Model 2 results from the Poisson arrival of the disastrous income shock at the constant disaster intensity δ. Compared to the basic model and Model 1, the agent in Model 2 should therefore reflect such extra risk consideration in the agent's expected risk premium on total wealth. The expected return compensation for the presence of the disastrous income shock is $\beta + \delta\Big(1 - \frac{A}{\lambda}e^{-\gamma r\left(G(\lambda)-\epsilon/r^{\epsilon}+k\epsilon/r^{\epsilon}\right)}\Big) + \theta^2 - r$ in the drift term of total wealth. The wedge $\delta\Big(1 - \frac{A}{\lambda}e^{-\gamma r\left(G(\lambda)-\epsilon/r^{\epsilon}+k\epsilon/r^{\epsilon}\right)}\Big)$ between expected risk premia in this model and in Model 1 then represents the disastrous income risk premium.

We conjecture that the convex-dual function $G(\lambda)$ that solves the HJB equation (22) has the following form: for $0 < \lambda < \overline{\lambda}$,

$$\begin{aligned} G(\lambda) = -\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta+\delta-r)\Big) + \eta(\lambda)\lambda^{-\alpha_\delta} + \eta^*(\lambda)\lambda^{-\alpha_\delta^*}, \\ \text{subject to } \ \eta'(\lambda)\lambda^{-\alpha_\delta} + (\eta^*(\lambda))'\lambda^{-\alpha_\delta^*} = 0, \end{aligned} \quad (23)$$

where $\alpha_\delta > 0$ and $-1 < \alpha_\delta^* < 0$ are the two roots of the following characteristic equation:

$$F(\alpha;\delta) \equiv -\frac{1}{2}\theta^2\alpha(\alpha-1) + \alpha(\beta+\delta-r) + r = 0.$$

Plugging (23) into (22), we get the following analytically tractable form: for $0 < \lambda < \overline{\lambda}$,

$$\begin{aligned} G(\lambda) = &-\frac{1}{\gamma r}\ln\lambda - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\delta-r)\Big) + B\lambda^{-\alpha_\delta^*} \\ &- \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)}\Big[(\alpha_\delta-1)\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta-2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu)-\frac{\epsilon}{r^\epsilon}+\frac{k\epsilon}{r^\epsilon}\right)}d\mu \\ &+ (\alpha_\delta^*-1)\lambda^{-\alpha_\delta^*}\int_\lambda^{\overline{\lambda}} \mu^{\alpha_\delta^*-2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu)-\frac{\epsilon}{r^\epsilon}+\frac{k\epsilon}{r^\epsilon}\right)}d\mu\Big], \end{aligned} \tag{24}$$

where

$$B \equiv \eta^*(\overline{\lambda}) + \frac{2\delta A}{\theta^2(\alpha_\delta-\alpha_\delta^*)\gamma r}\overline{\lambda}^{\alpha_\delta^*-1},$$

which is a constant to be determined with $\overline{\lambda}$. Using the boundary conditions in (13) and (14), we obtain the following two relations:

$$\begin{aligned} \frac{\epsilon}{r^\epsilon} = &-\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{\theta^2}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\delta-r)\Big) + B\overline{\lambda}^{\,-\alpha_\delta^*} \\ &- \frac{2\delta}{\theta^2(\alpha_\delta-\alpha_\delta^*)}(\alpha_\delta-1)\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu)-\frac{\epsilon}{r^\epsilon}+\frac{k\epsilon}{r^\epsilon}\right)}d\mu, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{\gamma r} + \frac{\alpha_\delta^*\epsilon}{r^\epsilon} + \frac{\theta^2\alpha_\delta^*}{2\gamma r^2}\Big(1+\frac{2}{\theta^2}(\beta+\delta-r)\Big) \\ &\quad = -\alpha_\delta^*\frac{1}{\gamma r}\ln\overline{\lambda} - \frac{2\delta}{\theta^2\overline{\lambda}}\frac{A}{\gamma r}e^{-\gamma r\frac{k\epsilon}{r^\epsilon}} + \frac{2\delta(\alpha_\delta-1)}{\theta^2}\overline{\lambda}^{\,-\alpha_\delta}\int_0^{\overline{\lambda}} \mu^{\alpha_\delta-2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu)-\frac{\epsilon}{r^\epsilon}+\frac{k\epsilon}{r^\epsilon}\right)}d\mu, \end{aligned}$$

and thereby the constants B and $\overline{\lambda}$ can be determined numerically.

Finally, the rewritten FOCs in (18) with the dual variable λ and the convex-dual function $G(\lambda)$ allow us to derive the following optimal consumption and investment strategies with substitution of the convex-dual function $G(\lambda)$ given in (24):

$$c = r\Big[x + \frac{\epsilon}{r^\epsilon} + \frac{\theta^2}{2\gamma r^2}\Big(1 + \frac{2}{\theta^2}(\beta + \delta - r)\Big) - B\lambda^{-\alpha_\delta^*} + \mathrm{PS}\Big], \tag{25}$$

$$\pi = \frac{\theta}{\gamma\sigma}\Big(\frac{1}{r} + \alpha_\delta^* B\lambda^{-\alpha_\delta^*} + \alpha_\delta \mathrm{PS1} + \alpha_\delta^* \mathrm{PS2} - \mathrm{RD}\Big), \tag{26}$$

where PS represents the precautionary savings driven by the disastrous income shock and it is given by

$$\mathrm{PS} = \mathrm{PS1} + \mathrm{PS2},$$

$$\mathrm{PS1} = \frac{2\delta(\alpha_\delta - 1)}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\lambda^{-\alpha_\delta}\int_0^\lambda \mu^{\alpha_\delta - 2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu) - \frac{\epsilon}{r^\epsilon} + \frac{k\epsilon}{r^\epsilon}\right)}d\mu > 0,$$

$$\mathrm{PS2} = \frac{2\delta(\alpha_\delta^* - 1)}{\theta^2(\alpha_\delta - \alpha_\delta^*)}\lambda^{-\alpha_\delta^*}\int_\lambda^{\bar{\lambda}} \mu^{\alpha_\delta^* - 2}\frac{A}{\gamma r}e^{-\gamma r\left(G(\mu) - \frac{\epsilon}{r^\epsilon} + \frac{k\epsilon}{r^\epsilon}\right)}d\mu < 0,$$

and RD represents the risk diversification demand driven by the disastrous income shock and it is given by

$$\mathrm{RD} = \frac{2\delta}{\theta^2\lambda}\frac{A}{\gamma r}e^{-\gamma r(x + k\epsilon/r^\epsilon)} > 0.$$

Model 2 can be thought of as a generalized consumption/savings and investment model with two-dimensional market incompleteness caused by borrowing constraints and the disastrous income shock. Model 1 cannot study the effects of the income shock on agents' optimal choices. Both borrowing constraints and the income shock affect agents' consumption and portfolio choice. We find a large precautionary savings motive represented by PS in the optimal consumption (25) for income-risk-exposed agents to consume less. The disastrous-income-shock-induced precautionary savings tend to be greater at lower levels of wealth x because the income shock is not spanned by the market and agents with low wealth find it much harder to buffer against the unspanned income shock in incomplete markets relying on their low liquidity position.

