Online Appendix

Term Limits and Bargaining Power in Electoral Competition

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Proof of Lemma 1. Consider a challenger i who first runs in period t against an incumbent of type (θ, k). Let $R(\theta')$ be i's expected lifetime rents from office, conditional on winning in period t and on her ability being θ'. Let $\gamma Q(\theta')$ be i's expected lifetime policy payoffs **excluding period** t, again conditional on winning in period t and her ability being θ'. Let $\gamma S_k(\theta, \theta')$ be i's policy payoff in period t, conditional on her ability being θ' and the incumbent being type (θ, k). (Note that $R(\theta')$, $Q(\theta')$ are independent of θ and k, and R, Q and S are *not* functions of γ.) Then

$$T_k(\theta) = \int_0^1 [R(\theta') + \gamma Q(\theta') + \gamma S_k(\theta, \theta')] \, r_k(\theta, \theta') f(\theta') d\theta'.$$

By Proposition 1, if the challenger wins and her ability is θ', then with probability $1-\mu$ she is unbiased, her policy is 0, and her policy payoff is 0; with probability μ she is biased, her policy is $\pm\sqrt{\frac{U_0(\theta')-U_k(\theta)}{\lambda}}$, and her policy payoff is $-\left(I - \sqrt{\frac{U_0(\theta')-U_k(\theta)}{\lambda}}\right)^2$. In other words, $S_k(\theta, \theta') = -\mu\left(I - \sqrt{\frac{U_0(\theta')-U_k(\theta)}{\lambda}}\right)^2$, which is a strictly decreasing function of $U_k(\theta)$. Furthermore, $r_k(\theta, \theta')$ is weakly decreasing as a function of $U_k(\theta)$ for each θ': if $U_k(\theta) < U_{\tilde{k}}(\tilde{\theta})$, then either $U_k(\theta) < U_0(\theta')$, implying $r_k(\theta, \theta') = 1 \geq r_{\tilde{k}}(\tilde{\theta}, \theta')$, or $U_0(\theta') < U_{\tilde{k}}(\tilde{\theta})$, implying $r_k(\theta, \theta') \geq r_{\tilde{k}}(\tilde{\theta}, \theta') = 0$. The result follows. □

Proof of Proposition 5–Pinning down θ_0. Under stationary limits, the expressions for

R and Q simplify to

$$R(\theta) = \frac{b}{1-\delta p(1-q(\theta)\kappa(\theta))} = \frac{b}{1-\delta p+\delta pq(\theta)\kappa(\theta)}$$
$$Q(\theta) = [q(\theta)y_1 + (1-q(\theta))y_0]\frac{\delta p}{1-\delta p+\delta pq(\theta)\kappa(\theta)},$$

where $\kappa(\theta) = \int_0^1 r(\theta,\theta')f(\theta')d\theta'$ is the probability that an incumbent of ability θ loses an election, conditional on the challenger running; and y_1, y_0 are the expected flow policy payoffs of an incumbent of ability θ if the challenger runs or does not run, respectively. Remember also that

$$\overline{T}_{\theta_0} = \int_0^1 (R(\theta)+\gamma Q(\theta)+\gamma S(\theta_0,\theta))\, r(\theta_0,\theta) f(\theta)\, d\theta.$$

Suppose first that the equilibrium is of type 2, and let $\theta_1 = \theta_1(\theta_0)$. Then $r(\theta_0,\theta) = 0$ for $\theta < \theta_0$, $r(\theta_0,\theta) = \frac{1}{2}$ for $\theta \in [\theta_0,\theta_1]$ and $r(\theta_0,\theta) = 1$ for $\theta > \theta_1$:

$$\overline{T}_{\theta_0} = \frac{1}{2}\int_{\theta_0}^{\theta_1} (R(\theta)+\gamma Q(\theta)+\gamma S(\theta_0,\theta))\, f(\theta)\, d\theta + \int_{\theta_1}^1 (R(\theta)+\gamma Q(\theta)+\gamma S(\theta_0,\theta))\, f(\theta)\, d\theta.$$

