The New Keynesian Model without Capital

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Objective

- Review the foundations of the basic New Keynesian model without capital.
- Derive the Equilibrium Conditions.
  - Small number of equations and a small number of variables, which summarize everything about the model (optimization, market clearing, gov't policy, etc.).
- Study some properties of the model.
  - Do this using Dynare and 'pencil and paper' methods.

Outline

- The model:
  - Individual agents: their objectives, what they take as given, what they choose.
    - Households, final good firms, intermediate good firms.
  - Economy-wide restrictions:
    - Market clearing conditions.
    - Relationship between aggregate output and aggregate factors of production, aggregate price level and individual prices.
- Properties of Equilibrium:
  - *Classical Dichotomy* - when prices flexible monetary policy irrelevant for real variables.
  - Monetary policy *essential* to determination of all variables when prices sticky.

Households

- There are many identical households.

- The problem of the typical ('representative') household:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right),$$

$$\text{s.t. } P_t C_t + B_{t+1}$$

$$\leq W_t N_t + R_{t-1} B_t$$

$$+\text{Profits net of government transfers and taxes}_t.$$

- Here, B_t denotes beginning-of-period t stock of bonds held by the household.

Households...

- Law of motion of the shock to preferences:

$$\tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}$$

- Preference shock is in the model for pedagogic purposes only,
  - Not an interesting shock from an empirical point of view.

- The household first order conditions:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

$$e^{\tau_t} C_t N_t^{\varphi} = \frac{W_t}{P_t}.$$

- All equations are derived by expressing the household problem in Lagrangian form, substituting out the multiplier on budget constraint and rearranging.

Consumption Smoothing

- Later, we'll see that *consumption smoothing* is an important principle for understanding the role of monetary policy in the New Keynesian model.
- Consumption smoothing is a characteristic of households' consumption decision when they expect a change in income and the interest rate is *not* expected to change.
- Peoples' current period consumption increases by the amount that can, according to their budget constraint, be maintained indefinitely.

Consumption Smoothing: Example

- Problem:

$$
\begin{aligned}
& max_{c_1, c_2} log\left(c_1\right) + \beta log\left(c_2\right) \\
\text{subject to :} \quad & c_1 + B_1 \leq y_1 + rB_0 \\
& c_2 \leq rB_1 + y_{2.}
\end{aligned}
$$

where y_1 *and* y_2 *are (given) income*

Consumption Smoothing: Example

- After imposing equality (optimality) and substituting out for B_1,

$$c_1 + \frac{c_2}{r} = y_1 + \frac{y_2}{r} + rB_0,$$

- FONC for B_1

$$\frac{1}{c_1} = \beta r \frac{1}{c_2}$$

- Suppose $\beta r = 1$:

$$c_1 = \frac{y_1 + \frac{y_2}{r}}{1 + \frac{1}{r}} + \frac{r}{1 + \frac{1}{r}} B_0$$

Consumption Smoothing: Example, cnt'd

- Solution to the problem:

$$c_1 = \frac{y_1 + \frac{y_2}{r}}{1 + \frac{1}{r}} + \frac{r}{1 + \frac{1}{r}} B_0.$$

- Consider three polar cases:
  - *temporary change in income*: $\Delta y_1 > 0$ and $\Delta y_2 = 0 \implies \Delta c_1 = \Delta c_2 = \frac{\Delta y_1}{1+\frac{1}{r}}$
  - *permanent change in income*: $\Delta y_1 = \Delta y_2 > 0 \implies \Delta c_1 = \Delta c_2 = \Delta y_1$
  - *future change in income*: $\Delta y_1 = 0$ and $\Delta y_2 > 0 \implies \Delta c_1 = \Delta c_2 = \frac{\frac{\Delta y_2}{r}}{1+\frac{1}{r}}$

- Common feature of each example:
  - When income rises, then - assuming r does not change - c_1 increases by an amount that can be maintained into the second period: **consumption smoothing**.

Production

- A homogeneous final good is produced using the following (Dixit-Stiglitz) production function:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}} .$$

- Each intermediate good, $Y_{i,t}$, is produced by a monopolist using the following production function:

$$Y_{i,t} = e^{a_t} N_{i,t}, \qquad a_t \sim \text{exogenous shock to technology.}$$

Final Good Producers

- Competitive firms maximize profits

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} dj,$$

subject to P_t, $P_{i,t}$ given, all $i \in [0,1]$, and the technology:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} .$$

Foncs:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon} \rightarrow \overbrace{P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}}^{\text{"aggregate price index"}}$$

$$Y_{i,t} = e^{a_t} N_{i,t}, \qquad a_t \sim \text{exogenous shock to technology.}$$

Aggregate Price Index

- To derive a price index, define nominal output as the sum of prices times quantities:

$$P_t Y_t = \int_0^1 P_{it} Y_{it} di$$

- Plugging in the demand for each variety, we have

$$P_t Y_t = \int_0^1 P_i^{1-\varepsilon} P_t^{\varepsilon} Y_t di$$

- Solving for P_t :

$$\overbrace{P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}}^{\text{"aggregate price index"}}$$

Intermediate Good Producers

- The i^{th} intermediate good is produced by a monopolist.