As opposed to the standard precautionary savings arguments, the secured substantial amount of precautionary savings is invested in the risky stock market rather than in the riskless bond market. Agents in our incomplete markets have an incentive to invest in risky assets that serve as a partial hedging tool against the disastrous income shock. This incentive is driven by the positive risk premium offered from risky investments for agents to promptly accumulate sufficient wealth to be in a more liquid position. Here, liquidity with far enough wealth buffers against the disastrous income shock and helps agents avoid binding borrowing constraints as well.

On the other hand, the disastrous income shock would have the opposite effects on the risky investment policy. Income-disaster-exposed agents are already exposed to high income risk on top of the stock market risk and hence, they show a demand for risk diversification represented by RD in the optimal investment (26). Put differently, the increased background risk due to the disastrous income shock leads agents to invest less in the stock market to reduce the overall risk exposure.

4.3 A consumption/savings and investment model with state-dependent and time-varying disastrous income risk (Model 3)

Model 2 we have considered in the previous subsection is limited by having the one-time disastrous income shock only. In this section, we overcome Model 2's limitation by reflecting the fact that in reality, agents still expect occurrence of disastrous income shocks even after the income shock takes place in the past. The disastrous income shocks can occur repeatedly. Thinking about large, negative income shocks as recurring events that repeat over time (e.g., the great depression, the 2008 global financial crisis, the recent COVID-19 pendemic), we importantly model and interpret the income shocks as state dependent disasters that fluctuate in extreme events. In light of such realistic ramifications that are present in the actual disastrous income risk, we consider a general Poisson jump process with state-dependent and stochastically time-varying disaster intensity δ_t instead of the previously considered constant intensity δ.

In the presence of state-dependent and stochastically time-varying disastrous income risk, the income dynamics ϵ_t are evolved by the following stochastic process: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_{t-} dt - (1-k)\epsilon_{t-} dN_t^G, \tag{27}$$

where N_t^G is the Poisson jump process with state-dependent and time-varying intensity δ_t.

We now model state-dependent disastrous income shocks by a two-state Markov chain: the good state G and the bad state B. For a small time period $(t, t+dt)$, the state switches from the good state G (B) to the bad state B (G) with probability $\phi^G dt$ ($\phi^B dt$) when the current state is G (B), and stays unchanged with the remaining probability $1 - \phi^G dt$ ($1-\phi^B dt$). We then introduce the dynamics of stochastically time-varying disaster intensity δ_t that is state dependent, so that we denote δ_t^i by the intensity dynamics in the state i ($i \in \{G, B\}$). We assume that the intensity dynamics δ_t^i in the state i are assumed to follow a GBM: $\delta_0^i = \delta^i > 0$,

$$d\delta_t^i = -\delta^i \delta_t^i dt + b^i \delta_t^i dZ_t, \tag{28}$$

where b^i is the volatility on the intensity growth rate and Z_t is a standard one-dimensional Brownian motion that is correlated with the market factor W_t considered in the stock price dynamics (1), i.e., $dW_t \cdot dZ_t = \rho^i dt$, $\rho^i \in [-1, 1]$. Notice that the negative intensity growth rate implies that probabilities of undergoing disastrous income shocks decrease over time as a reflection towards the reality that disasters will come to an end eventually after taking a long time.

Our GBM modeling for state-dependent and time-varying disastrous income risk has the convenient property that it nests Model 2 by setting $b^i = 0$. In this case, the intensity

δ_t^i reduces to $\delta_t^i = \delta^i e^{-\delta^i t}$, which is the probability density function of an exponential distribution with constant intensity δ^i that has been used to model the one-time disastrous income shock in Model 2.

Model 3 is to find the value function as follows: in the state i ($i \in \{G, B\}$),

$$V^i(x,\epsilon) \equiv \max_{(c,\pi)} E\Big[\int_0^{\tau^i} e^{-\beta t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t}\Big)dt + e^{-\beta\tau^i}V^j(x,\epsilon)\Big], \tag{29}$$

subject to the dynamic budget constraint (3), the borrowing constraints in (8), and the income dynamics (27) with state-dependent and time-varying disastrous income risk modeled by (28), where τ^i is the first jump time of state switching since the beginning of the state i and $V^j(x,\epsilon)$ is the value function in the state $j \neq i$.

To deal with both state dependency and time-varying severity of disastrous income risk in the expectation taken in the value function, we introduce a new probability measure in the state i as

$$\widetilde{P}(A) \equiv \int_A e^{b^i Z_t(\omega) - \frac{1}{2}(b^i)^2 t} dP(\omega) \quad \text{for all} \quad A \in \mathcal{F},$$

where $\mathcal{F}$ is the filtration generated by the stock and income dynamics. Then, by Girsanov's theorem

$$\widetilde{W}_t \equiv W_t - \rho^t b^i t$$

is the standard one-dimensional Brownian motion under the new probability measure. We can now restate the dynamic budget constraint (3) under the new probability measure as follows: $X_0 = x \geq 0$,

$$dX_t = \big(rX_t - c_t + \epsilon_t\big)dt + \pi_t\sigma\big\{d\widetilde{W}_t + (\theta + \rho^i b^i)dt\big\}. \tag{30}$$

The value function (29) can be restated under the new probability measure by integrating out δ^i and τ^i as follows: in the state i ($i \in \{G, B\}$),

$$V^i(x, \epsilon) = \max_{(c,\pi)} \widetilde{E}\Big[\int_0^\infty e^{-(\beta+\delta^i+\phi^i)t}\Big(-\frac{1}{\gamma}e^{-\gamma c_t} + \delta^i V^i\big(X_t, k\epsilon_t\big) + \phi^i V^j\big(X_t, \epsilon_t\big)\Big)dt\Big], \tag{31}$$

subject to the restated dynamic budget constraint (30) and the borrowing constraints in (8), where $\widetilde{E}$ is the expectation taken under the new probability measure.