Letting $R_* = \frac{\partial R}{\partial \theta_0}$ and so on, we then want to show that $\frac{\partial \overline{T}_{\theta_0}}{\partial \theta_0} < 0$ for all θ_0, where

$$\frac{\partial \overline{T}_{\theta_0}}{\partial \theta_0} = \frac{1}{2}\int_{\theta_0}^{\theta_1} (R_*(\theta)+\gamma Q_*(\theta)+\gamma S_*(\theta_0,\theta))\, f(\theta)\, d\theta + \int_{\theta_1}^1 (R_*(\theta)+\gamma Q_*(\theta)+\gamma S_*(\theta_0,\theta))\, f(\theta)\, d\theta$$
$$-\frac{1}{2}(R(\theta_0)+\gamma Q(\theta_0)+\gamma S(\theta_0,\theta_0))f(\theta_0) - \frac{1}{2}\theta_1'(\theta_0)(R(\theta_1)+\gamma Q(\theta_1)+\gamma S(\theta_0,\theta_1))f(\theta_1).$$

Note that $R_*(\theta) = 0$ for $\theta > \theta_1$ (because $q(\theta)\kappa(\theta) \equiv 0$), and $S(\theta_0,\theta) = S_*(\theta_0,\theta) = 0$ for $\theta \in [\theta_0,\theta_1]$. Then we need to show

$$\frac{1}{2}\int_{\theta_0}^{\theta_1} (R_*(\theta)+\gamma Q_*(\theta))\, f(\theta)\, d\theta + \int_{\theta_1}^1 (\gamma Q_*(\theta)+\gamma S_*(\theta_0,\theta))\, f(\theta)\, d\theta$$
$$-\frac{1}{2}(R(\theta_0)+\gamma Q(\theta_0))f(\theta_0) - \frac{1}{2}\theta_1'(\theta_0)(R(\theta_1)+\gamma Q(\theta_1))f(\theta_1) < 0$$

Because we want to show this holds for γ low enough, it is necessary and sufficient to prove that

$$\text{(B1)} \qquad \int_{\theta_0}^{\theta_1} R_*(\theta) f(\theta)\, d\theta < R(\theta_0) f(\theta_0) + R(\theta_1) f(\theta_1)\theta_1'(\theta_0)$$

and that $Q_*(\theta)$, $Q(\theta)$, and $S_*(\theta_0, \theta)$ ($\theta > \theta_1$) are bounded.[1] Before proceeding further, note that R_*, Q_* and S_* (hence also q_* and κ_*) must be well defined for our approach to be valid. This boils down to showing that $\theta_1'(\theta_0)$ exists, which follows from applying the Implicit Function Theorem to the characterization of θ_1 in Lemma B1.

We will first deal with office rents. We can calculate

$$R_*(\theta) = \frac{b\delta p q(\theta)\kappa(\theta)}{(1-\delta p + \delta p q(\theta)\kappa(\theta))^2}\left(-\frac{q_*(\theta)}{q(\theta)} - \frac{\kappa_*(\theta)}{\kappa(\theta)}\right).$$

Here $\kappa(\theta) = 1 - \frac{F(\theta_0)+F(\theta_1)}{2}$, so $\kappa_*(\theta) = -\frac{f(\theta_0)+f(\theta_1)\theta_1'(\theta_0)}{2}$, and $q(\theta) = \frac{\theta_1-\theta}{\theta_1-\theta_0}$, so $q_*(\theta) = \frac{\theta_1'(\theta_0)(\theta-\theta_0)+\theta_1-\theta}{(\theta_1-\theta_0)^2}$. A digression here will be necessary. Using our characterization of q' and θ_1 (Proposition 5—pinning down θ_1), we can show that $\theta_1 - \theta_0$ is bounded away from zero and θ_1' is bounded and bounded away from zero:

Lemma B1. *There are m, m', $M > 0$ dependent only on μ, δ, p and F such that $\theta_1(\theta_0) - \theta_0 \geq m'$ and $\theta_1'(\theta_0) \in [m, M]$.*

