Online Appendix to "Innovation-Led Transitions in Energy Supply" by Derek Lemoine

Appendix A derives some expressions useful elsewhere in the main text and appendix. Appendix B details the calibration and solution method. Appendix C reports further robustness checks. Appendix D contains a numerical example of the main analytic results. Appendix E contains additional theoretical results and proofs.

A Derivations of Useful Expression

The intermediate-good producer's first-order conditions for profit-maximization yield

$$p_{jXt} = (1-\kappa) p_{jt} \left[\frac{X_{jt}}{E_{jt}}\right]^{-1/\sigma} \quad \text{and} \quad p_{jRt} = \kappa\, p_{jt} \left[\frac{R_{jt}}{E_{jt}}\right]^{-1/\sigma}.$$

The relative incentive to research technologies for use in sector j increases in the relative price of the intermediates and decreases in the machine-intensity of sector j's output. Combining the first-order conditions, we have

$$p_{jXt} = \frac{1-\kappa}{\kappa} \left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma} p_{jRt}. \tag{A-1}$$

From equation (3) and the monopolist's markup, we have

$$x_{jit} = p_{jXt}^{\frac{1}{1-\alpha}} A_{jit}.$$

Substituting into the definition of X_{jt} and using the definition of A_{jt}, we have

$$X_{jt} = p_{jXt}^{\frac{\alpha}{1-\alpha}} A_{jt}. \tag{A-2}$$

Substituting into equation (A-1) and solving for equilibrium machine prices yields (7) and (11) in the main text.

The final-good producer's first-order condition for intermediates j is:

$$p_{jt} = \beta_E\, \nu_j \frac{Y_t}{\sum_{j=1}^N \nu_j E_{jt}^{\frac{\epsilon-1}{\epsilon}}} E_{jt}^{-\frac{1}{\epsilon}}. \tag{A-3}$$

Combining the intermediate-good producers' first-order condition for resources with the final-good producers' first-order conditions, we find demand for resource j:

$$p_{jRt} = \kappa\, \beta_E\, \nu_j \frac{Y_t^{\frac{\epsilon-1}{\epsilon}}}{\sum_{j=1}^N \nu_j E_{jt}^{\frac{\epsilon-1}{\epsilon}}} \left[\frac{E_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{E_{jt}}\right]^{-1/\sigma}.$$

Market-clearing for resource j then implies

$$\left[\frac{R_{jt}+\zeta_j Q_{jt}}{\Psi_j}\right]^{1/\psi_j} + \tau_t \xi_j = \kappa\, \beta_E\, \nu_j \frac{Y_t^{\frac{\epsilon-1}{\epsilon}}}{\sum_{j=1}^{N} \nu_j E_{jt}^{\frac{\epsilon-1}{\epsilon}}} \left[\frac{E_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{E_{jt}}\right]^{-1/\sigma}. \quad \text{(A-4)}$$

Demand for sector j's resources (for example) shifts inward as the share of those resources in the production of intermediate good j increases and also shifts inward as the share of intermediate good j in production of the final good increases. Equations (9) and (10) in the main text follow from dividing by the analogous equation for resource k.

Final-good producers' zero-profit condition is

$$Y_t = w_t L_t + r_t K_t + \sum_{j=1}^{N} p_{jt} E_{jt}, \quad \text{(A-5)}$$

where w_t is the wage paid to labor and r_t is the rental rate of capital. From final-good producers' first-order conditions, these are:

$$w_t = (1-\beta_K-\beta_E)\frac{Y_t}{L_t},$$
$$r_t = \beta_K \frac{Y_t}{K_t}.$$

B Calibration, Climate Change Modeling, and Solution Method

Table B-1 reports parameter values that are fixed across all specifications. Table B-2 reports market data used to calculate remaining parameters. I use a 10-year timestep and a policy horizon of 400 years. Let resources 1, 2, and 3 represent coal, natural gas, and renewables, respectively. I model coal and natural gas as depletable ($\zeta_1, \zeta_2 = 1$) and renewables as non-depletable ($\zeta_3 = 0$), as if renewable energy installations must be rebuilt every ten years. I set $Q_{j1} = 0$ for each j.

Begin by considering the supply of each type of resource. Marten et al. (2019) follow, among others, Haggerty et al. (2015) in using a long-run supply elasticity of 2.4 for coal. Marten et al. (2019) follow Arora (2014) in using a long-run supply elasticity of 0.5 for natural gas. Based on these, I use $\psi_1 = 2.4$ and $\psi_2 = 0.5$. The price-responsiveness of wind and solar derives from heterogeneity in resource sites' quality. Drawing in part on the work of others, Johnson et al. (2017) describe the supply of power from solar photovoltaics, concentrating solar power, onshore wind, and offshore wind available by region of the world and by resource quality. Costs are reported in dollars per unit power and resource potential is reported in units of energy. I convert costs to dollars per unit electrical energy by using the capacity factor reported for each resource quality bin in each region. This capacity factor adjusts for the fact that the power producible from renewable resources is not available throughout the day or throughout the year.[1] I then convert dollars per unit of electrical energy to dollars per units of energy in the resource by using the efficiency of each type of generator. From the Energy Information Administration's Annual Energy Review 2011, the efficiencies are 12% for solar photovoltaics, 21% for solar thermal, and 26% for wind. Aggregating across resource types and regions, I estimate $\psi_3 = 3.00$.

Next consider the elasticities of substitution in the final-good and intermediate-good production functions. Papageorgiou et al. (2017) estimate an elasticity of substitution between clean and dirty energy capacity of around 1.8, and Stern (2012) estimates an elasticity of substitution between coal and gas of 1.426, with a standard error of 0.387. Version 6 of the EPPA model uses an elasticity of substitution of 1.5 (Chen et al., 2016), and the ADAGE model uses an elasticity of substitution of 1.25 (Ross, 2009). In line with these, I fix $\epsilon = 1.8$.

Much literature has estimated the elasticity of substitution between energy and other inputs, but there is not much literature on the elasticity of substitution between resources and other inputs in the production of energy. I fix $\sigma = 0.4$ based on several lines of evidence. The most directly relevant calibration is the calibration of the energy supply sector's production function in Lemoine (2020). This calibration assigns an elasticity of substitution of 0.42 to the energy supply sector, based on estimates in Koesler and Schymura (2015) implemented by Marten and Garbaccio (2018).[2] As further evidence, some computable general equilibrium models of energy use assign an elasticity of substitution of 0.3 to nearly all sectors (see Turner, 2009), version 6 of the EPPA model uses an elasticity of substitution of 0.1 between resources and a capital-labor composite in electricity production (Chen et al., 2016), and ADAGE uses an elasticity of substitution of 0.6 between resources and a materials-value-added composite (Ross, 2009).

The inverse of α is the markup over marginal cost charged by machine producers. The average markup in 2016 was around 1.6 both in the U.S. (De Loecker et al., 2020) and globally (De Loecker and Eeckhout, 2018). I therefore fix $\alpha = 1/1.6 = 0.625$.

I fix $\kappa = 0.5$ and, following Golosov et al. (2014), fix $\beta_K = 0.3$ and $\beta_E = 0.04$. The theory showed that the critical share parameters were the ν_j, not the β or κ, and sensitivity tests support this conclusion.

Population L_t evolves as in DICE-2016R:

$$L_t = L_\infty \left(\frac{L_1}{L_\infty} \right)^{e^{-g_L (t-1)}},$$

where I convert the DICE-2016R equation into a differential equation (with time in decades) and solve it. The capital stock follows DICE-2016R. The initial value K_1 uses World Bank

[1] In my setting, capacity factors are implicitly captured by the calibration of the technology variables and the share parameters. Further, the elasticity of substitution σ can be interpreted as imposing a larger capacity factor penalty at higher penetrations.

[2] Koesler and Schymura (2015) use a nonlinear least squares estimator of a CES production function with a panel of countries. Marten and Garbaccio (2018) report those elasticities of substitution along with NAICS codes. Using these, Lemoine (2020) reports the average elasticity of substitution in a combined energy supply sector, weighted by gross output from the Bureau of Economic Analysis. The underlying elasticities are all similar.

GDP deflators to change the DICE-2016R initial value of 223 trillion year 2010 dollars to trillion year 2014 dollars. DICE uses an annual depreciation rate of 0.1. Converting to the decadal timestep yields

$$\delta = 1 - (1 - 0.1)^{10} = 0.6513.$$

The savings rate is endogenous in DICE-2016R but varies only between 0.24 and 0.26 over the 500-year horizon. I therefore fix $\Upsilon = 0.25$.

Now consider climate damages. The climate-economy integrated assessment literature typically models climate change as reducing total production. Letting T_t be surface temperature relative to 1900, we have, adapting Nordhaus (2017),

$$D(T_t) = 1 - d\,T_t^2$$

with $d = 0.00236$. The robustness check with higher damages increases d to 0.0228, from the mean of the calibration to Pindyck (2019) in Appendix C.1 of Lemoine (2021).

The evolution of total factor productivity A_{Yt} follows DICE-2016R (Nordhaus, 2017). It grows initially at 1.48% annually, with the growth rate declining at a rate of 0.5% annually:

$$A_{Y(t+1)} = A_{Yt} \prod_{s=0}^{9} \left[1 + (0.0148) e^{-0.005*(10*(t-1)+s)}\right].$$

Now consider the innovation function. Only the product of η and γ is important for improvements in technology over time. I therefore fix η at 1. Changes in γ do not affect the realized first-period technology, as the calibration of the A_{j0} (described below) adjusts to offset γ. Instead, changes in γ affect how rapidly technology evolves after the first period. Different values of γ can be interpreted as different step sizes for research advances, as different probabilities of research successes, and/or as different sizes for the population of researchers. I choose values of γ for the base case and the robustness check to generate a range of plausible futures, from relatively slow transitions in the base case ($\gamma = 1$) to relatively fast transitions in the "larger scientific advances" robustness check ($\gamma = 6$).