THe HJB equation associated with Model 3 is then given in the state i ($i \in \{G, B\}$) by

$$\begin{aligned} -(\beta + \delta^i + \phi^i)V^i(x, \epsilon) + (rx - c + \epsilon)V_x^i(x, \epsilon) + \frac{1}{2}\pi^2\sigma^2 V_{xx}^i(x, \epsilon) \\ + \pi\sigma(\theta + \rho^i b^i)V_x^i(x, \epsilon) - \frac{1}{\gamma}e^{-\gamma c} + \delta^i V^i(x, k\epsilon) + \phi^i V^j(x, \epsilon) = 0. \end{aligned} \tag{32}$$

Notice that the very last two terms involving the disaster intensity δ^i and the state-switching intensity ϕ^i represent the value functions post disaster and post state switching, respectively, and the post value functions $V^i(x, k\epsilon)$ and $V^j(x, \epsilon)$ directly influence the pre value function $V^i(x, \epsilon)$, thus affecting pre optimal policies.

As a slightly modified version of the convex-duality approach that has been adopted in Model 1, we introduce the state-dependent dual variable λ_i and the state-dependent convex-dual function $G_i(\lambda_i)$ as follows: in the state i ($i \in \{G, B\}$),

$$\lambda_i(x, \epsilon) \equiv V_x^i(x, \epsilon)$$

and

$$G_i(\lambda_i(x, \epsilon)) \equiv x + \frac{\epsilon}{r^\epsilon}.$$

The HJB equation (32) by taking its first derivative with respect to x then simplifies to the following dual HJB equation in the state i ($i \in \{G, B\}$): for $0 < \lambda_i < \overline{\lambda}_i$,

$$rG_i(\lambda_i) = \frac{1}{2}\tilde{\theta}_i^2\lambda_i^2 G_i''(\lambda_i)+\Big\{\beta+\delta^i\big(1-G_i\big(\lambda_i(x,k\epsilon)\big)\big)+\phi^i\big(1-\lambda_j(x,\epsilon)\big)+\tilde{\theta}_i^2-r\Big\}\lambda_i G_i'(\lambda_i)-\frac{1}{\gamma}\ln\lambda_i, \tag{33}$$

subject to the following boundary conditions:

$$G_i(\overline{\lambda}_i) = \frac{\epsilon}{r^{\epsilon}}, \quad G_i'(\overline{\lambda}_i) = 0, \tag{34}$$

where

$$\tilde{\theta}_i \equiv \theta + \rho^i b^i$$

is the disaster-risk-adjusted Sharpe ratio. Compared to Model 2, state-dependent and stochastically time-varying disastrous income risk has two major differences. On one hand, the Sharpe ratio θ is adjusted with the correlation ρ^i between the market and the disaster dynamics and the disaster volatility b^i. Put another way, the optimal decisions are affected accordingly by the disaster-risk-adjusted Sharpe ratio $\tilde{\theta}_i$ in the incorporation of hedging effectiveness of stock investment against disastrous income risk with the correlation ρ^i and in the reflection on different levels of deviations b^i from the expected disaster growth rate $-\delta^i$. On the other hand, the agent in Model 3 demands both time-varying disastrous income risk premium and state-dependent disastrous income risk premium in the agent's expected risk compensation. The expected return compensation for time-varying and state-dependent disastrous income risk is represented in the drift term of total wealth by $\delta^i\big(1 - G_i(\lambda_i, k\epsilon)\big)$ and $\phi^i\big(1 - \lambda_j(x,\epsilon)\big)$, respectively.

We conjecture that the solution of the dual HJB equation (33) has the following form in the state i ($i \in \{G, B\}$): for $0 < \lambda_i < \overline{\lambda}_i$,

$$G_i(\lambda_i) = -\frac{1}{\gamma r}\ln\lambda_i - \frac{\tilde{\theta}_i^2}{2\gamma r^2}\Big(1 + \frac{2}{\tilde{\theta}_i^2}(\beta + \delta^i + \phi^i - r)\Big) + \eta_i(\lambda_i)\lambda_i^{-\alpha_i} + \eta^*(\lambda_i)\lambda_i^{-\alpha_i^*}, \tag{35}$$

subject to

$$\eta_i'(\lambda_i)\lambda_i^{-\alpha_i} + (\eta^*(\lambda_i))'\lambda_i^{-\alpha_i^*} = 0,$$

where $\alpha_i > 1$ and $-1 < \alpha_i^* < 0$ are the two roots of the following characteristic equation:

$$F_i(\alpha_i; \delta^i, \phi^i) \equiv -\frac{1}{2}\tilde{\theta}_i^2 \alpha_i(\alpha_i - 1) + \alpha_i(\beta + \delta^i + \phi^i - r) + r = 0.$$

By putting the conjectured form (35) into the dual HJB equation (33), we obtain the following analytically tractable form in the state i ($i \in \{G, B\}$): for $0 < \lambda_i < \overline{\lambda}_i$,

$$\begin{aligned} G_i(\lambda_i) = &- \frac{1}{\gamma r}\ln\lambda_i - \frac{\tilde{\theta}_i^2}{2\gamma r^2}\Big(1 + \frac{2}{\tilde{\theta}_i^2}(\beta + \delta^i + \phi^i - r)\Big) + B_i\lambda_i^{-\alpha_i^*} \\ &+ \frac{2\phi^i\lambda_j}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\Big[(\alpha_i - 1)\lambda_i^{-\alpha_i}\int_0^{\lambda_i}\mu^{\alpha_i - 2}G_i(\mu)d\mu \\ &\qquad\qquad + (\alpha_i^* - 1)\lambda_i^{-\alpha_i^*}\int_{\lambda_i}^{\overline{\lambda}_i}\mu^{\alpha_i^* - 2}G_i(\mu)d\mu\Big] \\ &+ \frac{2\delta^i}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\Big[(\alpha_i - 1)\lambda_i^{-\alpha_i}\int_0^{\lambda_i}\mu^{\alpha_i - 2}V^i\big(G_i(\mu) - \epsilon/r^\epsilon, k\epsilon\big)d\mu \\ &\qquad\qquad + (\alpha_i^* - 1)\lambda_i^{-\alpha_i^*}\int_{\lambda_i}^{\overline{\lambda}_i}\mu^{\alpha_i^* - 2}V^i\big(G_i(\mu) - \epsilon/r^\epsilon, k\epsilon\big)d\mu\Big], \end{aligned} \tag{36}$$

where

$$B_i = \eta_i^*(\overline{\lambda}_i) - \frac{2\phi^i\overline{\lambda}_j}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}G_i(\overline{\lambda}_i)\overline{\lambda}_i^{\alpha_i^* - 1} - \frac{2\delta^i}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}V^i\big(G_i(\overline{\lambda}_i) - \epsilon/r^\epsilon, k\epsilon\big)\overline{\lambda}_i^{\alpha_i^* - 1},$$