Proof. Note that if $\theta_1(\theta_{0k}) - \theta_{0k} \xrightarrow[k\to\infty]{} 0$ for some sequence $(\theta_{0k})_k$, then in the limit we would have $|q'| \leq \frac{1}{\delta p \min(\mu,1-\mu)\int_0^1 \min\left(\frac{1-F(\theta)}{1-\delta p[\mu+(1-2\mu)F(\theta)]}, \frac{1-F(\theta)}{1-\delta p(1-\mu)}\right)d\theta} < \infty$, so $q'(\theta_1 - \theta_0) \to 0$, a contradiction. If $1 \geq \theta_1 - \theta_0 \geq m'$, then $1 \leq q' \leq \frac{1}{m'}$. θ_1' must solve $q'(\theta_1' - 1) + \left(\frac{\partial q'}{\partial \theta_1}\theta_1' + \frac{\partial q'}{\partial \theta_0}\right)(\theta_1 - \theta_0) = 0$, or $\theta_1' = \frac{q' - \frac{\partial q'}{\partial \theta_0}}{q' + \frac{\partial q'}{\partial \theta_1}}$. Here $-\frac{\partial q'}{\partial \theta_0} = q'^2 \frac{\delta p \mu(1-F(\theta_0))}{1-\delta p[\mu+(1-2\mu)F(\theta_0)]} \leq \frac{\delta p \mu}{(1-\delta p)m'^2}$ and $\frac{\partial q'}{\partial \theta_1} = q'^2 \frac{\delta p(1-\mu)(1-F(\theta_1))}{1-\delta p + \delta p \mu} \leq \frac{\delta p(1-\mu)}{(1-\delta p)m'^2}$. This yields the result. □

Using Lemma B1 and previous results, and denoting $\underline{m} = \min(m, 1)$,

$$-\frac{q_*(\theta)}{q(\theta)} = -\frac{1}{q(\theta)}\frac{\theta_1'(\theta_0)(\theta-\theta_0)+\theta_1-\theta}{(\theta_1-\theta_0)^2} \leq -\frac{1}{q(\theta)}\frac{\underline{m}(\theta-\theta_0)+(\theta_1-\theta)}{\theta_1-\theta_0} =$$
$$= -\frac{\underline{m}}{q(\theta)(\theta_1-\theta_0)} - \frac{1-\underline{m}}{\theta_1-\theta_0} \leq -\frac{1}{1-\theta_0}\left(\frac{\underline{m}}{q(\theta)} + 1 - \underline{m}\right)$$
$$-\frac{\kappa_*(\theta)}{\kappa(\theta)} = \frac{f(\theta_0)+f(\theta_1)\theta_1'(\theta_0)}{2-F(\theta_0)-F(\theta_1)} \leq \frac{f(\theta_0)+f(\theta_1)\theta_1'(\theta_0)}{1-F(\theta_0)}.$$

[1] Because both sides of (B1) are continuous in θ_0, if the inequality holds strictly for all θ_0, the difference between the two sides is bounded away from zero.

Then we can deal with the terms involving $f(\theta_1)$ as follows:

$$\int_{\theta_0}^{\theta_1} \frac{b\delta pq(\theta)\kappa(\theta)}{(1-\delta p+\delta pq(\theta)\kappa(\theta))^2} \frac{f(\theta_1)\theta_1'(\theta_0)}{(1-F(\theta_0))} f(\theta)d\theta < R(\theta_1)f(\theta_1)\theta_1'(\theta_0),$$

because $\frac{\delta pq(\theta)\kappa(\theta)}{(1-\delta p+\delta pq(\theta)\kappa(\theta))^2} < \frac{1}{1-\delta p}$ and $\int_{\theta_0}^{\theta_1} f(\theta)d\theta \leq 1-F(\theta_0)$. So it is enough to show

$$\int_{\theta_0}^{\theta_1} \frac{b\delta pq(\theta)\kappa(\theta)}{(1-\delta p+\delta pq(\theta)\kappa(\theta))^2} \left(-\frac{\frac{\underline{m}}{q(\theta)}+1-\underline{m}}{1-\theta_0} + \frac{f(\theta_0)}{1-F(\theta_0)}\right) f(\theta)d\theta < R(\theta_0)f(\theta_0).$$