These two values for γ are consistent with the range of values implied by prior literature. In the calibration of Acemoglu et al. (2019), each scientist expects to advance technology by 11% over 5 years at the initial level of renewable scientists used here, implying a γ of around 0.2 for our 10-year timestep.[3] This value is close to the base case. Ignoring spillovers between sectors, Fried (2018) estimates that marginally increasing the share of scientists improves technology by 426% over 5 years at the initial level of renewable scientists used here, implying a γ of around 8 for our 10-year timestep.[4] This estimate is close to the "larger

[3]In their paper, scientists improve technology by a factor γ: $A_{t+1} = \gamma * A_t$. The probability of success is $\eta s_t^{-\psi}$ (in practice, they fix their $\zeta = 0$). So the expected breakthrough per scientist is, in their notation, $\eta s_t^{-\psi}(\gamma - 1)$. Using their values of $\eta = 0.598$, $\psi = 0.67$, and $\gamma = 1.07$ yields an expected breakthrough per scientist of 0.1105.

[4]The increase in next period's technology A_{t+1} due to a marginal increase in scientists s_t is, using equation (4) in Fried (2018) and adjusting for the population of scientists being 1 for me and 0.01 for Fried (2018), $dA_{t+1}/ds_t = \gamma\eta(100\rho)(s_t/(100\rho))^{\eta-1} A_t$ (in her notation). Her Table 1 gives $\rho = 0.01$, $\gamma = 3.96$, and $\eta = 0.79$, implying that $dA_{t+1}/ds_t = 4.26\, A_t$.

scientific advances" case. (Acemoglu et al. (2016) also estimate an innovation production function, but the mapping to the present paper is less clear.)

The remaining parameters are each A_{j0}, each Ψ_j, each ν_j, and A_{Y1}. I calibrate these ten parameters so that the first period's equilibrium Y_1, R_{j1}, s_{j1}, and p_{j1} match data (see Table B-2). World Bank data for global output from 2011–2015 imply that the value of the final good produced over the first ten-year timestep is 765 trillion year 2014 dollars. Initial resource consumption comes from summing consumption from 2011–2015, as reported in the BP Statistical Review of World Energy.[5] The International Energy Agency's World Energy Investment 2017 gives R&D spending on clean energy, on thermal generation, on coal production, and on oil and gas production. I divide thermal expenditures equally between coal and gas and attribute all oil and gas spending to gas. The first period must therefore have 12% of scientists working on coal, 65% of them working on gas, and 23% of them working on renewables. I calibrate each p_{j1} to be consistent with levelized costs from IEA (2015). Using the market discount rate of 7%, the median cost for coal is around 80 \$/MWh, for natural gas combined cycle plants is around 100 \$/MWh, and for solar photovoltaics is around 150 \$/MWh.[6]

The initial conditions on the R_{j1} and the s_{j1} and the guesses for the A_{j0} and the Ψ_j combine to yield the E_{j1}. I then use the ratio of the final-good firms' first-order conditions (see equation (A-3)) and the adding-up constraint on the share parameters to solve for the ν_j:

$$\begin{aligned}
\nu_3 =& \frac{1}{1+\frac{p_{2,1}}{p_{3,1}}\left(\frac{E_{2,1}}{E_{3,1}}\right)^{1/\epsilon}\left(1+\frac{p_{1,1}}{p_{2,1}}\left(\frac{E_{1,1}}{E_{2,1}}\right)^{1/\epsilon}\right)},\\
\nu_2 =& \frac{(1-\nu_3)}{1+\frac{p_{1,1}}{p_{2,1}}\left(\frac{E_{1,1}}{E_{2,1}}\right)^{1/\epsilon}},\\
\nu_1 =& 1-\nu_2-\nu_3.
\end{aligned}$$

For the initial conditions and any given guesses for the A_{j0} and Ψ_j, I set A_{Y1} to ensure that initial final good production matches Y_0.[7]

We now have the ν_j, A_{Y1}, the initial conditions, and the guesses for the A_{j0} and the Ψ_j. The levels of the intermediate goods' prices then follow from the final-good firms' first-order conditions. We require six conditions to pin down the A_{j0} and the Ψ_j. The zero-profit conditions for intermediate-good firms provide three conditions. The conditions on the initial research allocation provide two more conditions, as $\Pi_{1,1}/\Pi_{2,1} = 1$ and $\Pi_{1,1}/\Pi_{3,1} = 1$. These

[5]Natural gas and coal are used for electricity generation, heating, and industrial processes. I here abstract from these differences. To obtain the energetic content of renewables from the reported tonnes of oil equivalent, use BP's assumed thermal efficiency of 38% to obtain the equivalent electrical energy and then use a 20% generator efficiency to convert electrical energy to energy in the renewable resource.

[6]These costs have changed over time and can be affected by pollution regulations. Further, costs for heating applications may be different from costs for electricity. Experiments suggest that results are not highly sensitive to these choices.

[7]Note that A_{Y1} absorbs any unit conversions between energy, other inputs, and output.

two conditions can be thought of as defining $A_{2,0}$ and $A_{3,0}$ as functions of $A_{1,0}$ and the Ψ_j. Final-good firms' zero-profit condition (equation (A-5)) provides the remaining condition. This zero-profit condition uses the calibrated intermediate prices, not the price implied by the final-good firm's first-order conditions (which would trivially satisfy the zero-profit condition by Euler's Homogeneous Function Theorem). This last condition can be thought of as pinning down the level of the final-good firms' first-order conditions. I solve for the A_{j0} and the Ψ_j via an optimizer that seeks to satisfy the nonlinear equality constraints subject to the implied share parameters being positive and summing to a value less than 1.[8]

Resource use generates carbon dioxide emissions that eventually cause warming. Time t emissions are

$$e_t = \bar{e} + \sum_{j=1}^{3} \xi_j R_{jt}.$$

I calculate the emission intensities of coal and gas by dividing emissions for each resource from 2010–2014 (from the Carbon Dioxide Information Analysis Center) by resource consumption over the initial timestep. Other emissions $\bar{e}$ come from summing emissions from all other reported categories, which includes emissions from oil.[9] The renewable resource does not generate emissions ($\xi_3 = 0$).

The carbon cycle and climate model update those in DICE-2016R. The carbon cycle follows Joos et al. (2013, Table 5), as recommended and compiled by Dietz et al. (2021). It represents 4 reservoirs. The transfer coefficients are

$$\mathbf{\Lambda} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.9975 & 0 & 0 \\ 0 & 0 & 0.9730 & 0 \\ 0 & 0 & 0 & 0.7927 \end{bmatrix}^{10}.$$

Emissions flow to each reservoir as

$$\boldsymbol{b} = \begin{bmatrix} 0.2173 \\ 0.2240 \\ 0.2824 \\ 0.2763 \end{bmatrix}.$$

The year 2015 values (in Gt C) are

$$\boldsymbol{M}_1 = \begin{bmatrix} 588 + 139.1 \\ 90.2 \\ 29.2 \\ 4.2 \end{bmatrix},$$

[8]The optimizer succeeds in satisfying the constraints to within 1% for all parameterizations used in the paper.

[9]In the base case's laissez-faire scenario, eliminating the (mostly oil) emissions $\bar{e}$ reduces global temperature by around 0.4°C in 400 years. In fact, projected oil use is not so far from constant in the base scenario of IEA (2021) and only slowly increasing in the reference case of EIA (2021). Fixing $\bar{e}$ may slightly understate future warming under laissez-faire but overstate future warming under optimal policy.

where 588 Gt C is the stock of preindustrial carbon.

The parameters of the climate model come from Geoffroy et al. (2013), as compiled by Dietz et al. (2021). Additional atmospheric carbon increases radiative forcing to $F_t(\boldsymbol{M}_t)$, which measures additional energy at the earth's surface due to CO_2 in the atmosphere. Forcing is

$$F_t(\boldsymbol{M}_t) = f_{2x} \frac{\ln\left(\sum_{i=1}^{4} M_t^i / 588\right)}{\ln(2)},$$

where M_t^i indicates element i of $\boldsymbol{M}_t$ and f_{2x} is forcing induced by doubling CO_2. Surface temperature evolves as

$$T_{t+1} = T_t + \frac{10}{5}\phi_1 \left[F_{t+1}(\boldsymbol{M}_{t+1}) - \lambda T_t - \phi_3 \left(T_t - T_t^o\right)\right].$$

Ocean temperature evolves as

$$T_{t+1}^o = T_t^o + \frac{10}{5}\phi_4 \left[T_t - T_t^o\right].$$

Steady-state warming from doubled carbon dioxide ("climate sensitivity") is $f_{2x}/\lambda = 3.1$°C.

The base specification's preferences follow DICE-2016R. Period utility takes the familiar power form in per-capita consumption, with elasticity of intertemporal substitution EIS. Converting a 1.5% per year utility discount rate to a per-decade rate yields:

$$\rho = (1 + 0.015)^{10} - 1 = 0.1605.$$

The policymaker seeks to maximize utilitarian welfare W:

$$W = \sum_{t=1}^{\hat{t}} \frac{L_t}{(1+\rho)^{t-1}} \frac{(c_t/L_t)^{1-1/EIS}}{1 - 1/EIS}.$$

I set $\hat{t} = 40$, implying a 400-year horizon.

In contrast to the DICE climate-economy model, abatement cost emerges endogenously within a period from the tradeoffs between fuels and evolves endogenously as technologies and resource depletion change over time. In the initial period, a tax of 1 \$/tCO_2 reduces emissions by 16%, a tax of 10 \$/tCO_2 reduces emissions by 19%, a tax of 50 \$/tCO_2 reduces emissions by 25%, and a tax of 100 \$/tCO_2 reduces emissions by 30%. In DICE-2016R, emission reductions of 25% require a tax of 59 \$/tCO_2 and emission reductions of 30% require a tax of 80 \$/tCO_2. These values are in the same ballpark as the present model even though there is nothing in the calibration that requires them to be.