which is a constant to be determined with $\overline{\lambda}_i$. Using the boundary conditions in (34), we obtain the following two relations in the state i ($i \in \{G, B\}$):

$$
\begin{aligned}
\frac{\epsilon}{r^{\epsilon}} = &- \frac{1}{\gamma r}\ln \overline{\lambda}_i - \frac{\tilde{\theta}_i^2}{2\gamma r^2}\Big(1 + \frac{2}{\tilde{\theta}_i^2}(\beta + \delta^i + \phi^i - r)\Big) + B_i \overline{\lambda}_i^{-\alpha_i^*} \\
&+ \frac{2\phi^i \overline{\lambda}_j}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}(\alpha_i - 1)\overline{\lambda}_i^{-\alpha_i}\int_0^{\overline{\lambda}_i} \mu^{\alpha_i - 2} G_i(\mu)d\mu \\
&+ \frac{2\delta^i}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}(\alpha_i - 1)\overline{\lambda}_i^{-\alpha_i}\int_0^{\overline{\lambda}_i} \mu^{\alpha_i - 2} V^i\big(G_i(\mu) - \epsilon/r^{\epsilon}, k\epsilon\big)d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{\gamma r} + \frac{\alpha_i^*\epsilon}{r^{\epsilon}} + \frac{\tilde{\theta}_i^2\alpha_i^*}{2\gamma r^2}\Big(1 + \frac{2}{\tilde{\theta}_i^2}(\beta + \delta^i + \phi^i - r)\Big) \\
&= -\alpha_i^*\frac{1}{\gamma r}\ln \overline{\lambda}_i - \frac{2\phi^i\overline{\lambda}_j}{\tilde{\theta}_i^2\overline{\lambda}_i}G_i(\overline{\lambda}_i) + \frac{2\phi^i\overline{\lambda}_j(\alpha_i - 1)}{\tilde{\theta}_i^2}\overline{\lambda}_i^{-\alpha_i}\int_0^{\overline{\lambda}_i}\mu^{\alpha_i - 2}G_i(\mu)d\mu \\
&\quad - \frac{2\delta^i}{\tilde{\theta}_i^2\overline{\lambda}_i}V^i\big(G_i(\overline{\lambda}_i) - \epsilon/r^{\epsilon}, k\epsilon\big) + \frac{2\delta^i(\alpha_i - 1)}{\tilde{\theta}_i^2}\int_0^{\overline{\lambda}_i}\mu^{\alpha_i - 2}V^i\big(G_i(\mu) - \epsilon/r^{\epsilon}, k\epsilon\big)d\mu,
\end{aligned}
$$

where the four constants B_G, B_B, $\overline{\lambda}_G$, and $\overline{\lambda}_B$ are determined numerically.

Now that the FOCs for consumption c and investment π given in (5) are restated with the dual variable λ_i and the convex-dual function $G_i(\lambda_i)$ in the state i ($i \in \{G, B\}$) as

$$
\begin{aligned}
c &= -\frac{1}{\gamma}\ln \lambda_i, \\
\pi &= -\frac{\theta}{\sigma}\lambda_i G_i'(\lambda_i),
\end{aligned}
$$

the following optimal consumption and investment strategies are obtained in the state i ($i \in \{G, B\}$):

$$
\begin{aligned}
c &= r\Big[x + \frac{\epsilon}{r^{\epsilon}} + \frac{\tilde{\theta}_i^2}{2\gamma r^2}\Big(1 + \frac{2}{\tilde{\theta}_i^2}(\beta + \delta^i + \phi^i - r)\Big) - B_i\lambda_i^{-\alpha_i} - \mathrm{PS}_i\Big], \\
\pi &= \frac{\tilde{\theta}_i}{\gamma\sigma}\Big(\frac{1}{r} + \alpha_i^* B_i \lambda_i^{-\alpha_i^*} - \alpha_i\mathrm{PS1}_i - \alpha_i^*\mathrm{PS2}_i - \mathrm{RD}_i\Big),
\end{aligned}
\tag{37}
$$

where PS_i represents the precautionary savings induced by state-dependent and time-varying disastrous income risk and it is given by

$$\mathrm{PS}_i = \mathrm{PS1}_i + \mathrm{PS2}_i,$$

$$\mathrm{PS1}_i = \frac{2\phi^i\lambda_j(\alpha_i - 1)}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\lambda_i^{-\alpha_i}\int_0^{\lambda_i}\mu^{\alpha_i-2}G_i(\mu)d\mu + \frac{2\delta^i(\alpha_i - 1)}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\lambda_i^{-\alpha_i}\int_0^{\lambda_i}\mu^{\alpha_i-2}V^i\big(G_i(\mu) - \epsilon/r^\epsilon, k\epsilon\big)d\mu,$$

$$\mathrm{PS2}_i = \frac{2\phi^i\lambda_j(\alpha_i^* - 1)}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\lambda_i^{-\alpha_i^*}\int_{\lambda_i}^{\bar{\lambda}_i}\mu^{\alpha_i^*-2}G_i(\mu)d\mu + \frac{2\delta^i(\alpha_i^* - 1)}{\tilde{\theta}_i^2(\alpha_i - \alpha_i^*)}\lambda_i^{-\alpha_i^*}\int_{\lambda_i}^{\bar{\lambda}_i}\mu^{\alpha_i^*-2}V^i\big(G_i(\mu) - \epsilon/r^\epsilon, k\epsilon\big)d\mu,$$

and RD_i represents the risk diversification demand induced by state-dependent and time-varying disastrous income risk and it is given by

$$\mathrm{RD}_i = -\frac{2\phi^i\lambda_j}{\tilde{\theta}_i^2\lambda_i}\Big(x + \frac{\epsilon}{r^\epsilon}\Big) - \frac{2\delta^i}{\tilde{\theta}_i^2\lambda_i}V^i(x, k\epsilon).$$

By allowing for state-dependent and stochastically time-varying disastrous income risk with Model 3, we have quantitatively identified in the optimal decisions in (37) state- and wealth-dependent precautionary savings and risk diversification demands. Provided that repeated disastrous income shocks with state-dependent and time-varying disaster intensity are even worse than the one-time-only disastrous income shock in Model 2, agents would show even more demands for precautionary savings and risk diversification. Repeated disastrous income shocks are much more challenging for agents to handle than the one-time-only shock as the likelihood and actual impact of future income shock do change according to whether the current state is good or bad and time-varying severity stemming from the stochastic nature of disastrous income risk.

5 Quantitative Analysis

In this section, we quantitatively discuss various properties of the agent's optimal consumption and investment decisions with disastrous income risk.

