Flexible Moral Hazard Problems

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Overview

Classic moral hazard model:

- Effort is either binary, or belongs to an interval.
- Main result: contracts are motivated by informativeness.
- Need strong assumptions for wage to increase in output.

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- Need strong assumptions for wage to increase in output.

Current paper:

- Allow agent to choose *any* output distribution.
- Contracts determined by agent's marginal costs.
- Wages are increasing whenever costs increase in FOSD.

Two Examples

Common Setup for Examples

A principal (she) contracts with an agent (he).

- Compact set $X \subset \mathbb{R}$ of possible outputs.
- Principal offers agent a (bounded) contract: $w : X \to \mathbb{R}$.
- Agent can opt out and get u_0.
- If opts in, agent covertly chooses $\alpha \in \mathcal{A} \subseteq \Delta(X)$.
- Effort costs: $C : \mathcal{A} \to \mathbb{R}_+$, increasing in FOSD.
- Payoffs:

$$\text{Principal: } x - w \qquad \text{Agent: } u(w) - C(\alpha).$$

u: strictly increasing, differentiable, unbounded, concave.

Standard Binary Effort Model

$$X = [L, H], \qquad \mathcal{A} = \{\alpha_l, \alpha_h\}.$$

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So: w is monotone $\iff$ MLRP holds.

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Implications:

- Cost minimization is trivial: min $w(L)$ s.t. IR.
- Shape of contract determined by C' and u.
- IC contracts are monotone:

$$w(H) = u^{-1}(u \circ w(L) + C'(\alpha)) \geq u^{-1}(u \circ w(L)) = w(L).$$

Our Main Model

A principal (she) contracts with an agent (he).

- Compact set $X \subset \mathbb{R}$ of possible outputs.
- Principal offers agent a (bounded) contract: $w : X \to \mathbb{R}_+$.
- Limited liability: $w(\cdot) \geq 0$.
- Agent covertly chooses $\alpha \in \mathcal{A} = \Delta(X)$.
- Effort costs: $C : \mathcal{A} \to \mathbb{R}_+$, continuous, increasing in FOSD.
- Payoffs:

$$\text{Principal: } x - w \qquad \text{Agent: } u(w) - C(\alpha),$$

u: increasing, continuous, unbounded & $u(0) = 0$.

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Assumption. (smoothness) C is **Gateaux differentiable**: every α admits a bounded $k_\alpha : X \to \mathbb{R}$ s.t.

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[C(\alpha + \epsilon(\beta - \alpha)) - C(\alpha) \right] = \int k_\alpha(x)\,(\beta - \alpha)(\mathrm{d}x)$$

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- $k_\alpha(x)$: MC of increasing probability of output x.
- If X is finite: smooth $\iff$ differentiable, which holds a.e.
- C increases in FOSD $\iff$ k_α increasing $\forall \alpha$.

First-Order Approach

Lemma. For a bounded $v : X \to \mathbb{R}$, and $\alpha \in \mathcal{A}$,

$$\alpha \in \arg\max_{\beta \in \mathcal{A}} \left[\int v(x)\beta\,(\mathrm{d}x) - C(\beta) \right]$$

if and only if

$$\alpha \in \arg\max_{\beta \in \mathcal{A}} \left[\int v(x)\beta(\mathrm{d}x) - \int k_\alpha(x)\beta(\mathrm{d}x) \right]$$

(the "only if" direction also works if C is not convex)

Relationship to Standard FOC

Consider the problem:

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Standard way of writing FOC for optimal $a^* \in (0,1)$ is

$$v - c'(a^*) = 0.$$

An equivalent way of writing the above condition is:

$$a^* \in \operatorname{argmax}_{a \in [0,1]} [av - ac'(a^*)].$$

The lemma generalizes the second formulation.

First-Order Approach

Lemma. For a bounded $v : X \to \mathbb{R}$, and $\alpha \in \mathcal{A}$,

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Characterization of IC

Say a contract-distribution pair (w, α) is **IC** if

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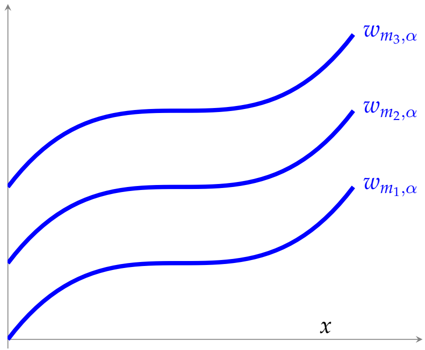

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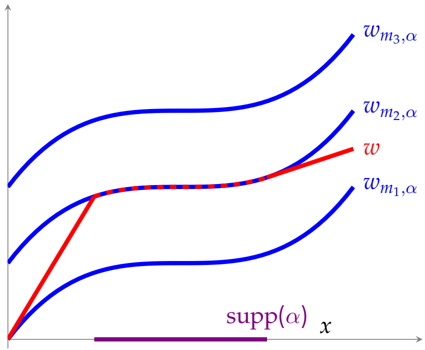