- Demand curve for i^{th} monopolist:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon}.$$

- Production function:

$$Y_{i,t} = e^{a_t} N_{i,t}, \ a_t \text{ ~ exogenous shock to technology.}$$

- Calvo Price-Setting Friction: rotemberg

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases}.$$

Marginal Cost of Production

- An important input into the monopolist's problem is its marginal cost:

$$MC_t = \frac{dCost}{dOutput} = \frac{\frac{dCost}{dWorker}}{\frac{dOutput}{dWorker}} = \frac{(1-\nu)\,W_t}{e^{a_t}}$$
$$= \frac{(1-\nu)\,e^{\tau_t}\,C_t N_t^{\varphi}}{e^{a_t}} P_t$$

- The tax rate, ν, represents a subsidy to hiring labor, financed by a lump-sum government tax on households.

- Firm's job sets prices whenever it has the opportunity to do so.

- Firm must always satisfy demand at its posted price.

Present Discounted Value of Intermediate Good Revenues

- i^{th} intermediate good firm's objective:

$$E_t^i \sum_{j=0}^{\infty} \beta^j \ \upsilon_{t+j} \overbrace{\left[\overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} s_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}^{\text{period } t+j \text{ profits sent to household}}$$

υ_{t+j} - Lagrange multiplier on household budget constraint

- Here, E_t^i denotes the firm's expectation over future variables, including the probability that the firm gets to reset its price at future dates.

Firms that Can Change Price at t

- Let $\widetilde{P}_t$ denote the price set by the $1-\theta$ firms who optimize at time t.

- Expected value of future profits sum of two parts:
  - future states in which price is still $\widetilde{P}_t$, so $\widetilde{P}_t$ matters.
  - future states in which the price is not $\widetilde{P}_t$, so $\widetilde{P}_t$ is irrelevant.
- That is,

$$E_t^i \sum_{j=0}^{\infty} \beta^j \upsilon_{t+j} \left[P_{i,t+j} Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right]$$

$$= \overbrace{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right]}^{Z_t} + X_t,$$

  - Z_t is the present value of future profits over all future states in which the firm's price is $\tilde{P}_t$.
  - X_t is the present value over all other states, so $dX_t / d\widetilde{P}_t = 0$.

Decision By Firm that Can Change Its Price

- Substitute out demand curve, $Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\varepsilon}$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon} \right].$$

- Differentiate with respect to $\tilde{P}_t$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[(1-\varepsilon) \left(\tilde{P}_t \right)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1} \right] = 0,$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

  - When $\theta = 0$, get standard result - price is fixed markup over marginal cost.

Decision By Firm that Can Change Its Price

- Substitute out the multiplier:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \overbrace{\frac{u'(C_{t+j})}{P_{t+j}}}^{=\upsilon_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

  - Using assumed log-form of utility,

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t},\ \bar{\pi}_t \equiv \frac{P_t}{P_{t-1}},\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}},\ j \geq 1 \\ 1,\ j = 0. \end{cases},$$

$$\text{'recursive property'}:\ X_{t,j} = X_{t+1,j-1}\frac{1}{\bar{\pi}_{t+1}},\ j > 0$$

Decision By Firm that Can Change Its Price

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0$$

- Solving for $\tilde{p}_t$, we conclude that prices are set as follows:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} \frac{Y_{t+j}}{C_{t+j}} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}.$$

- All firms who reset, face same MC, so they all choose the same reset price.

- Need convenient expressions for K_t, F_t.

Decision By Firm that Can Change Its Price

- After LOTS of algebra, we obtain

$$\tilde{P}_t = E_t \sum_{j=0}^{\infty} \omega_{t,j} \frac{\varepsilon}{\varepsilon - 1} MC_{t+j}, \ E_t \sum_{j=0}^{\infty} \omega_{t,j} = 1$$

where MC_t is nominal marginal cost, and

$$\omega_{t,j} = \frac{(\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{1-\varepsilon}}{F_t}, \ F_t = \frac{Y_t}{C_t} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{1-\varepsilon} F_{t+1} \ (2)$$

$$X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}}, \ j \geq 1 \\ 1, \ j = 0. \end{cases}, \ \bar{\pi}_t \equiv \frac{P_t}{P_{t-1}}.$$