Following the literature having a focus on disastrous income risk (Bensoussan *et al.*, 2016; Jang *et al.*, 2020), we set the baseline parameter values of the market as follows: the risk-free interest rate $r = 3.71\%$, the expected stock return $\mu = 11.23\%$, and the stock volatility $\sigma = 19.54\%$.

The subjective discount rate considered in this paper accounts for mortality risk as well, so it is higher than the risk-free rate of return. In light of this aspect, we set $\beta = r + 5\%$.[11] The risk aversion is set to 2, i.e., $\gamma = 2$. The annual rate of income is normalized as 1, i.e., $\epsilon = 1$.

The rare disastrous event would be especially problematic for the agent who works in a small firm. This is because the agent has some possibilities of a permanent loss of income as a result of the small firm's closure or business failure and costly bankruptcy (Picot, 1992; Morissette, 2004) unless there is appropriate protection for the small firm's businesses by limiting the long term economic impact of the disaster (Milne, 2020). Given a positive relation between firm size and credit ratings (Blume *et al.*, 1998; Amato and Furfine, 2004; Avramov *et al.*, 2009), we choose to relate the small firm's bankruptcy due to the disruptive effects of the disaster to the default of speculative-grade firms. More specifically, the random arrival of the disastrous income shock can be calibrated by Moody's (2012) historical data of average cumulative issuer-weighted global default rates by rating categories for the time period form 1983 to 2011. We select four categories Aaa, Aa, A, and B to calibrate the arrival intensity δ for the income shock. According to the calibration, disastrous income shock intensity δ is estimated as follows: 0.01% for Aaa, 0.12% for Aa,

[11]Farhi and Panageas (2007), Jang *et al.* (2019) have also adopted such a higher subjective discount rate than the risk-free rate.

0.30% for A, and 5.26% for B.[12]

Model 2 with the one-time-only disastrous income shock would serve as a benchmark against which what and how both state dependency and time-varying severity of disastrous income risk play an important role in determination of the agent's optimal decisions. So, we first focus on the Model 2's implications with the one-time-only large negative Poisson shock to labor income and move on Model 3 to further investigate the role of a general Poisson jump process with state-dependent and time-varying disaster intensity in the optimal strategies.

The recovery of income after the disastrous income shock represents our very reduced-form modeling for the role of insurance against disastrous income risk. The income recovery is determined by the parameter k ($0 \leq k < 1$). We consider three cases of the income recovery: 0%, 10%, and 25%. In the extreme case of no access to an insurance market for hedging against disastrous income risk, the agent's income is completely wiped out, i.e., $k = 0\%$. In this case, the agent has no income source in the disaster. In the other cases of any access to the insurance market, the agent's income can be partly recovered at the rates $k = 10\%$ and $k = 25\%$.

5.1 Optimal Consumption and Investment in the Stock Market

We now investigate the effects of the disastrous income shock in Model 2 on the agent's optimal consumption and investment in the stock market especially when the agent has no access to the insurance market for income recovery post the income shock, i.e., when the income recovery parameter $k = 0\%$.

[12]Our calibration result is consistent with Wang *et al.* (2016) in that the arrival rate of large discrete (jump) earnings shocks is chosen as 5%, which is close to our calibration result obtained from the worst rating category B. Our estimation of parameter δ is conservative as compared to Pindyck and Wang (2013) who have used 7.34% for the mean arrival rate of a catastrophic shock to the capital stock.

Effects of Disastrous Income Shock. The large, negative effects of the disastrous income shock are found to significantly lower the agent's amount of optimal consumption (Figure 3), which results from the agent's strong motive for precautionary savings quantified in the optimal strategies derived in (25) and (26) in Section 4.2. Such large, negative effects become stronger as the possibilities of undergoing the income shock represented by the credit ratings of the firm for which the agent currently works rise.

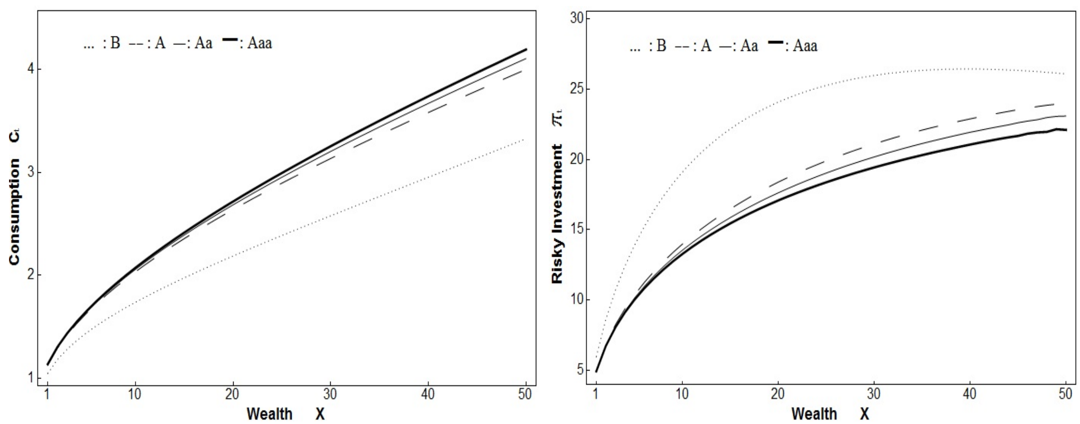

Figure 3: **Optimal consumption and investment in the stock market as a function of financial wealth** x.

Notably, the agent's optimal consumption shows a significant discontinuity and the dramatic change in the concavity of consumption when the credit rating falls from Aaa, Aa, A to B dramatically. We find that if the chance of experiencing the disastrous income shock is slim, i.e., when the credit rating of the firm that the agent works for lies in the range of Aaa to A, the agent's precautionary savings motive can be safely ignored with no significant consumption reductions as wealth decumulates. Comparatively, if the chance of having the disastrous income shock is sizable, i.e., when the credit rating of the firm is B, the agent's precautionary savings motive is quite strong as wealth decumulates. The highly likelihood of a rare event could change the consumption demand of low-wealth people, and thereby making them consume less substantially than without the income shock. Overall, how much agents do cut back their consumption is endogenously determined by the extent to which the disastrous income shock is quantitatively identified to agents.

Interestingly, the agent's equity demand can increase as the likelihood of occurrence of the disastrous income shock increases (Figure 3). Such an increased demand is driven by two main motives: the hedging demand and the precautionary savings demand. The effectiveness of hedging that has been quantified in the optimal strategies derived in (25) and (26) in Section 4.2 depends crucially on total available financial resources for future consumption. Hence, the agent can have an incentive to accumulate wealth by investing in the stock market, and taking advantage of positive risk premium. On the other hand, the precautionary savings are optimally allocated in both the riskless bond and the risky stock. It turns out that a part of the precautionary savings accounts for the increased equity demand, which is the result of optimal asset allocation between the riskless bond and the risky stock.