In the no-policy simulations, I solve each period's equilibrium by solving for the research allocation that maximizes scientists' expected profits (using equations (4) and (7)) within a search for the resource allocation that clears the market for resources (as in equation (A-4)). For any given resource allocation, I first check whether a case with all scientists in the renewable sector generates greater expected profits in that sector than in any other. If it does, the corner allocation is an equilibrium, but if it does not, I solve for the research allocation between the coal and gas sectors conditional on no scientists working in the renewable sector. If this allocation is also not an equilibrium, I solve for the equilibrium allocation between coal and gas conditional on any number of scientists in renewables and search for the number of scientists in working in renewables that equalizes that sector's expected profit to the expected profit from the other sectors that have nonzero scientists.

When working backwards in time from the year 2015, I solve for the time t equilibrium as follows. First, I guess a time t research allocation and a time t capital stock. Then I solve for the time t incoming technology implied by this allocation and the known time $t+1$ technology. The time t technology in turn implies a time t equilibrium, which includes the time t equilibrium research allocation and implies the time $t+1$ equilibrium capital stock. I search for the time t research allocation and time t capital stock at which the implied time t equilibrium research allocation matches the guess and the implied time $t+1$ equilibrium capital stock matches the known time $t+1$ capital stock. I simulate backwards with resource depletion fixed at its year 2015 value and with the realized history of global surface temperature from Zhang et al. (2021), adjusted slightly to ensure a match with T_1. I use the fitted population growth representation from Lemoine (2021, pgs A-7–A-8) to project population backwards, and I maintain the present calibration of growth in A_{Yt} when projecting total factor productivity backwards.

To optimize policy, I search for the policy and resource use trajectories that maximize welfare while clearing the market. This is a mathematical program with equilibrium constraints, which can be quite difficult to solve. There are 12 state variables: the capital stock, the two cumulative resource use trackers, the three average technology levels, the four carbon stocks, and the two temperature variables. The key to solving the model is to convert it to a form that allows for an analytic gradient. The trick is to have the solver guess not only the trajectories of the tax and/or research subsidy but also the trajectories of the 12 state variables and the three resource use trajectories, imposing constraints that the resource markets clear in every period (equation (A-4)) and the transition equations hold in every period.[10] For any given guess, I solve for each period's equilibrium allocation of scientists using equations (4) and (7)) and the algorithm described above. At a solution, the state variables' trajectories are as if the model were simulated forward with the chosen policies.[11]

This problem is still a difficult bilevel programming problem, with the lower level programming problem often finding corner solutions (i.e., it is often true that some sector has no scientists). But this form of the problem allows for the provision of analytic gradients for the objective and constraints: we essentially have a series of static problems once we condition

[10] In the cases with the research subsidy, the solver chooses the number of clean scientists directly, with the other two types of scientists clearing their markets conditional on this choice. The level of the subsidy is implied by the resulting research allocation. At a corner allocation with all scientists in the renewable sector, I define the subsidy as the smallest value compatible with the corner allocation.

[11] In effect, the policymaker gets to simultaneously choose the trajectories of all states and all policy controls subject to constraints imposed by the market and by physical laws. If I did not impose the market constraints, then I would have the social planner's problem.

on the full set of state variables, because the partial derivative of the objective (and also of the constraints) with respect to any element of the solver's guess needs to account only for effects on same-period payoffs and on the same-period transition equations (observing that the partial derivative holds later states fixed because they are also elements of the solver's guess). Within those analytic gradients, I obtain the derivatives of equilibrium scientists by applying the implicit function theorem to the system of equations defined by equalized expected profits (for those sectors for which scientists are interior) and by the constraint on total scientists. I solve the model using the active-set algorithm in the Knitro solver for Matlab (Byrd et al., 2006).

B.1 Replicating with Matlab's built-in solver

In case the user does not have access to the Knitro solver, the replication package permits the user to use Matlab's built-in fmincon solver. The code uses the interior-point algorithm in order to force the solver to honor constraints at every iteration (which avoids errors). Most results with the fmincon solver are quite similar to those reported in the main text and in Appendix C, but some do differ notably. The substantive differences are of two types.

First, in some cases the fmincon algorithm does not calibrate the model properly, failing to fit the market data targeted in Table B-2. A poor calibration throws off welfare, states, and controls in all policy scenarios and in the no-policy scenario. This issue is notable in the "Cobb-Douglas" and "Larger Scientific Advances" cases, in which the fmincon calibration fails to achieve the targeted share of initial research activity in renewables.

Second, the fmincon algorithm does not find the optimal policy in some cases, especially when the optimal policy includes a research subsidy that generates a corner solutions in scientists and when higher damages justify a higher optimal emission tax. Two such cases that also have a roughly similar calibrated initial research share (so that welfare comparisons are meaningful) are "Resource-Saving Machines" and "Higher Damages". In the "Resource-Saving Machines" specification, the reported results have all scientists initially working in the renewable sector when the policymaker uses both an emission tax and a research subsidy, but fmincon finds only 84% of scientists initially working in the renewable sector. In the "Higher Damages" specification, the reported results have an emission tax of around \$261 per tCO_2 whether or not there is also a research subsidy, whereas fmincon finds an emission tax of \$109 per tCO_2 in the absence of a research subsidy (the same as in the base scenario, even though one should have expected the tax to incease upon raising damages) and \$9,768,667 per tCO_2 in the presence of a research subsidy (even though one should have expected the tax to weakly decrease upon introducing a research subsidy). In each of these cases, the reported results achieve higher welfare than does the fmincon solution, which is further evidence of the suboptimality of the fmincon solution.

Table B-1: Parameters fixed across specifications.

Parameter	Value	Description
Market parameters		
ϵ	1.8	Elasticity of substitution in final-good production
σ	0.4	Elasticity of substitution in intermediate-good production
β_K	0.3	Factor share of capital in final-good production
β_E	0.04	Factor share of energy in final-good production
κ	0.5	Share parameter in intermediate-good production
α	0.625	Inverse of machine producers' markup
ψ_1, ψ_2, ψ_3	2.4, 0.5, 3	Resource supply elasticities
$\zeta_1, \zeta_2, \zeta_3$	1, 1, 0	Indicators for resource depletion
$Q_{1,1}, Q_{2,1}, Q_{3,1}$	0, 0, 0	Year 2015 depletion adjustment
η	1	Probability of research success
γ	1	Innovation step size
L_1	7403	Year 2015 population (millions)
L_∞	11500	Asymptotic population (millions)
g_L	0.7	Rate of approach to asymptotic population level
δ	0.6513	Depreciation rate of capital per decade
Υ	0.25	Capital savings rate
K_1	238.6	Year 2015 capital (trillion year 2014 dollars)
Welfare parameters		
ρ	0.1605	Utility discount rate per decade
EIS	1/1.45	Elasticity of intertemporal substitution
$\hat{t}$	40	Horizon (decades)
Climate parameters		
d	0.00236	Damage parameter
ξ_1, ξ_2, ξ_3	0.0250, 0.0139, 0	Emission intensity of resources (Gt C per EJ)
$\bar{e}$	37.7	Exogenous emissions per timestep (Gt C per decade)
ϕ_1	0.386	Warming delay parameter
ϕ_3	0.73	Parameter governing transfer of heat from ocean to surface
ϕ_4	0.034	Parameter governing transfer of heat from surface to ocean
f_{2x}	3.503	Forcing from doubling CO_2 (W/m^2)
λ	1.13	Forcing per degree warming ([W/m^2]/°C)
$\boldsymbol{M}_1$	see text	Year 2015 carbon reservoirs (Gt C)
T_1	0.85	Year 2015 surface temperature (°C, relative to 1900)
T_1^o	0.0068	Year 2015 lower ocean temperature (°C, relative to 1900)

Table B-2: Market data matched by the first period's equilibrium (2011–2020). Resources are ordered as coal, gas, renewable.

Endogenous Outcome	Target	Description
Y_1	765	Global output in trillion year 2014 dollars
$\{R_{1,1}, R_{2,1}, R_{3,1}\}$	$\{1617, 1278, 224\}$	Resource consumption in EJ
$\{p_{1,1}, p_{2,1}, p_{3,1}\}$	$\{80, 100, 150\}$	Energy prices in \$/MWh
$\{s_{1,1}, s_{2,1}, s_{3,1}\}$	$\{0.12, 0.65, 0.23\}$	Shares of research

C Additional Robustness Results

Table C-1 reports the data underlying Table 1 in the main text. The first rows in each panel of Table C-1 repeat results familiar from the main text. I here discuss the fourth through final rows in more detail than in the main text.

The fourth row delays policy by 50 years. Whereas a policymaker again uses a research subsidy to shift all scientists to the renewable sector as soon as she can, the optimal emission tax is actually less effective at redirecting scientists to the renewable sector than in the base case. The delay reduces the benefits of each type of policy by around half, but the policies' relative value is largely unchanged.

The fifth row applies a lower utility discount rate. Each policy is now nearly ten times more valuable than before because the present-day policymaker is more sensitive to future damages from warming. The level of the standalone research subsidy is unchanged because the policymaker maxed it out even in the base case, but the initial emission tax increases to \$188 per tCO_2. The magnitude of the standalone emission tax's advantage over the standalone research subsidy is now larger than in the base case, but its relative benefit is now smaller. Adding a research subsidy to an emission tax does not generate any additional value because the emission tax is large enough to switch all scientists to the renewable sector with or without the complementary research subsidy.