- $\tilde{P}_t$ is a weighted average of current and future marginal costs, where weights depend on expected future demand and inflation.
- Note that $\theta = 0$ implies $\omega_{t,0} = 1, \omega_{t,j} = 0$, for $j > 0$, so

$$\tilde{P}_t = \frac{\varepsilon}{\varepsilon - 1} MC_t$$

Scaling the Marginal Price Setter's Price

- Let
$$s_t \equiv \frac{MC_t}{P_t} = \frac{(1-\nu)\frac{W_t}{P_t}}{e^{a_t}} = (1-\nu)\, e^{\tau_t} C_t N_t^{\varphi} / e^{a_t}..$$
- Denoting $p_t \equiv \tilde{P}_t / P_t$:
$$\tilde{p}_t = \frac{K_t}{F_t}$$
where
$$K_t = E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}$$
$$= \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} \frac{(1-\nu)\, e^{\tau_t} C_t N_t^{\varphi}}{e^{a_t}} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} K_{t+1} \ (1)$$

▸ derive

Calibrating the Calvo Parameter ϕ

- The bigger is ϕ,the stickier are prices
  - The bigger will be the effects of nominal shocks and the more distorted will be the response of variables to real shocks

- There exists a close mapping between ϕ and the expected duration of a price change.

Calibrating the Calvo Parameter ϕ

- Consider a firm that gets to update its price in a period.

- Probability of getting to adjust its price one period from now is $(1-\phi)$.
- Probability of adjusting in two periods is $\phi(1-\phi)$.
- Probability of adjusting in three periods is $\phi^2(1-\phi)$

$$\text{Expected duration}=(1-\phi)\sum_{j=1}^{\infty}\phi^j j$$

$$S = 1 + 2\phi + 3\phi^2 + 4\phi^3 + ...$$

$$\phi S = \phi + 2\phi^2 + 3\phi^3 ...$$

$$(1-\phi)S = 1 + \phi + \phi^2 + ...$$

$$(1-\phi)S = \frac{1}{1-\phi}$$

Moving On to Aggregate Restrictions

- Link between aggregate price level, P_t, and $P_{i,t}$, $i \in [0,1]$.
  - Potentially complicated because there are MANY prices, $P_{i,t}$, $i \in [0,1]$.
  - Important: *Calvo result*.

- Link between aggregate output, Y_t, and aggregate employment, N_t.
  - Complicated, because Y_t depends not just on N_t but also on how employment is allocated across sectors.
  - Important: *Tack Yun distortion*.

- Market clearing conditions.
  - Bond market clearing.
  - Labor and goods market clearing.

Aggregate Price Index: Calvo Result

- Trick: rewrite the aggregate price index.
  - let $p \in (0, \infty)$ the set of logically possible prices for intermediate good producers.
  - let $g_t(p) \geq 0$ denote the measure (e.g., 'number') of producers that have price, p, in t
  - let $g_{t-1,t}(p) \geq 0$, denote the measure of producers that had price, p, in $t-1$ and could not re-optimize in t
  - Then,

$$P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^\infty g_t(p) \, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

- Note:

$$P_t = \left((1-\theta) \tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t}(p) \, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

Aggregate Price Index: Calvo Result

- Calvo randomization assumption:

$$\overbrace{g_{t-1,t}(p)}^{\text{measure of firms that had price, } p\text{, in } t-1 \text{ and could not change}}$$

$$= \theta \times \overbrace{g_{t-1}(p)}^{\text{measure of firms that had price } p \text{ in } t-1}$$

Aggregate Price Index: Calvo Result

- Using $g_{t-1,t}(p) = \theta g_{t-1}(p)$:

$=$

$$P_t = \left((1-\theta)\tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t}(p)\, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}$$

$=$

$$P_t = \left((1-\theta)\tilde{P}_t^{1-\varepsilon} + \theta \overbrace{\int_0^\infty g_{t-1}(p)\, p^{(1-\varepsilon)} dp}^{=P_{t-1}^{1-\varepsilon}} \right)^{\frac{1}{1-\varepsilon}}$$

This is the **Calvo result:**

$$P_t = \left((1-\theta)\tilde{P}_t^{1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$$

- Simple!: Only two variables: $\tilde{P}_t$ and P_{t-1}.

Inflation and Marginal Price Setter

- Calvo result: derive

$$P_t = \left((1-\theta)\,\tilde{P}_t^{1-\varepsilon} + \theta P_{t-1}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$$

- Divide by P_t :

$$1 = \left((1-\theta)\,\tilde{p}_t^{1-\varepsilon} + \theta \left(\frac{1}{\bar{\pi}_t} \right)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}$$

$\tilde{p}_t$ is relative price of marginal price setter.