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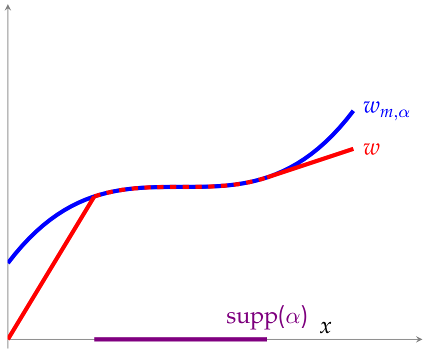

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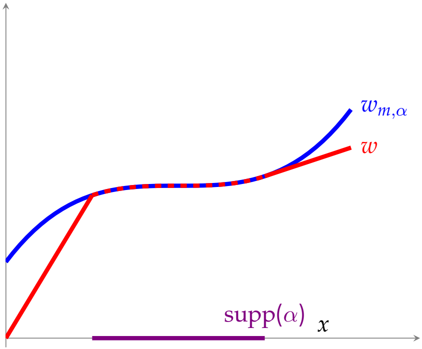

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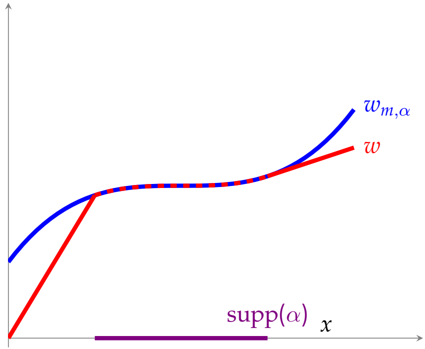

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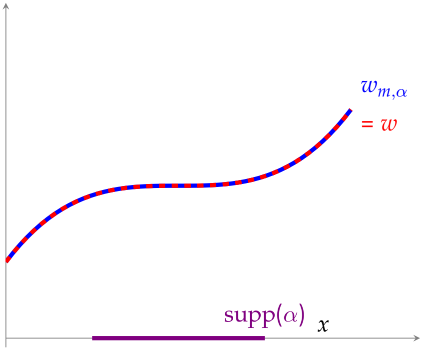

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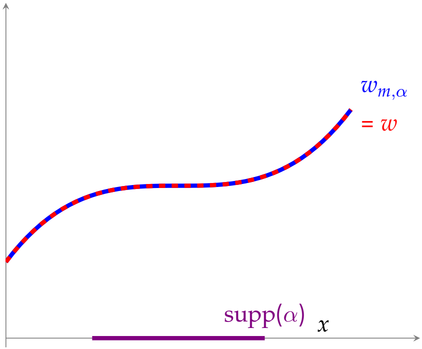

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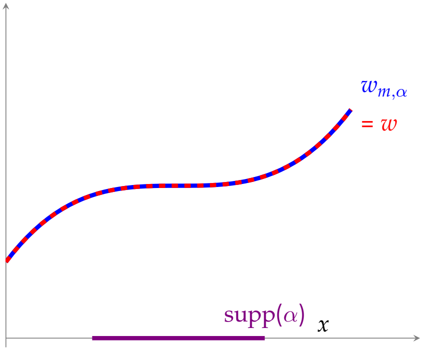

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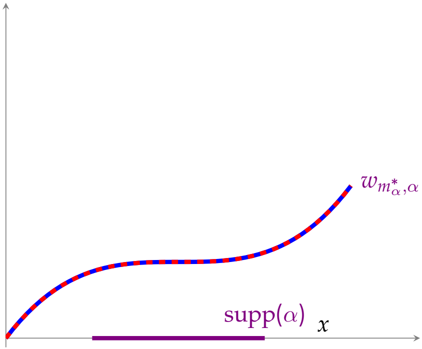

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Therefore,

$$w_{m,\alpha} \text{ is increasing } \forall \alpha, m \iff k_\alpha \text{ is increasing } \forall \alpha$$

Explanation: because u^{-1} is increasing.

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Therefore,

$$\begin{aligned} w_{m,\alpha} \text{ is increasing } \forall \alpha, m &\iff k_\alpha \text{ is increasing } \forall \alpha \\ &\iff C \text{ is FOSD monotone.} \end{aligned}$$

Explanation: by the Fact .

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for all x, and with equality α-almost surely.

Implications:

(i) Without loss for principal to offer $w_{m,\alpha}$ for some m.

(ii) A cheapest contract implementing α is $w_{m^*_\alpha,\alpha}$ for

$$m^*_\alpha = -\inf k_\alpha(X).$$

(iii) C FOSD increasing $\implies$ wage is increasing without loss.

Related Literature

- **Less general flexible models**: Holmstrom and Milgrom (1987), Diamond (1998), Mirrlees and Zhou (2006), Hebert (2018), Bonham (2021), Mattsson and Weibull (2022), Bonham and Riggs-Cragun (2023).

- **Flexible Monitoring:** Georgiadis and Szentes (2020), Mahzoon, Shourideh, and Zetlin-Joines (2022), Wong (2023).

- **Robust contracting:** Carroll (2015), Antic (2022), Antic and Georgiadis (2022), Carroll and Walton (2022).

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We show that in smooth & flexible moral hazard problems:

- Parameters driving contract: k_α and u.
- Cost minimization is trivial.
- Every distribution can be implemented.
- FOSD monotonicity $\implies$ wages increase in output.

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In paper, we also have results about principal optimality:

- FOC for the principal (1st order approach is valid).
- Optimality of single, binary, and discrete distributions.

Thanks!