The sixth row considers a case in which each unit of climate change reduces output to a larger degree. The initial emission tax is now much higher, increasing from \$132 to \$261 per tCO_2. It is also insensitive to the presence of the research subsidy: when emission reduction motivations justify a tax so large as to immediately shift all researchers to the renewable sector, the policymaker does not care whether she also has access to a research subsidy or not. The emission tax is again around twice as valuable as the research subsidy, and now the optimal portfolio of the two policies provides exactly the same value as the optimal standalone emission tax.

The seventh row studies a case in which energy intermediates are more substitutable for each other, as with an improved electric grid or improved battery technology. Laissez-faire is qualitatively consistent with the base case. Because policy now more quickly shifts resource supply towards renewables, it limits warming to lower levels and provides greater value than in the base case. The increased ease of shifting resource supply narrows the wedge (as a percentage of policy value) between the emission tax and the research subsidy, and the optimal portfolio of the two policies provides exactly the same value as the optimal standalone emission tax because the standalone emission tax shifts all research to renewables in the first period.

The eighth row reports an alternate parameterization of the research process, increasing the innovation step size γ from 1 to 6 (discussed in Appendix B). The laissez-faire transition to renewable resource use occurs around a century earlier than in the base case because innovation is so much more effective (in particular, the supply expansion effect pushes researchers to renewables sooner), and a standalone research subsidy advances that by another eighty years. Renewables now dominate resource supply by midcentury whether or not the policymaker can also use an emission tax. The standalone emission tax is still more valuable than the standalone research subsidy, but the gap is narrower than in the base case. Further, the standalone emission tax does not shift researchers towards the renewable resource as effectively as in the base case. As a result, the benefits from combining the two policies are larger than in the base case.

The ninth row considers a policymaker who optimally subsidizes production of machines in order to overcome market power.[12] Correcting this additional market failure increases welfare when the policymaker can use an emission tax, but the aditional value created is only a tiny fraction of the value created by the emission tax. Further, an initial emission tax of \$122 per tCO_2 now suffices to redirect all research to the renewable sector, which eliminates the gap in value between the standalone emission tax and the portfolio of the two instruments. However, in a demonstration of the theory of the second-best (Lipsey and Lancaster, 1956), correcting the market failure in machine production actually reduces welfare when the policymaker can use only a research subsidy. Allowing the policymaker to subsidize machine production strengthens the importance of the emission tax, and this machine production subsidy is itself far less important than either the emission tax or the research subsidy.

The final row assesses the importance of resource depletion. Now a laissez-faire transition to renewable research occurs only near the end of the policy horizon and a laissez-faire transition to renewable resources happens just after the policy horizon. Relative to the base model, turning off depletion increases laissez-faire temperature in 2115 (2415) from 3.2°C (8.5°C) to 3.9°C (12.9°C). The optimal year 2015 emission tax falls from \$132 to \$74 per tCO_2. Instead of shifting 95% of scientists to the renewable sector, this tax shifts only 75% of scientists. As a result, the wedge between the value of the standalone emission tax and the standalone research subsidy is narrower than in the base model, and adding a research subsidy to the emission tax now creates more value (and substantially lowers the optimal initial emission tax, to \$13 per tCO_2).[13] However, the main story is unchanged, as the emission tax is still more valuable than the research subsidy and still provides nearly as much value as the portfolio of the two.

D Numerical Example

A numerical example will make the analytic results more concrete. Ignore climate damages, depletion, and growth in productivity, and set $\beta_E = 1$ so that energy is the only input to

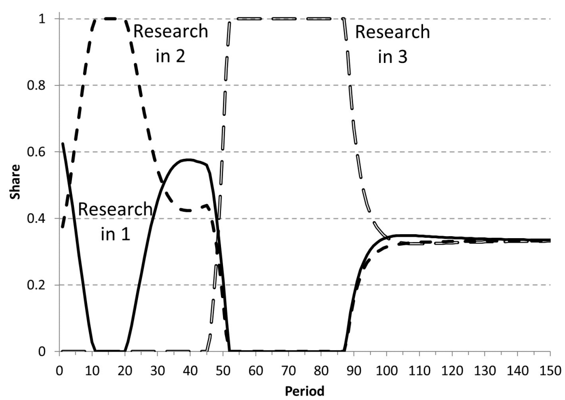
[12] This subsidy reduces the consumer price p_{jxit} of machines from α to α^2. It is not applied when calibrating the model. It is also not applied in laissez-faire, so the reported balanced growth equivalent benefit of policy includes the benefits of the machine subsidy.

[13] The much smaller emission tax in the absence of depletion likely reflects two factors. First, consumption per capita reaches extraordinary levels, which leads to very high long-run consumption discount rates via Ramsey discounting intuition. Second, the marginal effect of emissions on long-run warming is smaller in the absence of depletion because the "forcing" that determines warming is concave in the stock of atmospheric carbon (see Appendix B). This concavity becomes especially relevant because laissez-faire carbon dioxide increases from 394 ppm in 2015 to a staggering 8730 ppm in 2415, as opposed to "only" 2500 ppm under laissez-faire in the base case.

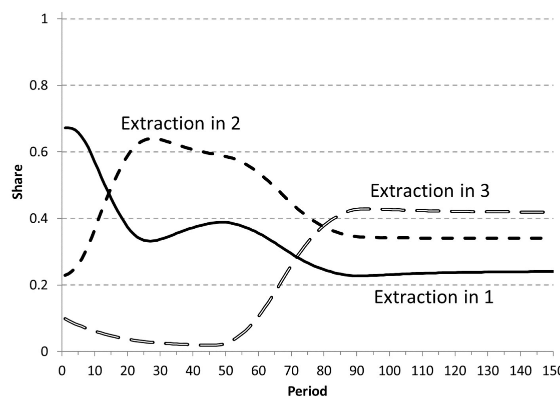
Table C-1: Additional results for alternate model versions.

	Policy Tools Available			
Specification	**No policy**	**Emission tax**	**Research subsidy**	**Both instruments**
Emission Tax in 2015 ($ per tCO_2)				
Base	-	131.8	-	122.3
Resource-Saving Machines[a]	-	98.6	-	90.4
Cobb-Douglas Machines[b]	-	99.0	-	73.3
50-Year Delay	-	0	-	0
Less Discounting[c]	-	188.2	-	188.2
Higher Damages[d]	-	260.9	-	260.9
More Substitutable Energy Types[e]	-	117.0	-	117.0
Larger Scientific Advances[f]	-	163.0	-	118.3
Optimal Machine Subsidy[g]	-	121.9	-	121.8
No Depletion[h]	-	74.4	-	12.9
Renewables' Share of Resources in 2015 (%)				
Base	7.2	22.3	8.8	21.8
Resource-Saving Machines[a]	7.2	15.1	7.3	14.7
Cobb-Douglas Machines[b]	7.2	17.6	7.6	15.9
50-Year Delay	7.2	7.2	7.2	7.2
Less Discounting[c]	7.2	26.2	8.8	26.2
Higher Damages[d]	7.2	31.0	8.8	31.0
More Substitutable Energy Types[e]	7.2	28.4	10.1	28.4
Larger Scientific Advances[f]	7.2	25.3	11.5	25.3
Optimal Machine Subsidy[g]	7.2	26.8	10.5	26.8
No Depletion[h]	7.2	17.9	8.8	14.6
Renewables' Share of Scientists in 2015 (%)				
Base	22.6	95.0	100	100
Resource-Saving Machines[a]	23.3	31.7	100	100
Cobb-Douglas Machines[b]	23.2	38.3	100	100
50-Year Delay	22.6	22.6	22.6	22.6
Less Discounting[c]	22.6	100	100	100
Higher Damages[d]	23.0	100	100	100
More Substitutable Energy Types[e]	23.4	100	100	100
Larger Scientific Advances[f]	23.3	56.4	100	100
Optimal Machine Subsidy[g]	22.6	100	100	100
No Depletion[h]	22.6	75.1	100	100
Temperature in 2115 (°C, relative to 1900)				
Base	3.2	2.5	2.9	2.5
Resource-Saving Machines[a]	2.7	2.3	2.7	2.3
Cobb-Douglas Machines[b]	2.7	2.2	2.7	2.2
50-Year Delay	3.2	2.7	3.0	2.7
Less Discounting[c]	3.2	2.4	2.9	2.4
Higher Damages[d]	3.1	2.2	2.8	2.2
More Substitutable Energy Types[e]	3.7	2.3	2.7	2.3
Larger Scientific Advances[f]	3.8	2.5	2.7	2.4
Optimal Machine Subsidy[g]	3.2	2.5	2.9	2.5
No Depletion[h]	3.9	3.0	3.4	3.0

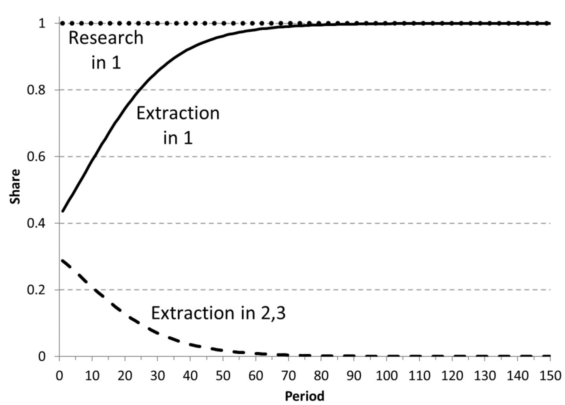
[a] σ increased from 0.4 to 1.5.
[b] σ increased from 0.4 to 1.
[c] ρ reduced from 1.5% to 0.01% per year, as in Stern (2007).
[d] Damages increased to calibration of Lemoine (2021), from survey evidence in Pindyck (2019).
[e] ϵ increased from 1.8 to 5.
[f] Innovation step size increased from $\gamma = 1$ to $\gamma = 6$.
[g] p_{jxit} reduced from α to α^2 in policy scenarios but not in laissez-faire.
[h] Each ζ_j set to zero.