Using that $\frac{f(\theta_0)}{1-F(\theta_0)} \leq \frac{\phi}{1-\theta_0}$, it is enough to show that for any $0 \leq q \leq 1$

$$\left(\frac{b\delta pq\kappa}{(1-\delta p+\delta pq\kappa)^2}\left(1-\frac{\underline{m}}{\phi q}-\frac{1-\underline{m}}{\phi}\right)\frac{f(\theta_0)}{1-F(\theta_0)}\right)(F(\theta_1)-F(\theta_0)) < \frac{bf(\theta_0)}{1-\delta p+\delta p\kappa}$$
$$\frac{\delta pq\kappa}{(1-\delta p+\delta pq\kappa)^2}\left(1-\frac{\underline{m}}{\phi q}-\frac{1-\underline{m}}{\phi}\right) < \frac{1}{1-\delta p+\delta p\kappa}$$

The left-hand side is single-peaked in q with a maximum at $q^* = \frac{1-\delta p}{\delta pk} + \frac{2\underline{m}}{\phi-1+\underline{m}}$. If this q^* is greater than 1, then we need

$$\frac{\delta p\kappa}{(1-\delta p+\delta p\kappa)^2}\left(1-\frac{1}{\phi}\right) < \frac{1}{1-\delta p+\delta p\kappa},$$

which always holds. If $0 < q^* < 1$, then the maximized value of the left-hand side is $\frac{1}{\frac{4\phi}{\phi-1+\underline{m}}(1-\delta p)+\frac{4\underline{m}\phi}{(\phi-1+\underline{m})^2}\delta p\kappa}$. Since $\underline{m} \leq 1$, $\frac{4\phi}{\phi-1+\underline{m}} \geq 4 > 1$, so the required inequality is guaranteed to hold if $\frac{4\underline{m}\phi}{(\phi-1+\underline{m})^2}$ is at least 1. This expression is decreasing in ϕ (again given $\underline{m} \leq 1$) and equals $\frac{4}{\underline{m}} > 1$ if $\phi = 1$, so there is $\phi^*(\underline{m}) > 1$ such that the inequality holds whenever $\phi \leq \phi^*(\underline{m})$.

We now turn to policy payoffs. For $\theta \in [\theta_0, \theta_1]$,

$$\begin{aligned} Q(\theta) &= [q(\theta)y_1 + (1-q(\theta))y_0]\frac{\delta p}{1-\delta p+\delta pq(\theta)\kappa(\theta)} \\ \implies Q_*(\theta) &= -\frac{q(\theta)y_1+(1-q(\theta))y_0}{(1-\delta p+\delta pq(\theta)\kappa(\theta))^2}\delta^2p^2q(\theta)\kappa(\theta)\left(\frac{q_*(\theta)}{q(\theta)}+\frac{\kappa_*(\theta)}{\kappa(\theta)}\right) \\ &\quad -\delta p\frac{q_*(\theta)(y_0-y_1)}{1-\delta p+\delta pq(\theta)\kappa(\theta)} + \delta p\frac{q(\theta)y_{1*}+(1-q(\theta))y_{0*}}{1-\delta p+\delta pq(\theta)\kappa(\theta)}. \end{aligned}$$

f is bounded by assumption and $q, \kappa \leq 1$. Also $|Q(\theta)|, |y_0|, |y_1| \leq \frac{I^2}{1-\delta p}$. It remains to bound y_{0*} and y_{1*}. Using that $y_0 = S(0,\theta)$, $y_1 = \int_0^1 S(\theta',\theta)f(\theta')d\theta$, and $S(\theta',\theta) = \mu\left(-\frac{U(\theta)-U(\theta')}{\lambda} + 2\sqrt{\frac{U(\theta)-U(\theta')}{\lambda}}I - I^2\right)$ for any $\theta' \leq \theta$ (see Lemma 1), we obtain:

$$y_0 = \mu\left(-I^2 + 2I\sqrt{\frac{\tilde{U}(\theta_0)}{\lambda}} - \frac{\tilde{U}(\theta_0)}{\lambda}\right), \quad y_{0*} = \mu U'(\theta_0)\left[I\sqrt{\frac{1}{\tilde{U}(\theta_0)\lambda}} - \frac{1}{\lambda}\right],$$

$$y_1 = \mu\int_0^{\theta_0}\left(-I^2 + 2I\sqrt{\frac{\tilde{U}(\theta_0)-\tilde{U}(\theta)}{\lambda}} - \frac{\tilde{U}(\theta_0)-\tilde{U}(\theta)}{\lambda}\right)f(\theta)d\theta,$$

$$y_{1*} = \mu U'(\theta_0)\int_0^{\theta_0}\left(I\sqrt{\frac{1}{(\tilde{U}(\theta_0)-\tilde{U}(\theta))\lambda}} - \frac{1}{\lambda}\right)f(\theta)d\theta.$$

Now, using that $1 \leq U'(\theta) \leq \frac{1}{1-\delta p}$ for $\theta < \theta_0$, and denoting $\max f = \overline{f}$,

$$-\frac{\mu}{\lambda(1-\delta p)} \leq y_{1*} \leq \frac{\mu}{1-\delta p}\int_0^{\theta_0} I\sqrt{\frac{1}{(\theta_0-\theta)\lambda}}\overline{f}d\theta = \frac{\mu}{1-\delta p}\frac{2I\overline{f}\sqrt{\theta_0}}{\sqrt{\lambda}} \leq \frac{\mu}{1-\delta p}\frac{2I\overline{f}}{\sqrt{\lambda}}$$

$$-\frac{\mu}{\lambda(1-\delta p)} \leq y_{0*} \leq \frac{\mu}{1-\delta p}\frac{I}{\sqrt{\theta_0}\sqrt{\lambda}}.$$

$Q_*(\theta)$ for $\theta > \theta_1$ and $S_*(\theta_0,\theta)$ for $\theta > \theta_1$ can be bounded with similar arguments. All of our bounds are uniform in θ_0 except for the upper bound on y_{0*}, which is proportional to $\frac{1}{\sqrt{\theta_0}}$ and explodes as $\theta_0 \to 0$.

We finish our proof of equilibrium uniqueness in this region with the following argument. If $\gamma = 0$, given values of all other parameters, there is a unique equilibrium whenever $\phi < \phi^*(\underline{m})$. Let θ^* be the value of θ_0 in this equilibrium. If $\theta^* > 0$, the *marginal* policy payoffs that show up in $\frac{\partial\overline{T}}{\partial\theta_0}$ are bounded in a neighborhood of θ^*, and the *total* policy payoffs in $\overline{T}(\theta)$ are bounded everywhere (i.e., $\overline{T}$ may be nonmonotonic near 0, but this is far from θ^*, where $\overline{T}$ crosses c). If $\theta^* = 0$, then $\overline{T}(\theta^*) < c$ for any $\gamma > 0$ (because policy payoffs are negative), so the equilibrium is type 3, which does not have these issues.

Next, suppose the equilibrium is type 1. Then

$$\overline{T}_{\theta_0} = \frac{1}{2}\int_{\theta_0}^{1} \left(R(\theta) + \gamma Q(\theta) + \gamma S(\theta_0, \theta)\right) f(\theta)\, d\theta$$

$$\frac{\partial \overline{T}_{\theta_0}}{\partial \theta_0} = \frac{1}{2}\int_{\theta_0}^{1} \left(R_*(\theta) + \gamma Q_*(\theta) + \gamma S_*(\theta_0, \theta)\right) f(\theta)\, d\theta - \frac{1}{2}(R(\theta_0) + \gamma Q(\theta_0) + \gamma S(\theta_0, \theta_0))f(\theta_0)$$

Bounding the policy payoffs in this case is not hard (the issues that arise as θ_0 approaches zero do not apply here). We then have to show

$$\int_{\theta_0}^{1} R_*(\theta) f(\theta)\, d\theta < R(\theta_0) f(\theta_0).$$