Notably, the agent's equity demand may be even stronger especially when wealth is low than when it is high. This can be understood as a relative comparison of the optimal decisions between low-wealth people and high-wealth people. Comparatively, the optimal consumption decision of low-wealth people could be affected substantially by the disastrous income shock (and thereby, they are willing to consume less and save more) and hence, their reduced consumption and the resulting extra savings may give rise to further increase in the equity demand for both saving and hedging purposes.

Effects of Risk Aversion. The levels of risk aversion of the agent have an influence on the agent's optimal consumption/savings decision. As expected, more risk averse agents naturally consume less and save more in the riskless bond than less risk averse agents do as the likelihood of the disastrous income shock increases, i.e., as the credit rating of the firm that agents work for worsens from Aaa to Aa, A, and B (Figure 4).

When it comes to the agent's optimal risky investment decision, the agent's high risk aversion obviously reduces the agent's equity demand significantly for all four possibilities of the disastrous income shock (Aaa, Aa, A, and B). Given that the agent's equity demand rather rises as the chance of the disastrous income shock increases (Figure 3), an assumption of low risk aversion strengthens such effects of the disastrous income shock on risky investments. A low risk aversion can increase the attractiveness of risky stocks as a partial hedging against the disastrous income shock.

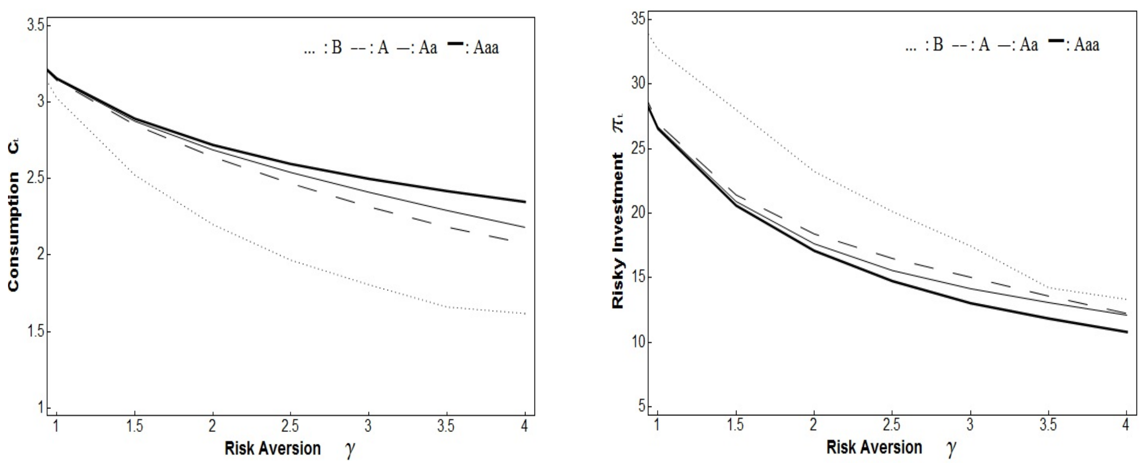

Figure 4: **Optimal consumption and investment in the stock market as a function of risk aversion**.

5.2 Discussion on Role of Insurance

Optimal Consumption and Investment Decisions. The role of insurance in the agent's optimal consumption and investment decisions in the presence of the disastrous income shock will be discussed with an emphasis on the agent's income recovery k post disaster. When the probability of the disastrous income shock is slim, i.e., when the credit rating of the firm for which agents work lies in the range of Aaa to A, the agent's optimal decisions are not likely to be affected by how much income is recovered in the income shock, thus suggesting a marginal role of insurance in the agent's optimal strategies (Table 2).

When the probability of the income shock is highly likely, i.e., when the credit rating

$x \setminus \delta$	$k=0$				$k=0.1$				$k=0.25$			
	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	1.1371	1.1300	1.1260	1.0469	1.1326	1.1311	1.1288	1.0930	1.1327	1.1323	1.1322	1.1664
10	2.0755	2.0596	2.0366	1.7398	2.0748	2.0636	2.0461	1.8023	2.0752	2.0681	2.0576	1.8894
20	2.7194	2.6871	2.6410	2.1895	2.7195	2.6945	2.6575	2.2612	2.7203	2.7030	2.6778	2.3641
30	3.2571	3.2064	3.1379	2.5806	3.2578	3.2173	3.1606	2.6578	3.2590	3.2301	3.1892	2.7841
40	3.7427	3.6732	3.5845	2.9552	3.7439	3.6875	3.6125	3.0359	3.7456	3.7044	3.6484	3.1890
50	4.1971	4.1093	4.0034	3.3295	4.1988	4.1268	4.0357	3.4116	4.2009	4.1477	4.0777	3.5735

(A) consumption

$x \setminus \delta$	$k=0$				$k=0.1$				$k=0.25$			
	Aaa	Aa	A	B	Aaa	Aa	A	B	Aaa	Aa	A	B
1	4.9360	4.9024	4.9514	5.9681	4.8716	4.8936	4.9301	5.9793	4.8701	4.8843	4.9078	6.2262
10	13.2873	13.5563	13.9905	19.1176	13.2675	13.4948	13.8481	18.6576	13.2612	13.4244	13.6769	18.0992
20	17.0904	17.6332	18.3955	24.0844	17.0719	17.5228	18.1680	23.6951	17.0598	17.3932	17.8819	22.6893
30	19.4330	20.1824	21.1097	25.9751	19.4125	20.0400	20.8520	25.6721	19.3955	19.8677	20.5128	23.9466
40	21.0595	21.9355	22.8843	26.4341	21.0367	21.7729	22.6365	26.2535	21.0165	21.5840	22.2962	24.7958
50	22.1026	23.0891	24.0704	26.0817	22.0784	22.9668	23.8046	26.0728	22.0596	22.8175	23.5572	26.6611

(B) risky investment

Table 2: **Optimal consumption and investment in the stock market as a function of initial wealth x for different levels of disastrous income risk and income recovery.**.

of the firm is B, the story is quite different. The agent's optimal decisions are significantly affected by how much income is recovered post disaster depending upon levels of the income recovery parameter k. Without any access to insurance for the income recovery in the disastrous income shock, i.e., when $k = 0$, the agent's income will be completely wiped out when the income shock takes place. In this case, the optimal consumption decision is for the agent with little wealth ($x = 1$) to cut back by 7.93% as the probability of the income shock sharply increases (i.e., as the credit rating of the firm plummets from Aaa to B). The poor would be better off saving current wealth as much as possible in the preparation for future consumption post disaster, which is an accurate reflection towards substantial precautionary saving. However, with access to an insurance market for the agent to partly recover from the disastrous income shock, i.e., when $k = 0.25$, the optimal consumption decision is for the agent with $x = 1$ to even increase by 2.98% for the same large chance of the the income shock with the credit rating B.