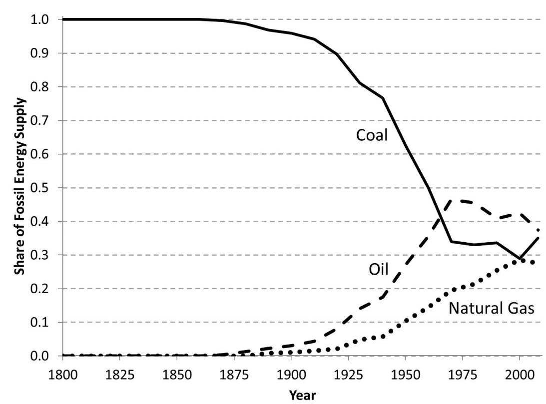
(a) Research Shares with $\sigma = 0.5$



(b) Resource Use Shares with $\sigma = 0.5$



(c) Research and Resource Use Shares with $\sigma = 1.5$



(d) Historical Resource Use Shares

Figure D-1: Top: An example of an innovation-led transition, with $\sigma = 0.5$. Bottom left: An example of lock-in, with $\sigma = 1.5$. Resources 2 and 3 have nearly identical resource use shares. Bottom right: Shares of global fossil energy supply, from Smil (2010).

final-good production (or, equivalently, capital and labor are fixed over time). Let there be three types of energy ($N = 3$), which differ only in their quality ν and in their initial technology. Let the first type of energy represent coal, the second represent oil, and the third represent gas. Looking back two hundred years, technologies for using coal were far more advanced than technologies for using oil, which in turn were more developed than technologies for using gas. I therefore fix the initial average quality of technology at 0.5 for coal, at 1% of this value for oil, and at 0.1% of this value for gas. We can think of the quality of fossil fuel resources as largely determined by the ratio of carbon to hydrogen bonds.[14] Energy derives from breaking hydrogen bonds. Fuels with a lot of carbon and little hydrogen are considered to be of lower quality because they are bulkier and more polluting. Coal is mostly carbon, oil has more hydrogen bonds per unit carbon, and natural gas has the most hydrogen bonds per unit carbon. I therefore set $\nu_1 = 0.27$ (for coal), $\nu_2 = 0.34$ (for oil), and $\nu_3 = 0.39$ (for gas).[15]

The top panels of Figure D-1 plot a case with $\sigma = 0.5$, and the lower left panel plots a case with $\sigma = 1.5$. The "coal" sector 1 begins with the majority of resource use and research activity. In the case of resource-saving technologies (bottom left), research activity and resource use are locked-in to the "coal" sector 1, which attracts all research effort in all periods and increases its share of resource use over time. In the case of resource-using technologies, we see innovation-led transitions. Research begins transitioning immediately towards the "oil" sector 2 (top left panel), and resource use eventually follows (top right panel). The "gas" sector 3 does not attract any research effort for a while and maintains a very small share of resource use even as oil displaces coal. However, after 20 periods, research effort shifts strongly towards the gas sector, and resource use shifts towards the gas sector after 60 periods. In the long run, all sectors attract identical shares of research effort and maintain stable shares of resource use, with their ordering determined by the quality ν of each resource.

The endogenous dynamics of our setting with resource-using machines are qualitatively similar to historical patterns. The bottom right panel of Figure D-1 plots resource shares since 1800. The historical patterns in these shares are similar to the patterns that emerge from our numerical simulations with resource-using machines: resource shares change rapidly as a transition occurs, and transitions do not drive formerly dominant resources out of the market. In fact, resource shares have been fairly stable since 1970. The historical patterns are nothing like the patterns that emerge from our simulations with resource-saving machines.

[14] Smil (2017, 245) describes how oil is of higher quality than coal because it has higher energy density, is cleaner, and is more transportable and storable. On page 270, he writes: "There has been a clear secular shift toward higher-quality fuels, that is, from coals to crude oil and natural gas, a process that has resulted in relative decarbonization (a rising H:C ratio) of global fossil fuel extraction..."

[15] The remaining parameters are $D(\cdot) = 0$, $\zeta_1 = \zeta_2 = \zeta_3 = 0$, $A_{Y1} = 1$, $\epsilon = 3$, $\alpha = 0.5$, $\kappa = 0.5$, $\psi_1 = \psi_2 = \psi_3 = 3$, $\Psi_1 = \Psi_2 = \Psi_3 = 1$, $\eta = 1$, and $\gamma = 0.5$. The qualitative results are not sensitive to the choice of these parameters.

E Proofs and Derivations for Section 3

This appendix derives useful intermediate results before providing proofs and derivations omitted from the main text.

E.1 Tâtonnement Stability

One may be concerned that interior equilibria are not "natural" equilibria in the presence of positive feedbacks from resource use to innovation and of potential complementarities. Indeed, Acemoglu (2002) and Hart (2012) have emphasized the role of knowledge spillovers in allowing interior research allocations to be stable in the long run. This appendix shows that interior equilibria are in fact "natural" equilibria in the present setting.

Rearranging equation (12) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation.[16] Substituting into the versions of equation (A-4) corresponding to each resource then gives us two equations in two unknowns. This system defines the equilibrium R_{jt} and R_{kt} that clear the markets for each resource.

Define the tâtonnement adjustment process and stability as follows:

Definition E-1. *A* tâtonnement adjustment process *increases R_{jt} if equation* (A-4) *is not satisfied and its right-hand side is greater, decreases R_{jt} if equation* (A-4) *is not satisfied and its left-hand side is greater, and obeys analogous rules for R_{kt}. I say that an equilibrium (R^*_{jt}, R^*_{kt}) is* tâtonnement-stable *if and only if the tâtonnement adjustment process leads to (R^*_{jt}, R^*_{kt}) from (R_{jt}, R_{kt}) sufficiently close to (R^*_{jt}, R^*_{kt}).*

The tâtonnement process changes R_{jt} and R_{kt} so as to eliminate excess supply or demand, and tâtonnement stability requires that this adjustment process converge to an equilibrium point from values close to the equilibrium. This process is the same as that in Samuelson (1941) and Arrow and Hurwicz (1958), except expressed in quantities rather than prices. The following proposition shows that our equilibrium is tâtonnement-stable:

Proposition E-1. *The equilibrium is tâtonnement-stable.*

Proof. See Appendix E.3. □

Now use the versions of equation (A-4) corresponding to each resource to define R_{jt} and R_{kt} as functions of s_{jt},[17] and then restate equation (12) as a function only of s_{jt}:

$$\frac{\Pi_{jt}}{\Pi_{kt}} = \frac{A_{j(t-1)}}{A_{k(t-1)}} \left(\frac{A_{j(t-1)} + \eta\gamma s_{jt} A_{j(t-1)}}{A_{k(t-1)} + \eta\gamma(1 - s_{jt}) A_{k(t-1)}} \right)^{\frac{-1}{\sigma+\alpha(1-\sigma)}} \left(\frac{R_{jt}(s_{jt})}{R_{kt}(s_{jt})} \right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k} \right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} . \tag{E-1}$$

The following corollary gives us the total derivative of Π_{jt}/Π_{kt} with respect to s_{jt}:

[16] Technically, this function should be written to allow for corner solutions in the research allocation. The proof of stability will account for corner solutions.

[17] Rearrange the versions of equation (A-4) corresponding to each resource to put all terms on the right-hand side. For given s_{jt}, the Jacobian of this system in R_{jt} and R_{kt} is negative definite.

Corollary E-2. *The right-hand side of equation* (E-1) *strictly decreases in* s_{jt}*.*

Proof. See Appendix E.4 □

The supply expansion effect makes the relative incentive to research in sector j decline in the number of scientists working in sector j. However, when sector j's share of resource use increases in the relative quality of its technology, a positive feedback between research and resource use maintains sector j's research incentives even as more scientists move to sector j. The proof shows, as is intuitive, that whether the relative incentive to research in sector j declines in the number of scientists working in sector j is identical to whether the equilibrium is tâtonnement-stable: tâtonnement-stability is not consistent with positive feedbacks that are strong enough to overwhelm the supply expansion effect. And we have already seen that interior equilibria are in fact tâtonnement-stable.

E.2 Useful Lemmas

First, note that equations (A-2) and (7) imply

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} p_{jRt}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}. \tag{E-2}$$

Rearranging equation (12) and using $s_{jt} + s_{kt} = 1$, we obtain s_{jt} as an explicit function of $A_{j(t-1)}/A_{k(t-1)}$ and of R_{jt}/R_{kt} at an interior allocation:

$$s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right) = \frac{(1+\eta\gamma)\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} - 1}{\eta\gamma + \eta\gamma\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)} \frac{R_{jt}}{R_{kt}} \left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}. \tag{E-3}$$

Let $\Sigma_{x,y}$ represent the elasticity of x with respect to y, and let $\Sigma_{x,y|z}$ represent the elasticity of x with respect to y holding z constant. The following lemma establishes signs and bounds for elasticities that will prove useful:

Lemma E-3. *The following hold, with analogous results for sector k:*

1. $\Sigma_{Y_t,E_{jt}}, \Sigma_{Y_t,E_{kt}} \in [0,1]$ *and* $\Sigma_{Y_t,E_{jt}} + \Sigma_{Y_t,E_{kt}} = 1$.
2. $\Sigma_{E_{jt},R_{jt}|X_{jt}}, \Sigma_{E_{jt},X_{jt}} \in [0,1]$ *and* $\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}} = 1$.
3. *If* $\sigma < 1$*, then* $\Sigma_{E_{jt},X_{jt}} \to 0$ *as* $A_{j(t-1)} \to \infty$ *and* $\Sigma_{E_{kt},X_{kt}} \to 0$ *as* $A_{k(t-1)} \to \infty$.
4. $\Sigma_{X_{jt},A_{jt}} = \frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha} \in (0,1)$
5. $\Sigma_{X_{jt},R_{jt}} = \frac{\alpha\sigma/\psi+\alpha}{\sigma(1-\alpha)+\alpha} \in (0,1]$
6. $\Sigma_{A_{jt},s_{jt}} = \frac{\eta\gamma s_{jt}}{1+\eta\gamma s_{jt}} \in [0,1)$

7. $\Sigma_{s_{jt},R_{jt}} = \frac{\psi+\sigma}{\psi}\frac{2+\eta\gamma}{\eta\gamma s_{jt}} Z_t > 0$, *where* $Z_t \in \left[\frac{1+\eta\gamma}{(2+\eta\gamma)^2}, \frac{1}{4}\right]$. $\Sigma_{s_{jt},R_{kt}} = -\Sigma_{s_{jt},R_{jt}}$.