- Then,

$$\tilde{p}_t = \left[\frac{1-\theta\,(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}$$

Tack Yun Distortion (JME1996)

- Define Y_t^*:

$$
\begin{aligned}
Y_t^* &\equiv \int_0^1 Y_{i,t} di \qquad \left(= \int_0^1 e^{a_t} N_{i,t} di = e^{a_t} N_t \right) \\
&\overbrace{=}^{\text{demand curve}} \quad Y_t \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di = Y_t P_t^{\varepsilon} \int_0^1 \left(P_{i,t} \right)^{-\varepsilon} di \\
&= Y_t P_t^{\varepsilon} \left(P_t^* \right)^{-\varepsilon}.
\end{aligned}
$$

$$
P_t^* = \left(\int_0^1 P_{i,t}^{-\varepsilon} di \right)^{\frac{-1}{\varepsilon}}
$$

So,

$$
Y_t = p_t^* Y_t^*, \; p_t^* = \left(\frac{P_t^*}{P_t} \right)^{\varepsilon} = \text{‘Tack Yun Distortion’}
$$

- Then:

$$
Y_t = p_t^* e^{a_t} N_t.
$$

Understanding the Tack Yun Distortion

- Relationship between aggregate inputs and outputs:

$$Y_t = p_t^* e^{a_t} N_t.$$

- Note that p_t^* is a function of the ratio of two averages (with different weights) of $P_{i,t}$, $i \in (0,1)$:

$$p_t^* = \left(\frac{P_t^*}{P_t} \right)^{\varepsilon},$$

where

$$P_t^* = \left(\int_0^1 P_{i,t}^{-\varepsilon} di \right)^{\frac{-1}{\varepsilon}}, \; P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}$$

- The Tack Yun distortion, p_t^*, is a measure of *dispersion* in prices, $P_{i,t}$, $i \in [0,1]$.

Understanding the Tack Yun Distortion

- Why is a ratio of two different weighted averages of prices a measure of dispersion?
  - Example

$$\frac{\bar{x}}{\tilde{x}} = \frac{\frac{1}{2}x_1 + \frac{1}{2}x_2}{\frac{1}{4}x_1 + \frac{3}{4}x_2} = \begin{cases} 1 & \text{if } x_1 = x_2 \\ \neq 1 & x_1 \neq x_2. \end{cases}$$

- But, the Tack Yun distortion is not the ratio of just *any* two different weighted averages.
  - In fact, simple Jensen's inequality argument shows: proof

$$p_t^* \leq 1, \text{ with equality iff } P_{i,t} = P_{j,t} \text{ for all } i, j.$$

  - Actually, it must be that proof

$$Y_t = \left[\overbrace{\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj}^{\text{average of concave functions of } Y_{i,t},\, i \in [0,1]} \right]^{\frac{\varepsilon}{\varepsilon-1}} \leq e^{a_t} N_t.$$

Law of Motion of Tack Yun Distortion

- We have, using the Calvo result:

$$P_t^* = \left[(1-\theta)\,\tilde{P}_t^{-\varepsilon} + \theta\left(P_{t-1}^*\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}$$

- Dividing by P_t:

$$p_t^* \equiv \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left[(1-\theta)\,\tilde{p}_t^{-\varepsilon} + \theta\frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1}$$

$$= \left((1-\theta)\left[\frac{1-\theta\left(\bar{\pi}_t\right)^{\varepsilon-1}}{1-\theta}\right]^{\frac{-\varepsilon}{1-\varepsilon}} + \theta\frac{\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right)^{-1} \qquad (4)$$

using the restriction between $\tilde{p}_t$ and aggregate inflation developed earlier.

Market Clearing

- We now summarize the market clearing conditions of the model.
- Labor, bond and goods markets.

Other Market Clearing Conditions

- Bond market clearing:

$$B_{t+1} = 0, \; t = 0, 1, 2, ...$$

- Labor market clearing:

$$\overbrace{N_t}^{\text{supply of labor}} = \overbrace{\int_0^1 N_{i,t} di}^{\text{demand for labor}}$$

- Goods market clearing:

$$\overbrace{C_t + G_t}^{\text{demand for final goods}} = \overbrace{Y_t}^{\text{supply of final goods}},$$

and, using relation between Y_t and N_t:

Equilibrium Conditions

- 6 equations in 7 unknowns: $C_t, p_t^*, F_t, K_t, N_t, R_t, \bar{\pi}_t$

$$K_t = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} \frac{(1-\nu)\, e^{\tau_t} C_t N_t^{\varphi}}{A_t} + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \text{ (1)}$$

$$F_t = \frac{Y_t}{C_t} + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \text{ (2)}, \quad \frac{K_t}{F_t} = \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \quad \text{(3)}$$

$$p_t^* = \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1} \quad \text{(4)}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \text{ (5)}, \quad C_t + G_t = p_t^* e^{a_t} N_t \text{ (6)}$$

- System underdetermined! Flexible price case, $\theta = 0$ is interesting.