The importance of insurance when the highly likely disastrous income shock facing agents remains significant for the wealthy as well. Without access to the insurance market ($k = 0$), the optimal consumption decision of the agent with high wealth ($x = 20$) is to significantly reduce by 19.49% as the probability of the income shock surges (as the credit rating worsens from *Aaa* to *B*), thus demonstrating quite strong savings motives of the wealthy.[13] With access to the insurance market ($k = 0.25$), the optimal consumption decision of the agent with $x = 20$ is to reduce by 13.09%, which is smaller than without availability of insurance.

A similar analysis can be applied to the agent's optimal investment decision. With no access to the insurance market for income recovery post disaster, i.e., when $k = 0$, the optimal investment decision is for the agent with little wealth ($x = 1$) to increase by 20.91% in a response to the significant fall of the firm's credit rating from *Aaa* to *B*. However, with access to the insurance market, i.e., when $k = 0.25$, the optimal investment decision is for the same agent to even more increase by 27.85% for the same credit rating reduction of the firm. For the wealthy with $x = 20$, the optimal investment decision is to increase 40.92% and 33.00%, respectively, when $k = 0$ and $k = 0.25$ in a response to the same credit rating reduction. The equity demand of the wealthy caused by the disastrous income shock becomes attenuated as their income is more recovered post disaster with higher levels of k because their far enough wealth already secured would buffer against the income shock with the help of income recovery as well.

Overall, our results highlight that availability of insurance can be particularly important for both the poor and the wealthy in the sense that they could even consume more, save less, and invest more post disaster as long as their future income can be (partly) recovered with the help of insurance.

Human Capital Value. Following Koo (1998), we measure the human capital value as the marginal rate of substitution between income and financial wealth. It can be viewed

[13] The average ratio of wealth-to-income between 1952 and 2016 in the U.S. was 6.6. Since labor income is normalized as one in this paper, the wealthy can be regarded as people having wealth much higher than 6.6.

as a proxy for the agent's subjective marginal value of income relative to financial wealth.

Definition 5.1 *The agent's human capital value is measured as the marginal rate of substitution between income and financial wealth as follows:*

$$\frac{V_{\epsilon}(x,\epsilon)}{V_x(x,\epsilon)},$$

where $V(x,\epsilon)$ is the agent's value function of Model 2 in Section 4.2.

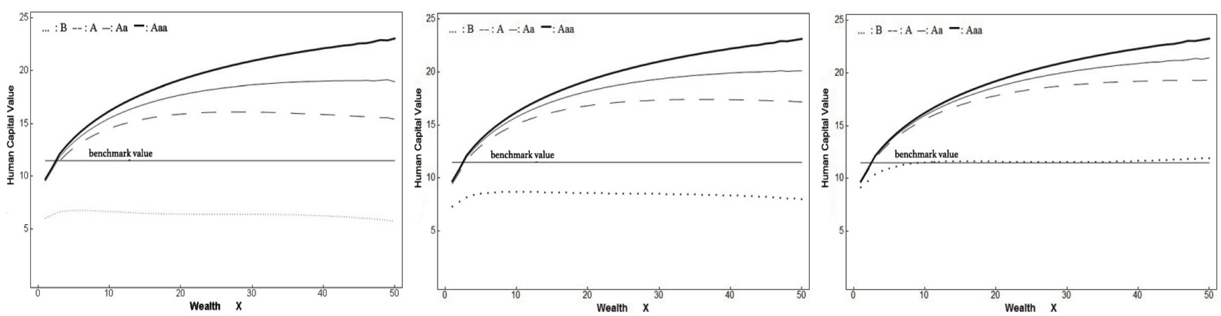

Figure 5: **Human capital value as a function of financial wealth** x. We have used $k = 0$ (left panel), $k = 0.1$ (middle panel), and $k = 0.25$ (right panel).The benchmark value, ϵ/r, is the present value of future income without the disastrous income shock ($\delta = 0$) discounted by the risk-free interest r.

The human capital plays a vital role in the agent's optimal consumption and investment decisions (Figure 5). For a relatively poor agent, the human capital is a major source for financing investment and consumption, so she should concern herself with relying on the extent of the income recovery in the disastrous income shock. Without any access to the insurance market with $k = 0$, the human capital value of the poor agent plummets substantially (which is much lower than the benchmark value[14]) if the probability of the disastrous income shock is very high (as the credit rating of the firm for which the agent works drops significantly from Aaa to B). The poor agent's optimal decision is therefore to reduce both her consumption and risky investment dramatically to preserve available financial resources for future consumption post disaster. However, with access to the

[14]The benchmark value, ϵ/r, is the present value of future income without the disastrous income shock ($\delta = 0$) discounted by the risk-free interest r.

insurance market with $k = 0.25$, the human capital value of the poor agent is rather quite close to the benchmark value even if the credit rating of the firm falls sharply from Aaa to B. This would suggest that the poor agent can even consume well and virtually save nothing in the disastrous income shock as long as the agent's income can be recovered from the income shock thanks to the insurance access.

Both quantitative and qualitative properties of the human capital value are altered with the disastrous income shock compared to the case for which there does not exist the income shock. When the disastrous income shock rarely takes place, i.e., when the credit rating of the firm lies in the range of Aaa to A, the human capital value shows an increasing and concave pattern with respect to levels of financial wealth x. However, when the income shock occurs imminently, i.e., when the credit rating of the firm is B, the human capital value becomes nearly constant and it is no longer the increasing and concave function of wealth.

The role of insurance providing the agent with income recovery post disastrous income shock is crucial especially when the income shock is highly likely to take place soon. Without access to the insurance market with no income recovery ($k = 0$), the human capital value is far lower than the benchmark value for all levels of wealth. This result implies that the disastrous income shock remain significant not only for the wealth poor, but also for the wealthy. The substantially reduced human capital value would lead both the poor and the rich to concern themselves with diversifying their significant exposure to the disastrous income shock, thus consuming less, saving less, and investing more as identified in the analysis of agent's equity demand with the income shock. However, with access to the insurance market with income recovery ($k = 0.25$), the value of human capital increases a lot and becomes almost the same as the benchmark value for all levels of wealth. Thanks to the increased human capital value with the help of insurance, all agents tend to consume more, save less, and invest more comparatively, having smaller savings motives for precautionary purposes than without access to the insurance market.