8. $\Sigma_{s_{jt},A_{j(t-1)}} = -\frac{(1-\sigma)(1-\alpha)}{A_{j(t-1)}}\frac{(2+\eta\gamma)}{\eta\gamma} Z_t$, *which is* < 0 *if and only if* $\sigma < 1$. Z_t *is as above.* $\Sigma_{s_{jt},A_{k(t-1)}} = -\Sigma_{s_{jt},A_{j(t-1)}}$.

9. $\Sigma_{s_{jt},s_{kt}} = -s_{kt}/s_{jt} \leq 0$

Proof. Most of the results follow by differentiation and the definition of an elasticity. #1 follows from differentiating the final-good production function $Y_t(E_{jt}, E_{kt})$; #2 follows from differentiating the intermediate-good production function $E_{jt}(R_{jt}, X_{jt})$; #4 follows from differentiating equation (E-2); #5 follows from differentiating equation (E-2) after using equation (2) to substitute for p_{jRt} and using $\psi \geq \alpha/(1-\alpha)$; #6 follows from differentiating equation (5); #7 and #8 follow from differentiating equation (E-3); and #9 follows from the research constraint.

To derive #3, note that

$$\Sigma_{E_{jt},X_{jt}} = \frac{(1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}{\kappa R_{jt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}}.$$

From (A-1), (A-2), and (2), we have:

$$\begin{aligned} X_{jt} =& A_{jt}\left(\frac{1-\kappa}{\kappa}\left[\frac{R_{jt}}{X_{jt}}\right]^{1/\sigma}\Psi_j^{-1/\psi}R_{jt}^{1/\psi}\right)^{\frac{\alpha}{1-\alpha}} \\ =& A_{jt}\left(\frac{1-\kappa}{\kappa}\Psi_j^{-1/\psi}R_{jt}^{\frac{1}{\psi}+\frac{1}{\sigma}}\right)^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}}. \end{aligned}$$

$X_{jt} \to \infty$ as $A_{j(t-1)} \to \infty$, which implies with $\sigma < 1$ that $\Sigma_{E_{jt},X_{jt}} \to 0$ as $A_{j(t-1)} \to \infty$. Analogous results hold for sector k.

To derive #7 and #8, define

$$Z_t \triangleq \frac{\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}}{\left[1+\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma}\right]^2}$$

and recognize that $s_{jt} \in (0,1)$ implies

$$\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-(1-\sigma)(1-\alpha)}\frac{R_{jt}}{R_{kt}}\left[\frac{[R_{jt}/\Psi_j]^{1/\psi}}{[R_{kt}/\Psi_k]^{1/\psi}}\right]^{\sigma} \in \left(\frac{1}{1+\eta\gamma},\ 1+\eta\gamma\right)$$

from equation (12).

□

Note that $\Sigma_{X,A}$ and $\Sigma_{X,R}$ are the same in each sector. I therefore often omit the sector subscripts on these terms.

Using $s_{jt}\left(\frac{R_{jt}}{R_{kt}}, \frac{A_{j(t-1)}}{A_{k(t-1)}}\right)$, the equilibrium is defined by the versions of equation (A-4) corresponding to each resource, which are functions only of R_{jt} and R_{kt}. Rewrite these equations as (suppressing the predetermined technology arguments in s_{jt}):

$$1 = \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{E_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{E_{jt}\left(R_{jt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq G_j(R_{jt}, R_{kt}),$$

$$1 = \kappa\, (1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{E_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}\right]^{1/\epsilon} \left[\frac{E_{kt}\left(R_{kt}, s_{jt}\left(R_{jt}/R_{kt}\right)\right)}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq G_k(R_{jt}, R_{kt}).$$

We have:

Lemma E-4. *$\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$ and $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt} < 0$.*

Proof. Differentiating yields:

$$\begin{aligned}
\frac{\partial G_j(R_{jt}, R_{kt})}{\partial R_{jt}} =& G_j \Bigg\{ -\left(\frac{1}{\psi} + \frac{1}{\sigma}\right)\frac{1}{R_{jt}} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\frac{1}{E_{jt}}\left[\frac{\partial E_{jt}}{\partial R_{jt}} + \frac{\partial E_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right] \\
&+ \frac{1}{\epsilon}\frac{1}{Y_t}\left[\frac{\partial Y_t}{\partial E_{jt}}\frac{\partial E_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial E_{jt}}\frac{\partial E_{jt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} + \frac{\partial Y_t}{\partial E_{kt}}\frac{\partial E_{kt}}{\partial s_{kt}}\frac{\partial s_{kt}}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\right] \Bigg\} \\
=& \frac{G_j}{R_{jt}} \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\left[1 - \Sigma_{E_{jt},R_{jt}|X_{jt}} - \Sigma_{E_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
&- \frac{1}{\epsilon}\Bigg[\left(1 - \Sigma_{Y_t,E_{jt}}\right)\left(\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
&- \Sigma_{Y_t,E_{kt}}\Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg]\Bigg\}.
\end{aligned}$$

If the economy is at a corner in s_{jt}, then $\Sigma_{s_{jt},R_{jt}} = 0$ and, using Lemma E-3, the above expression is clearly negative. So consider a case with interior s_{jt}. The final two lines are negative. So the overall expression is negative if the third-to-last line is negative, which is the case if and only if

$$\begin{aligned}
0 \geq& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\left(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\left[-1 + \Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\left(\frac{\sigma+\psi}{\psi}\frac{\alpha + \sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right)\right] \\
=& -\frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\left[-1 + \frac{\sigma+\psi}{\psi}\frac{\alpha + \sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}}Z_t}{\sigma(1-\alpha)+\alpha}\right], \qquad \text{(E-4)}
\end{aligned}$$

where I use results from Lemma E-3. Note that $\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t \leq 3/4$, which implies that $\Sigma_{E_{jt},X_{jt}} \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\sigma(1-\alpha)+\alpha} < 1$. Using this, inequality (E-4) holds if and only if

$$\frac{\sigma}{\psi} \geq \Sigma_{E_{jt},X_{jt}} \frac{-1 + \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)}}{1 - \Sigma_{E_{jt},X_{jt}} \frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)}}. \tag{E-5}$$

$\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t \leq 3/4$ implies that $\frac{\alpha+\sigma(1-\alpha)\frac{2+\eta\gamma}{1+\eta\gamma s_{jt}} Z_t}{\alpha+\sigma(1-\alpha)} < 1$, which implies that the right-hand side of inequality (E-5) is negative. Thus, inequality (E-5) always holds and $\partial G_j(R_{jt}, R_{kt})/\partial R_{jt} < 0$.

The analysis of $\partial G_k(R_{jt}, R_{kt})/\partial R_{kt}$ is virtually identical.

□

Now define the matrix G:

$$G \triangleq \begin{bmatrix} \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}} \\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}} \end{bmatrix}.$$

We have:

Lemma E-5. *The determinant of G is positive.*

Proof. Analyze $det(G)$:

$$
\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}} \Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg) \Bigg] \Bigg\} \\
& \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}} \Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg) \Bigg] \Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}} \Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg) \\
& \qquad - \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg] \Bigg\} \\
& \Bigg\{ \frac{1}{\epsilon} \Bigg[\Sigma_{Y_t,E_{kt}} \Bigg(\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},R_{kt}} + \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg) \\
& \qquad + \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg] \Bigg\} \\
& + \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma} + \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right) \Bigg[\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}} \Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg) \\
& \qquad - \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Bigg] \Bigg\} \\
& \Bigg\{ \frac{1}{\epsilon} \Bigg[\Sigma_{Y_t,E_{jt}} \Bigg(\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},R_{jt}} + \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg) \\
& \qquad + \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}} \Bigg] \Bigg\} \\
& - \left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)^2 \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} \Sigma_{A_{jt},s_{jt}} \Sigma_{s_{jt},R_{kt}} \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} \Sigma_{A_{kt},s_{kt}} \Sigma_{s_{kt},s_{jt}} \Sigma_{s_{jt},R_{jt}},
\end{aligned}
$$

where I factored $G_j G_k / R_{jt} R_{kt}$. Use $\Sigma_{Y_t,Yjt} + \Sigma_{Y_t,E_{kt}} = 1$ from Lemma E-3 and cancel terms with $1/\epsilon^2$ to obtain:

$$
\begin{aligned}
det(G) \propto & \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{E_{jt},R_{jt}|X_{jt}} - \Sigma_{E_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg]\Bigg\} \\
& \qquad \Bigg\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Bigg[1 - \Sigma_{E_{kt},R_{kt}|X_{kt}} - \Sigma_{E_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg]\Bigg\} \\
& - \frac{1}{\sigma}\left(\frac{1}{\sigma} - \frac{1}{\epsilon}\right)\left(\Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right) \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}} \\
& \quad \Bigg[-\Bigg(\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg) \\
& \qquad + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
& + \left\{ -\frac{1}{\psi} - \frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}} \\
& \quad \Bigg[-\Bigg(\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& \qquad + \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg] \\
& + \frac{1}{\epsilon}\frac{1}{\sigma}\Bigg[\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Bigg(\Sigma_{X_{jt},R_{jt}} + \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg)\Bigg] \\
& \quad \Bigg[\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Bigg(\Sigma_{X_{kt},R_{kt}} + \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg)\Bigg].
\end{aligned}
\tag{E-6}
$$