We now have

$$q_*(\theta) \geq \frac{1 - q(1)}{1 - \theta_0},\ \kappa(\theta) = \frac{1 - F(\theta_0)}{2} \Longrightarrow \kappa_*(\theta) = -\frac{1}{2} f(\theta_0),\ -\frac{\kappa_*(\theta)}{\kappa(\theta)} \leq \frac{f(\theta_0)}{1 - F(\theta_0)}.$$

(The bound on $q_*(\theta)$ uses the fact that, when $\theta_1 = 1$, $|q'(\theta)|$ is decreasing in θ_0—see Proposition 5.) Arguing as before, it is enough to show

$$\frac{b\delta pq\kappa}{(1 - \delta p + \delta pq\kappa)^2}\left(1 - \frac{1 - q(1)}{\phi q}\right) \frac{f(\theta_0)}{1 - F(\theta_0)}(1 - F(\theta_0)) < \frac{bf(\theta_0)}{1 - \delta p + \delta p\kappa}$$

$$\Longleftrightarrow \frac{\delta pq\kappa}{(1 - \delta p + \delta pq\kappa)^2}\left(1 - \frac{1 - q(1)}{\phi q}\right) < \frac{1}{1 - \delta p + \delta p\kappa}$$

subject to $q \geq q(1)$.

Again $\frac{\delta pq\kappa}{(1-\delta p+\delta pq\kappa)^2}\left(1 - \frac{1-q(1)}{\phi q}\right)$ is single peaked in q with a maximum at $q^* = \frac{1-\delta p}{\delta pk} + \frac{2(1-q(1))}{\phi}$. There are three cases. If $q^* > 1$, then we need

$$\frac{\delta p\kappa}{(1 - \delta p + \delta p\kappa)^2}\left(1 - \frac{1 - q(1)}{\phi}\right) < \frac{1}{1 - \delta p + \delta p\kappa},$$

which always holds. If $1 > q^* > q(1)$, then $q^* > \frac{\frac{1-\delta p}{\delta pk}+\frac{2}{\phi}}{1+\frac{2}{\phi}} > q(1)$, and

$$\frac{\delta p q^* \kappa}{(1-\delta p+\delta p q^* \kappa)^2}\left(1-\frac{1-q(1)}{\phi}\right) =$$
$$=\frac{1}{4\left(1-\delta p+\frac{\delta p\kappa}{\phi}(1-q(1))\right)} < \frac{1}{4\left(1-\delta p+\frac{\delta p\kappa}{\phi}\left(1-\frac{\frac{1-\delta p}{\delta pk}+\frac{2}{\phi}}{1+\frac{2}{\phi}}\right)\right)} =$$
$$=\frac{1}{4\left((1-\delta p)\left(1-\frac{1}{\phi+2}\right)+\delta p\kappa\frac{1}{\phi+2}\right)}$$

which is always smaller than $\frac{1}{1-\delta p+\delta p\kappa}$ if $\phi < 2$.

Finally, if $q(1) > q^*$, then we need

$$\frac{\delta p q(1)\kappa}{(1-\delta p+\delta p q(1)\kappa)^2}\left(1-\frac{1-q(1)}{\phi q(1)}\right) < \frac{1}{1-\delta p+\delta p\kappa}$$
$$\iff \frac{\delta p q(1)\kappa}{(1-\delta p+\delta p q(1)\kappa)^2}\frac{\phi+1}{\phi}\left(1-\frac{1}{(\phi+1)q(1)}\right) < \frac{1}{1-\delta p+\delta p\kappa}$$

The value of $q(1)$ that maximizes the left-hand side is $\frac{1-\delta p}{\delta p\kappa}+\frac{2}{\phi+1}$, and the maximized value of the left-hand side is $\frac{\phi+1}{\phi}\frac{1}{4(1-\delta p)+\frac{4}{\phi+1}\delta p\kappa}$. This expression is decreasing in ϕ and always less than $\frac{1}{1-\delta p+\delta p\kappa}$ for $\phi = 1$, so there is again a threshold $\phi^* > 1$ such that the inequality holds if $\phi < \phi^*$.