Welfare Cost. Neglecting the disastrous income shock that is even one-time-event in the agent's life cycle can be significantly costly to the agent who aims to make optimal consumption/savings and investment choices if the income shock is soon to come. We calculate the agent's welfare cost of ignoring the disastrous income shock by comparing two opposing situations (two value functions) with and without the income shock. The welfare cost is then quantitatively identified as the agent's certainty equivalent in wealth units. That is, the welfare cost would be the maximum possible wealth that the agent is willing to pay for to eliminate the disastrous income shock.

Definition 5.2 $\Delta(x)$ *is the certainty equivalent measured in wealth unities at the level* x *of financial wealth if it solves*

$$V(x - \Delta(x), \epsilon; \delta = 0) = V(x, \epsilon; \delta > 0),$$

where $V(x, \epsilon)$ *is the value function of Model 2 in Section 4.2.*

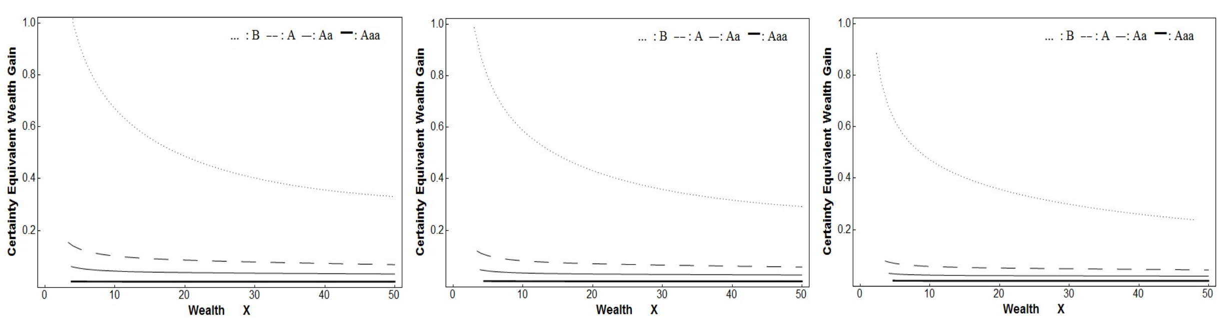

Figure 6: **Welfare cost (%) as a function of financial wealth** x. We have used $k = 0$ (left panel), $k = 0.1$ (middle panel), and $k = 0.25$ (right panel).

The economic significance of clearly taking the disastrous income shock into account the agent's optimal decisions is well recognized especially when the likelihood of the income shock is large (Figure 6). If the chance of undergoing the income shock is slim, i.e., when the credit rating of the firm for which the agent works lies in the range of Aaa to A, the welfare costs are all below 20% of the agent's initial endowment for all levels of wealth. However, if that chance is highly likely, the welfare costs surge and are even more than 100% of initial wealth for the poor with no access to the insurance market for their income recovery in the aftermath of income disaster.

Insurance helping all agents who can be poor or rich to effectively recover in part their lost income from the disastrous income shock will allow the agents to rationally respond to the income shock with optimal choices and hence, lower the welfare costs. In particular, the poor would benefit from their access to the insurance market by recovering their income in the disaster, and thereby the poor's welfare costs incurred with no consideration of the disastrous income shock could be reduced and lower than 100% of initial wealth with no access to the insurance market.

5.3 Effects of Time-Varying Severity of Disastrous Income Risk

We now move on the Model 3's implications with an emphasis on the time-varying severity of the disastrous income shock. To abstract from other factors affecting the agent's optimal choices, we consider the one-time-only disastrous income shock in this section, but having the stochastically time-varying disaster intensity δ_t that is not state dependent. That is, the volatility b on the intensity growth and the correlation ρ between the stock market and the dynamics of disaster intensity are the extra dimensions considered in this section. Provided that the value of human capital is of utmost importance to the characterization of the agent's optimal strategies, we explore the effects of the intensity growth volatility b and the correlation ρ on the human capital value.

Remarkably, the time-varying severity itself does not seem to affect the agent's human capital value dramatically (Table 3). The slightly larger human capital values are observed

	A-rated firm			*B*-rated firm		
	$b = 0$	$b = 0.1$		$b = 0$	$b = 0.1$	
$x \setminus \rho$		0.1	-0.1		0.1	-0.1
1	9.32452	9.9536	9.3245	6.0474	6.1378	6.0473
10	14.5331	14.5217	14.5331	6.6683	6.9284	6.6686
20	15.8903	16.0435	15.8911	6.4155	6.5633	6.4170
30	16.0815	16.5805	16.0822	6.3911	6.4006	6.3928
40	15.8504	16.8132	15.8528	6.2640	6.2085	6.2639
50	15.4103	16.9610	15.5094	5.7710	5.7728	5.7714

Table 3: **Human capital value as a function of financial wealth x with respect to changes in b and ρ.**

than with no disaster intensity volatility ($b = 0$). Provided nearly the same human capital values with and without disaster intensity volatility, the stochastic nature of the disastrous income shock cannot change the agent's optimal choices significantly. Rather, the presence of the income shock does alter the agent's optimal decisions. Indeed, the high likelihood of the occurrence of the income shock substantially reduces the human capital value.

The similar analysis can be applied to the effects of the correlation ρ between the stock market and the income disaster intensity process. The correlation ρ determines the effectiveness of hedging by dynamically trading in the stock market against the income shock. When the correlation is positive, the effectiveness of hedging with stock investment is reduced. However, the human capital value does not seem to show major differences between positive and negative correlations. Once again, the likelihood of the disastrous income shock taking place is a dominating factor affecting the human capital value and hence, the agent's optimal policies.

6 Conclusion

We have developed the analytically tractable optimal consumption/savings and investment model with disastrous income risk facing every individual who suffers from today's cost of living crisis due to high inflation. In the context of Rietz's rare disaster risk hypothesis, the low-probability, depression-like additional state in the agent's income caused by disastrous income risk significantly affects the agent's optimal choices. Consistent with the standard precautionary savings argument, the agent finds it optimal to consume less and save more. However, the precautionary savings required turn out to contribute to an increase in risky investments surprisingly. The agent's ability to trade a risky asset in the market helps manage the effects of disastrous income risk by taking advantage of positive risk premia obtained from risky investments compared to riskless savings in the bond market only. Finally, the role of insurance providing income recovery post disaster becomes ever important in the disaster as access to the insurance market for income recovery allows the agent to consume more than with no access to insurance.

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