All lines after the first three are positive by results from Lemma E-3. Expanding the products in those first three lines and rearranging, those first three lines become:

$$
\begin{aligned}
& \frac{1}{\psi^2} \\
& + \frac{1}{\sigma^2}\Bigg[1 - \Sigma_{X,R}\Bigg]\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\Bigg(1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg) \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}\Bigg[1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Bigg] \\
& + \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\Bigg[1 - \Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\Bigg] \\
& + \frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right),
\end{aligned}
\tag{E-7}
$$

where I write $\Sigma_{X,R}$ because this elasticity is the same in each sector. At corner allocations of research, $\Sigma_{s_{jt},R_{jt}} = \Sigma_{s_{jt},R_{kt}} = 0$. In this case, (E-7) is clearly positive. Now assume an interior allocation of research, so that $\Pi_{jt} = \Pi_{kt}$. Note that

$$\begin{aligned}
&1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} \\
=&\frac{1}{\psi}\frac{\sigma}{\sigma(1-\alpha)+\alpha}\left\{\psi[1-\alpha] - \alpha - (1-\alpha)[\sigma+\psi]\frac{(2+\eta\gamma)^2}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})}Z_t\right\}.
\end{aligned} \tag{E-8}$$

Substituting for Z_t and using equation (12) at $\Pi_{jt}/\Pi_{kt} = 1$, we have

$$\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}.$$

Equation (E-8) then becomes

$$1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} = -\frac{\sigma}{\psi}.$$

Substituting into (E-7), the first three lines of (E-6) are equal to

$$\begin{aligned}
&\frac{1}{\psi^2} \\
&- \frac{1}{\psi}\frac{1}{\sigma}\left[1 - \Sigma_{X,R}\right]\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}} \\
&+ \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}\left[1 - \Sigma_{X,R} - \Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right] \\
&+ \frac{1}{\psi}\frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\left[1 - \Sigma_{X,R} - \Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right] \\
&+ \frac{1}{\sigma}\frac{1}{\epsilon}\left(\Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\right)\left(\Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{jt}}\right).
\end{aligned} \tag{E-9}$$

The final line is positive. Factoring $1/\psi$, the first four lines are jointly positive if and only if:

$$\begin{aligned}
0 \leq& \frac{1}{\psi} + \frac{1}{\sigma}\Big[(1-\Sigma_{X,R})\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\
&\qquad - \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}\Sigma_{A_{jt},s_{jt}}\Sigma_{s_{jt},R_{jt}} - \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}\Sigma_{A_{kt},s_{kt}}\Sigma_{s_{kt},s_{jt}}\Sigma_{s_{jt},R_{kt}}\Big] \\
=& \frac{1}{\psi} + \frac{1}{\sigma}\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\
&- \frac{1}{\sigma}\frac{\sigma+\psi}{\psi}\frac{1}{\sigma(1-\alpha)+\alpha}\Big[\alpha\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\
&\qquad + \sigma(1-\alpha)\Big(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\Big)\frac{1}{2+\eta\gamma}\Big],
\end{aligned} \tag{E-10}$$

where we use $\frac{Z_t}{(1+\eta\gamma s_{jt})(1+\eta\gamma s_{kt})} = \frac{1}{[2+\eta\gamma]^2}$. Note that $\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}$ increases in $\Sigma_{E_{jt},X_{jt}}$ and thus reaches a maximum at $\Sigma_{E_{jt},X_{jt}} = 1$. Therefore,

$$\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}} \leq 1 + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{kt},X_{kt}} = 1.$$

Also note that $\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})$ increases in each elasticity, and each elasticity is ≤ 1. Thus,

$$\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt}) \leq (1+\eta\gamma s_{kt}) + (1+\eta\gamma s_{jt}) = 2+\eta\gamma,$$

which implies

$$\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma} \leq 1.$$

These results together imply that

$$\begin{aligned} &\alpha + \sigma(1-\alpha) \\ \geq &\alpha\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) + \sigma(1-\alpha)\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}. \end{aligned} \tag{E-11}$$

Using this, we have that inequality (E-10) holds if and only if

$$\begin{aligned} \frac{\sigma}{\psi} \geq &\Bigg\{ -\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) + \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\} \\ &\Bigg\{1 - \frac{1}{\sigma(1-\alpha)+\alpha}\Bigg[\alpha\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\ &\qquad + \sigma(1-\alpha)\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)\frac{1}{2+\eta\gamma}\Bigg]\Bigg\}^{-1}. \end{aligned} \tag{E-12}$$

The denominator on the right-hand side is positive via inequality (E-11). The numerator on the right-hand side is equal to:

$$\begin{aligned} &\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right) \\ &\left\{-1 + \frac{1}{\sigma(1-\alpha)+\alpha}\left[\alpha + \sigma(1-\alpha)\frac{\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right)}{(2+\eta\gamma)\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right)}\right]\right\}. \end{aligned} \tag{E-13}$$

Consider the fraction in brackets. If that fraction is ≤ 1, then the whole expression is negative and we are done. I will now prove that the fraction cannot be > 1. Assume that the fraction is > 1. Then:

$$\left(\Sigma_{E_{jt},X_{jt}}(1+\eta\gamma s_{kt}) + \Sigma_{E_{kt},X_{kt}}(1+\eta\gamma s_{jt})\right) > (2+\eta\gamma)\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\right)$$
$$\Leftrightarrow \eta\gamma s_{kt}\Sigma_{E_{jt},X_{jt}} + \eta\gamma s_{jt}\Sigma_{E_{kt},X_{kt}} \geq (1+\eta\gamma)\left(\Sigma_{E_{jt},X_{jt}} + \Sigma_{E_{kt},X_{kt}}\right) - (2+\eta\gamma)\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}.$$

Assume without loss of generality that $\Sigma_{E_{jt},X_{jt}} > \Sigma_{E_{kt},X_{kt}}$. Then the left-hand side of the last line attains its largest possible value when $s_{kt} = 1$. The inequality on the last line is then satisfied only if

$$0 > \Sigma_{E_{jt},X_{jt}} + (1+\eta\gamma)\Sigma_{E_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}. \tag{E-14}$$

The right-hand side is monotonic in $\Sigma_{E_{jt},X_{jt}}$. At $\Sigma_{E_{jt},X_{jt}} = 1$, the right-hand side is

$$1 + (1+\eta\gamma)\Sigma_{E_{kt},X_{kt}} - (2+\eta\gamma)\Sigma_{E_{kt},X_{kt}} = 1 - \Sigma_{E_{kt},X_{kt}} \geq 0.$$

But this contradicts inequality (E-14). Now consider the other extremum: $\Sigma_{E_{jt},X_{jt}} = 0$. The right-hand side of inequality (E-14) becomes:

$$(1+\eta\gamma)\Sigma_{E_{kt},X_{kt}} \geq 0,$$

which again contradicts inequality (E-14). Because the right-hand side of inequality (E-14) was monotonic in $\Sigma_{E_{jt},X_{jt}}$ and was not satisfied for either the greatest or smallest possible values for $\Sigma_{E_{jt},X_{jt}}$, the inequality is not satisfied for any values of $\Sigma_{E_{jt},X_{jt}}$. Thus, the fraction in brackets in (E-13) is ≤ 1, which means that the right-hand side of inequality (E-12) is ≤ 0 and inequality (E-12) is satisfied. As a result, the first three lines of (E-6) are positive, which means that $det(G) > 0$.

□

The next two lemmas establish how relative resource use and relative profit change with the average quality of technology in sector j:

Lemma E-6. *Define* $\mathbf{R}(A_{jt}, A_{kt}) \triangleq [R_{jt}(A_{jt}, A_{kt})/R_{kt}(A_{jt}, A_{kt})]$. *Then (i)* $\partial\mathbf{R}/\partial A_{jt} > 0$ *and (ii)* $\partial\mathbf{R}/\partial A_{jt} \to 0$ *as* $A_{jt} \to \infty$.

Proof. I begin by using the implicit function theorem on the two-dimensional system obtained from the versions of equation (A-4) corresponding to each resource. Rewriting previous expressions for G_j and G_k to hold s_{jt} fixed at some value s, the two-dimensional system becomes:

$$1 = \kappa\,\nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}=s\right)}{E_{jt}\left(R_{jt}, s_{jt}=s\right)}\right]^{1/\epsilon} \left[\frac{E_{jt}\left(R_{jt}, s_{jt}=s\right)}{R_{jt}}\right]^{1/\sigma} \left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} \triangleq H_j(R_{jt}, R_{kt}; s_{jt}=s),$$
$$1 = \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{Y_t\left(R_{jt}, R_{kt}, s_{jt}=s\right)}{E_{kt}\left(R_{kt}, s_{jt}=s\right)}\right]^{1/\epsilon} \left[\frac{E_{kt}\left(R_{kt}, s_{jt}=s\right)}{R_{kt}}\right]^{1/\sigma} \left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} \triangleq H_k(R_{jt}, R_{kt}; s_{jt}=s).$$

Fixing $s_{jt} = s$ makes A_{jt} a parameter. I analyze the following:

$$\begin{aligned}
\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} =& \frac{R_{jt}}{R_{kt}} \left\{ \frac{\partial R_{jt}}{\partial A_{jt}} \frac{1}{R_{jt}} - \frac{\partial R_{kt}}{\partial A_{jt}} \frac{1}{R_{kt}} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \left\{ \frac{1}{R_{jt}} \frac{-\frac{\partial H_j}{\partial A_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}} \frac{\partial H_k}{\partial A_{jt}}}{det(H)} - \frac{1}{R_{kt}} \frac{-\frac{\partial H_k}{\partial A_{jt}} \frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}} \frac{\partial H_j}{\partial A_{jt}}}{det(H)} \right\} \\
=& \frac{R_{jt}}{R_{kt}} \frac{1}{det(H)} \left\{ - \frac{\partial H_j}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} \right] + \frac{\partial H_k}{\partial A_{jt}} \left[\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} \right] \right\}.
\end{aligned} \tag{E-15}$$