The case of a type 3 equilibrium is the simplest one. The policy payoffs can be handled as before. For office rents, we need to show that

$$\int_0^{\theta_1} R_*(\theta)f(\theta)d\theta < R(\theta_1)f(\theta_1),$$

where $R_*(\theta)$ now represents $\frac{\partial R(\theta)}{\partial\theta_1}$. (We can't use θ_0 as the parameter since it is 0, and θ_1 is more convenient than $q(0)$.) We can, as before, show that $q_*(\theta) > 0$, and $\kappa(\theta) = 1-\frac{F(\theta_1)}{2}$, so $\kappa_*(\theta) = -\frac{f(\theta_1)}{2}$ and $-\frac{\kappa_*(\theta)}{\kappa(\theta)} = \frac{f(\theta_1)}{2-F(\theta_1)} < f(\theta_1)$. Then

$$R_*(\theta) = \frac{b\delta p q(\theta)\kappa(\theta)}{(1-\delta p+\delta p q(\theta)\kappa(\theta))^2}\left(-\frac{q_*(\theta)}{q(\theta)}-\frac{\kappa_*(\theta)}{\kappa(\theta)}\right) < \frac{b}{1-\delta p}f(\theta_1)$$
$$\implies \int_0^{\theta_1} R_*(\theta)f(\theta)d\theta < \frac{b}{1-\delta p}f(\theta_1)F(\theta_1) < \frac{b}{1-\delta p}f(\theta_1) = R(\theta_1)f(\theta_1).$$

□

Proof of Corollary 1. Parts (i) and (ii) are immediate consequences of Proposition 6. For part (iii), note that $U_1(\theta) = \theta + \delta V$ and $U_0(\theta) = \theta + \delta V_1(\theta)$, so $U_1'(\theta) = 1$ and $U_0'(\theta) = 1 + \delta V_1'(\theta)$. For $\theta < \theta_0$, $V_1(\theta) = \mu E(\min(U_1(\theta), U_0(\theta'))|\theta' \sim F) + (1 - \mu)E(\max(U_1(\theta), U_0(\theta'))|\theta' \sim F)$. $U_1'(\theta) = 1$then implies $V_1'(\theta)$, so $U_0'(\theta) > U_1'(\theta)$. For $\theta > \theta_0$, $V_1(\theta) = \mu \min(U_1(\theta), U_0(0)) + (1 - \mu) \max(U_1(\theta), U_0(0))$. $U_1'(\theta) = 1$ again implies $V_1'(\theta) > 0$ and $U_0'(\theta) > U_1'(\theta)$ unless $\mu = 1$, in which case $V_1(\theta) = U_0(0)$ and $U_0'(\theta) = 1 = U_1'(\theta)$.

For part (iv), if $\mu = 1$, we will argue that $U_0(0) < U_1(\theta)$ for all θ. This follows since $U_0(0) = \delta V_1(0) \leq V_1(0) = V_1(0) = E(\min(U_1(0), U_0(\theta')|\theta' \sim F) \leq U_1(0)$, and U_1 is increasing. (Note that V, U_0, U_1, $V_1 \geq 0$, since electing the weaker candidate always gives a nonnegative flow payoff.) Hence $V_1(\theta) = U_0(0)$ for $\theta > \theta_0$. It also follows that $U_0(0) \leq V$, as $U_0(0) \leq U_1(0) = \delta V$. Hence $U_1(\theta) \geq U_0(\theta)$ for $\theta > \theta_0$, as $V \geq V_1(\theta) = U_0(0)$ for $\theta > \theta_0$. Both inequalities are strict unless $V = 0$, which happens iff $q_0 = 0$. This argument also goes through for μ in a neighborhood of 1.

There are two degenerate cases. If U^* is above $U_1(1)$, there always is competition. This is possible in under classic limits if c is low enough, since in an open election there is always a positive probability of winning, and in a closed election the challenger can always defeat the incumbent with non-negligible probability, since $U_0(1) > U_1(1)$ (see part (iv) of Proposition 2). If U^* is below $U_1(0)$, there never is competition in a closed election. This is possible if c is high enough. □