Classical Dichotomy Under Flexible Prices

- *Classical Dichotomy*: when prices flexible, $\theta = 0$, then real variables determined.
  - Equations (2),(3) imply:
    $$F_t = K_t = \frac{Y_t}{C_t},$$
    which, combined with (1) implies
    $$\frac{\varepsilon(1-\nu)}{\varepsilon - 1} \times \overbrace{e^{\tau_t} C_t N_t^{\varphi}}^{\text{Marginal Cost of work}} = \overbrace{e^{a_t}}^{\text{marginal benefit of work}}$$
  - Expression (6) with $p_t^* = 1$ (since $\theta = 0$) is
    $$C_t + G_t = e^{a_t} N_t.$$
- Thus, we have two equations in two unknowns, N_t and C_t.

Classical Dichotomy: No Uncertainty

- Real interest rate, $R_t^* \equiv R_t / \bar{\pi}_{t+1}$, is determined:

$$R_t^* = \frac{\frac{1}{C_t}}{\beta \frac{1}{C_{t+1}}}.$$

- So, with $\theta = 0$, the following are determined:

$$R_t^*, C_t, N_t, \ t = 0, 1, 2, ...$$

- What about the nominal variables?
  - Suppose the central bank wants a given sequence of inflation rates, $\bar{\pi}_t$, $t = 0, 1, ...$.
  - Then it must produce the following sequence of interest rates:

$$R_t = \bar{\pi}_{t+1} R_t^*, \ t = 0, 1, 2, ...$$

How Does the CB Set the Interest Rate?

- When NK model leaves out money demand, modeler implicitly has in mind that money enters preferences additively separably:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} + \gamma log \left(\frac{M_{t+1}}{P_t} \right) \right),$$

$$\text{s.t. } P_t C_t + B_{t+1} + M_{t+1}$$

$$\leq W_t N_t + R_{t-1} B_t + M_t$$

where M_{t+1} is the beginning of period $t+1$ stock of money.

- Labor and bond first order conditions same as before.
- Money first order condition: proof

$$\frac{M_{t+1}}{P_t} = \left(\frac{R_t}{R_t - 1} \right) \gamma C_t,$$

which looks like a standard undergrad money demand equation.

Classical Dichotomy versus New Keynesian Model

- When $\theta = 0$, then the Classical Dichotomy occurs.
  - In this case, Central Bank cannot affect R_t^*, C_t, N_t.
  - Monetary policy simply affects the split in the real interest rate between nominal and real rates:
  $$R_t^* = \frac{R_t}{\bar{\pi}_{t+1}}.$$
  - For a careful treatment when there is uncertainty, see.
- When $\theta > 0$ (NK model) we can't pin down any of the 7 endogenous variables using the 6 available equations.
  - In this case, monetary policy matters for R_t^*, C_t, N_t.

Monetary Policy in New Keynesian Model

- Suppose $\theta > 0$, so that we're in the NK model and monetary policy matters.
- The standard assumption is that the monetary authority sets money growth to achieve an interest rate target, and that that target is a function of inflation:

$$R_t/R = (R_{t-1}/R)^{\alpha} \exp\{(1-\alpha)[\phi_\pi(\bar{\pi}_t - \bar{\pi}) + \phi_x x_t]\} \text{ (7)'},$$

  where x_t denotes the log deviation of actual output from target (more on this later).
- This is a *Taylor rule*, and it satisfies the *Taylor Principle* when $\phi_\pi > 1$.
- Smoothing parameter: α.
  - Bigger is α the more persistent are policy-induced changes in the interest rate.

Equilibrium Conditions of NK Model with Taylor Rule

$$K_t = \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} \frac{(1-\nu)\, e^{\tau_t} C_t N_t^{\varphi}}{A_t} + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} \text{ (1)}$$

$$F_t = \frac{Y_t}{C_t} + \beta\theta E_t \bar{\pi}_{t+1}^{\varepsilon-1} F_{t+1} \text{ (2)}, \quad \frac{K_t}{F_t} = \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}} \quad \text{(3)}$$

$$p_t^* = \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*}\right]^{-1} \quad \text{(4)}$$

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \text{ (5)}, \quad C_t + G_t = p_t^* e^{a_t} N_t \text{ (6)}$$

$$R_t/R = (R_{t-1}/R)^{\alpha} \exp\left\{(1-\alpha)\left[\phi_\pi(\bar{\pi}_t - \bar{\pi}) + \phi_x x_t\right]\right\} \text{ (7).}$$

Natural Equilibrium

- When $\theta = 0$, then

$$\frac{\varepsilon(1-\nu)}{\varepsilon-1} \times \overbrace{e^{\tau_t} C_t N_t^{\varphi}}^{\text{Marginal Cost of work}} = \overbrace{e^{a_t}}^{\text{marginal benefit of work}}$$

$$N_t = \left(e^{\tau_t}\right)^{-(+}$$

  so that we have a form of efficiency when ν is chosen to that $\varepsilon(1-\nu)/(\varepsilon-1) = 1$.

- In addition, we have allocative efficiency in the flexible price equilibrium. proof

-

$$N_t = e^{-\tau_t/(1+\varphi)}$$

  So, the flexible price equilibrium with the efficient setting of ν represents a natural benchmark for the New Keynesian model, the version of the model in which $\theta > 0$.

  - We call this the *Natural Equilibrium*

Natural Equilibrium

- With $G_t = 0$, equilibrium conditions for C_t and N_t:

$$\overbrace{e^{\tau_t} C_t N_t^{\varphi}}^{\text{Marginal Cost of work}} = \overbrace{e^{a_t}}^{\text{marginal benefit of work}}$$

Aggregate production relation: $C_t = e^{a_t} N_t$.

- Substituting,

$$e^{\tau_t} e^{a_t} N_t^{1+\varphi} = e^{a_t} \rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$

$$C_t = \exp\left(\mathrm{a}_t - \frac{\tau_t}{1+\varphi}\right)$$

$$R_t^* = \frac{\frac{1}{C_t}}{\beta E_t \frac{1}{C_{t+1}}} = \frac{1}{\beta E_t \frac{C_t}{C_{t+1}}} = \frac{1}{\beta E_t \exp\left(-\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1+\varphi}\right)}$$

Natural Equilibrium, cnt'd

- Natural rate of interest:

$$R_t^* = \frac{\frac{1}{C_t}}{\beta E_t \frac{1}{C_{t+1}}} = \frac{1}{\beta E_t \exp\left(-\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1+\varphi}\right)}$$

- Two models for a_t :

$$DS: \ \Delta a_{t+1} = \rho \Delta a_t + \varepsilon_{t+1}^a$$

$$TS: \ a_{t+1} = \rho a_t + \varepsilon_{t+1}^a$$

- Model for τ_t :

$$\tau_{t+1} = \lambda \tau_t + \varepsilon_{t+1}^\tau$$

Natural Equilibrium, cnt'd

- Suppose the ε_t's are Normal. Then,

$$E_t \exp\left(-\Delta a_{t+1} + \frac{\Delta \tau_{t+1}}{1+\varphi}\right) = \exp\left(-E_t \Delta a_{t+1} + E_t \frac{\Delta \tau_{t+1}}{1+\varphi} + \frac{1}{2} V\right)$$

$$V = \sigma_a^2 + \frac{\sigma_\tau^2}{(1+\varphi)^2}$$

- Then, with $r_t^* \equiv \log R_t^* : r_t^* = -\log \beta + E_t \Delta a_{t+1} - E_t \frac{\Delta \tau_{t+1}}{1+\varphi} - \frac{1}{2} V.$
- Useful: consider how natural rate responds to ε_t^a shocks under DS and TS models for a_t and how it responds to ε_t^τ shocks.
  - To understand how r_t^* responds, consider implications of consumption smoothing in absence of change in r_t^*.
  - Hint: in natural equilibrium, r_t^* steers the economy so that natural equilibrium paths for C_t and N_t are realized.

Conclusion

- Described NK model and derived equilibrium conditions.
  - The usual version of model represents monetary policy by a Taylor rule.
- When $\theta = 0$, so that prices are flexible, then monetary policy has no impact on C_t, N_t, R_t^*.
  - Changes in money growth move prices and wages in such a way that real wages do not change and employment and output don't change.

Conclusion...

- When prices are sticky, then a policy-induced reduction in the interest rate encourages more nominal spending.
  - The increased spending raises W_t more than P_t because of the sticky prices, thereby inducing the increased labor supply that firms need to meet the extra demand.
  - Firms are willing to produce more goods because:
    - The model assumes they *must* meet all demand at posted prices.
    - Firms make positive profits, so as long as the expansion is not too big they still make positive profits, even if not optimal.

Calvo versus Rotemberg

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    - Why? From the perspective of Calvo, the parameter ϕ is a function of the Calvo parameter, θ, and that is something that people have strong views about because it can be directly estimated from observed micro data.
  - Similarity of Calvo and Rotemberg only reflects that people have the habit of linearizing around zero inflation. Different in empirically plausible case of inflation. Difference is small in the simple model, but not in more plausible models.

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  - Another example (besides HANK) is the finding that the network nature of production (see Christiano et. al. 2011 and Christiano (2016)) matters for key properties of the New Keynesian model, including (i) the cost of inflation, (ii) the slope of the Phillips curve and (iii) the value of the Taylor Principle for stabilizing inflation. This is a growing area of research in macroeconomics.

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  - Another example is the importance of networks in financial firms for the possibility of financial crisis.
- Calvo's interesting implications for the distribution of prices in micro data has launched an enormous literature (see Eichenbaum, et al, Nakamura and Steinsson and many more papers). It is generating a picture of what kind of model is needed to eventually replace the Calvo model. Go Back

Firms that Can Change Price at t

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- Expected value of future profits sum of two parts:
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  - future states in which the price is not $\widetilde{P}_t$, so $\widetilde{P}_t$ is irrelevant.
- That is,

$$E_t^i \sum_{j=0}^{\infty} \beta^j \upsilon_{t+j} \left[P_{i,t+j} Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}\right]$$

$$= \overbrace{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}\right]}^{Z_t} + X_t,$$

where

- Z_t is the present value of future profits over all future states in which the firm's price is $\tilde{P}_t$.
- X_t is the present value over all other states, so $dX_t / d\widetilde{P}_t = 0$.

Decision By Firm that Can Change Its Price

- Substitute out demand curve:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \, \upsilon_{t+j} \left[\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j} \right]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \, \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon} \right].$$

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- Differentiate with respect to $\tilde{P}_t$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} \left[(1-\varepsilon) \left(\tilde{P}_t \right)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1} \right] = 0,$$

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$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon-1} s_{t+j}\right] = 0.$$

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$$\to E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

  - When $\theta = 0$, get standard result - price is fixed markup over marginal cost.

Decision By Firm that Can Change Its Price

- Substitute out the multiplier:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \overbrace{\frac{u'\left(C_{t+j}\right)}{P_{t+j}}}^{=\upsilon_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

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  - Using assumed log-form of utility,

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t \equiv \frac{\tilde{P}_t}{P_t},\ \bar{\pi}_t \equiv \frac{P_t}{P_{t-1}},\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}},\ j \geq 1 \\ 1,\ j=0. \end{cases},$$

$$\text{'recursive property'}:\ X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}},\ j > 0$$

Decision By Firm that Can Change Its Price

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j}\right] = 0$$

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- Solving for $\tilde{p}_t$, we conclude that prices are set as follows:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} \frac{Y_{t+j}}{C_{t+j}} (\beta\theta)^j \left(X_{t,j}\right)^{1-\varepsilon}} = \frac{K_t}{F_t}.$$

Decision By Firm that Can Change Its Price

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$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0$$

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- Need convenient expressions for K_t, F_t.

Decision By Firm that Can Change Its Price

- Recall,

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{1-\varepsilon}} = \frac{K_t}{F_t}$$

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The numerator has the following simple representation:

$$\begin{aligned} K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\ &= \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} \frac{(1-\nu) e^{\tau_t} C_t N_t^{\varphi}}{e^{a_t}} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1} \; (1), \end{aligned}$$

after using $s_t = (1-\nu) e^{\tau_t} C_t N_t^{\varphi} / e^{a_t}$.

Decision By Firm that Can Change Its Price

- Recall,

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

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- Similarly,

$$F_t = \frac{Y_t}{C_t} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{1-\varepsilon} F_{t+1} \text{ (2)}$$

Simplifying Numerator

$$K_t = E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}$$

Simplifying Numerator

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$$= \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} s_t$$

Simplifying Numerator

$$
\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \, \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} \frac{Y_t}{C_t} s_t \\
&\quad + \beta\theta E_t \sum_{j=1}^{\infty} \frac{Y_{t+j}}{C_{t+j}} (\beta\theta)^{j-1} \left(\overbrace{\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1}}^{=X_{t,j}, \text{ recursive property}} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j}
\end{aligned}
$$

Simplifying Numerator

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K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} \frac{Y_t}{C_t} s_t \\
&+ \beta\theta E_t \sum_{j=1}^{\infty} \frac{Y_{t+j}}{C_{t+j}} (\beta\theta)^{j-1} \left(\overbrace{\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1}}^{=X_{t,j}, \text{ recursive property}} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon-1} \frac{Y_t}{C_t} s_t + \mathcal{Z}_t,
\end{aligned}
$$

where

$$
\mathcal{Z}_t = \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \frac{Y_{t+j}}{C_{t+j}} \left(\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}
$$

Simplifying Numerator, cnt'd

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Simplifying Numerator, cnt'd

$$K_t = E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{Y_{t+j}}{C_{t+j}} \left(X_{t,j}\right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} = \frac{\varepsilon}{\varepsilon - 1} s_t + \mathcal{Z}_t$$

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Aggregate Price Index: Calvo Result

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  - Then,

$$P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} = \left(\int_0^\infty g_t(p)\, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

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- Note:

$$P_t = \left((1-\theta) \tilde{P}_t^{1-\varepsilon} + \int_0^\infty g_{t-1,t}(p) \, p^{(1-\varepsilon)} dp \right)^{\frac{1}{1-\varepsilon}}.$$

Aggregate Price Index: Calvo Result

- Calvo randomization assumption:

$$\overbrace{g_{t-1,t}(p)}^{\text{measure of firms that had price, } p\text{, in } t-1 \text{ and could not change}}$$

$$= \theta \times \overbrace{g_{t-1}(p)}^{\text{measure of firms that had price } p \text{ in } t-1}$$

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This is the **Calvo result:**

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- Wow, simple!: Only two variables: $\tilde{P}_t$ and P_{t-1}.

Go Back

Tack Yun Distortion

- Let $f(x) = x^4$, a convex function. Then,

$$\text{convexity: } \alpha x_1^4 + (1-\alpha)\, x_2^4 > (\alpha x_1 + (1-\alpha)\, x_2)^4$$

for $x_1 \neq x_2$, $0 < \alpha < 1$.

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  for $x_1 \neq x_2$, $0 < \alpha < 1$.

- Applying this idea:

$$\begin{aligned}
\text{convexity:} \quad & \int_0^1 \left(P_{i,t}^{(1-\varepsilon)}\right)^{\frac{\varepsilon}{\varepsilon-1}} di \geq \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
& \Longleftrightarrow \left(\int_0^1 P_{i,t}^{-\varepsilon} di\right) \geq \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\
& \Longleftrightarrow \overbrace{\left(\int_0^1 P_{i,t}^{-\varepsilon} di\right)^{\frac{-1}{\varepsilon}}}^{P_t^*} \leq \overbrace{\left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{1}{1-\varepsilon}}}^{P_t}
\end{aligned}$$

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Efficient Sectoral Allocation of Resources

- Consider the following problem

$$\max_{N_{i,t}, i \in [0,1]} \left[\int_0^1 \left(e^{a_t} N_{i,t} \right)^{\frac{\varepsilon - 1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon - 1}}$$

subject to a given amount of total employment:

$$N_t = \int_0^1 N_{i,t} di$$

- In Lagrangian form:

$$\max_{N_{i,t}, i \in [0,1]} \overbrace{\left[\int_0^1 \left(e^{a_t} N_{i,t} \right)^{\frac{\varepsilon - 1}{\varepsilon}} \, di \right]^{\frac{\varepsilon}{\varepsilon - 1}}}^{Y_t} + \lambda \left[N_t - \int_0^1 N_{i,t} \right],$$

where $\lambda \geq 0$ denotes the Lagrange multiplier.

Efficient Sectoral Allocation of Resources

- Lagrangian problem:

$$\max_{N_{i,t}, i\in[0,1]} \overbrace{\left[\int_0^1 \left(e^{a_t} N_{i,t}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}}}^{Y_t} + \lambda \left[N_t - \int_0^1 N_{i,t}\right],$$

  where $\lambda \geq 0$ denotes the Lagrange multiplier.

- First order necessary condition for optimization:

$$\left(\frac{Y_t}{N_{i,t}}\right)^{\frac{1}{\varepsilon}} \left(e^{a_t}\right)^{\frac{\varepsilon-1}{\varepsilon}} = \lambda \to N_{i,t} = N_{j,t} = N_t, \text{for all } i, j,$$

  so Y_t is as big as it possibly can be for given aggregate employment, when

$$Y_t = \left[\int_0^1 \left(e^{a_t} N_{i,t}\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = e^{a_t} N_t.$$

- Result is obvious because $\left(N_{i,t}\right)^{\frac{\varepsilon-1}{\varepsilon}}$ is strictly concave in $N_{i,t}$. Go Back

Money Demand

- The Lagrangian representation of the household problem is ($\lambda_t \geq 0$ is multiplier):

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log C_t - \exp\left(\tau_t\right) \frac{N_t^{1+\varphi}}{1+\varphi} + \gamma log \left(\frac{M_{t+1}}{P_t} \right) \right)$$
$$+\lambda_t \left(W_t N_t + R_{t-1} B_t + M_t - P_t C_t - B_{t+1} - M_{t+1} \right) \}.$$

- First order conditions:

$$C_t : u'\left(C_t\right) = \lambda_t P_t; \qquad B_{t+1} : \lambda_t = \beta E_t \lambda_{t+1} R_t$$
$$M_{t+1} : \lambda_t = \frac{\gamma}{M_{t+1}} + \beta E_t \lambda_{t+1}$$

  - Substitute out for $\beta E_t \lambda_{t+1}$ in M_{t+1} equation from B_{t+1} equation; then substitute out for λ_t from C_t equation and rearrange, to get

$$\frac{M_{t+1}}{P_t} = \left(\frac{R_t}{R_t - 1} \right) \gamma C_t.$$

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