Differentiation and algebraic manipulations (including applying relationships from Lemma E-3) yield:

$$-\frac{\partial H_j}{\partial A_{jt}} = - H_j \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t, E_{kt}} \right\} \Sigma_{E_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\frac{\partial H_k}{\partial A_{jt}} = H_k \frac{1}{\epsilon} \Sigma_{Y_t, E_{jt}} \Sigma_{E_{jt}, X_{jt}} \Sigma_{X_{jt}, A_{jt}} \frac{1}{A_{jt}},$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_k}{\partial R_{jt}} =& \frac{H_k}{R_{jt} R_{kt}} \left\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{E_{kt}, X_{kt}} \left[1 - \Sigma_{X,R} \right] \right. \\
& \left. + \frac{1}{\epsilon} \Sigma_{Y_t, E_{jt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{E_{jt}, X_{jt}} - \Sigma_{E_{kt}, X_{kt}} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
\frac{1}{R_{jt}} \frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}} \frac{\partial H_j}{\partial R_{jt}} =& \frac{H_j}{R_{jt} R_{kt}} \left\{ - \frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{E_{jt}, X_{jt}} \left[1 - \Sigma_{X,R} \right] \right. \\
& \left. + \frac{1}{\epsilon} \Sigma_{Y_t, E_{kt}} \left[\Sigma_{X,R} - 1 \right] \left[\Sigma_{E_{kt}, X_{kt}} - \Sigma_{E_{jt}, X_{jt}} \right] \right\}.
\end{aligned}$$

Using these in equation (E-15), we obtain:

$$\frac{\partial \mathbf{R}(A_{jt}, A_{kt})}{\partial A_{jt}} = \frac{1}{A_{jt}} \frac{1}{det(H)} \frac{R_{jt}}{R_{kt}} \frac{H_j H_k}{R_{jt} R_{kt}} \Sigma_{X,A} \left(\frac{1}{\sigma} - \frac{1}{\epsilon} \right) \Sigma_{E_{jt}, X_{jt}} \left(\frac{1}{\psi} + \frac{1}{\sigma} \Sigma_{E_{kt}, X_{kt}} [1 - \Sigma_{X,R}] \right). \tag{E-16}$$

Now consider $det(H)$. It follows from our analysis of $det(G)$ with $\Sigma_{s,R} = 0$. Make this change in equation (E-6):

$$det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(\Big\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Big[1 - \Sigma_{E_{jt},R_{jt}|X_{jt}} - \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Big]\Big\}$$
$$\Big\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Big[1 - \Sigma_{E_{kt},R_{kt}|X_{kt}} - \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Big]\Big\}$$
$$+ \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}}\Big[-\Big(\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Big)\Big]$$
$$+ \Big\{ -\frac{1}{\psi} - \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\Big[-\Big(\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Big)\Big]$$
$$+ \frac{1}{\epsilon}\frac{1}{\sigma}\Big[\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},R_{jt}}\Big]\Big[\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},R_{kt}}\Big]\Bigg).$$

Now analyze, using relations in Lemma E-3:

$$det(H) = \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(\Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\Big[1 - \Sigma_{X,R}\Big]\Big\}\Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}\Big[1 - \Sigma_{X,R}\Big]\Big\}$$
$$+ \Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}}\Big(\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X,R}\Big)$$
$$+ \Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\Big(\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X,R}\Big)$$
$$+ \frac{1}{\epsilon}\frac{1}{\sigma}\Big[\Sigma_{E_{jt},R_{jt}|X_{jt}} + \Sigma_{E_{jt},X_{jt}}\Sigma_{X,R}\Big]\Big[\Sigma_{E_{kt},R_{kt}|X_{kt}} + \Sigma_{E_{kt},X_{kt}}\Sigma_{X,R}\Big]\Bigg)$$
$$= \frac{H_j H_k}{R_{jt} R_{kt}} \Bigg(\Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\Big[1 - \Sigma_{X,R}\Big]\Big\}\Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}\Big[1 - \Sigma_{X,R}\Big]\Big\}$$
$$+ \Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}}\Big[1 - \Sigma_{E_{kt},X_{kt}}(1 - \Sigma_{X,R})\Big]$$
$$+ \Big\{ \frac{1}{\psi} + \frac{1}{\sigma}\Big\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\Big[1 - \Sigma_{E_{jt},X_{jt}}(1 - \Sigma_{X,R})\Big]$$
$$+ \frac{1}{\epsilon}\frac{1}{\sigma}\Big[1 - \Sigma_{E_{jt},X_{jt}}(1 - \Sigma_{X,R})\Big]\Big[1 - \Sigma_{E_{kt},X_{kt}}(1 - \Sigma_{X,R})\Big]\Bigg).$$

From Lemma E-3, $1 - \Sigma_{X,R} = \frac{\sigma}{\psi}\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}$. Substituting $det(H)$ into equation (E-16), we have:

$$\begin{aligned}\frac{\partial \mathbf{R}(A_{jt},A_{kt})}{\partial A_{jt}} =& \frac{1}{A_{jt}}\frac{R_{jt}}{R_{kt}}\Sigma_{X,A}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\Sigma_{E_{jt},X_{jt}}\left(\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}[1-\Sigma_{X,R}]\right)\\ &\left(\left\{\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{E_{jt},X_{jt}}\left[1-\Sigma_{X,R}\right]\right\}\left\{\frac{1}{\psi}+\frac{1}{\sigma}\Sigma_{E_{kt},X_{kt}}\left[1-\Sigma_{X,R}\right]\right\}\right.\\ &+\left\{\frac{1}{\psi}+\frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}}\left[1-\Sigma_{E_{kt},X_{kt}}(1-\Sigma_{X,R})\right]\\ &+\left\{\frac{1}{\psi}+\frac{1}{\sigma}\right\}\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\left[1-\Sigma_{E_{jt},X_{jt}}(1-\Sigma_{X,R})\right]\\ &\left.+\frac{1}{\epsilon}\frac{1}{\sigma}\left[1-\Sigma_{E_{jt},X_{jt}}(1-\Sigma_{X,R})\right]\left[1-\Sigma_{E_{kt},X_{kt}}(1-\Sigma_{X,R})\right]\right)^{-1} \qquad \text{(E-17)}\\ >&0.\end{aligned}$$

We have established the first part of the lemma. To establish the second part, use Lemma E-3 in equation (E-17).

□

Lemma E-7. *Fix $s_{jt} = s$. If $\sigma > 1$ or σ is not too much smaller than 1, then Π_{jt}/Π_{kt} increases in $A_{j(t-1)}$. As $A_{j(t-1)} \to \infty$, Π_{jt}/Π_{kt} decreases in $A_{j(t-1)}$ for all $\sigma < 1$.*

Proof. To a first-order approximation, we have, with s_{jt} fixed at s,

$$\begin{aligned}&\frac{\mathrm{d}\ln[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}A_{j(t-1)}}\\ \approx&\frac{1}{A_{j(t-1)}}\left[1-\frac{1}{\sigma+\alpha(1-\sigma)}\right]+\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}\frac{\partial A_{jt}}{\partial A_{j(t-1)}}\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}\\ =&\frac{1}{A_{j(t-1)}}\left[1-\frac{1}{\sigma+\alpha(1-\sigma)}\right]+\frac{1}{\psi}\frac{\psi+\sigma}{\sigma+\alpha(1-\sigma)}(1+\eta\gamma s)\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}\\ =&\frac{1}{A_{j(t-1)}}\frac{(1-\alpha)(\sigma-1)}{\sigma+\alpha(1-\sigma)}+\frac{1}{\psi}\frac{\psi+\sigma}{\sigma+\alpha(1-\sigma)}(1+\eta\gamma s)\frac{\partial[R_{jt}/R_{kt}]}{\partial A_{jt}}\frac{R_{kt}}{R_{jt}}.\end{aligned}$$

The first term is positive if and only if $\sigma > 1$ and, using Lemma E-6, the second term is positive. Therefore the whole expression is positive if $\sigma > 1$. The first term becomes small for σ close to 1. Therefore the second term dominates (and the whole expression is positive) for σ not too much smaller than 1. Finally, Lemma E-6 shows that the second term goes to 0 as $A_{j(t-1)} \to \infty$ if $\sigma < 1$. Therefore the whole expression is negative if $\sigma < 1$ and $A_{j(t-1)} \to \infty$.

□

Finally, consider the evolution of relative resource use and thus of market size and resource cost effects. From equation (13), R_{jt}/R_{kt} increases in s_{jt}. Define $\hat{s}_{t+1}$ as the unique value of

$s_{j(t+1)}$ such that sector j's share of resource resource use increases from time t to $t+1$ if and only if $s_{j(t+1)} \geq \hat{s}_{t+1}$. Lemma E-6 implies that $\hat{s}_{t+1} \in (0,1)$.

Lemma E-8. *If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \geq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$. If $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if $A_{j(t-1)}/A_{k(t-1)} \leq [\Psi_j/\Psi_k]^{1/[(1-\alpha)(1+\psi)]}$.*

Proof. The change in R_{jt}/R_{kt} from time t to $t+1$ is

$$\begin{aligned}\frac{R_{j(t+1)}}{R_{k(t+1)}} - \frac{R_{jt}}{R_{kt}} &= \frac{(R_{j(t+1)} - R_{jt})R_{kt} - (R_{k(t+1)} - R_{kt})R_{jt}}{R_{k(t+1)}R_{kt}} \\ &\propto \frac{R_{j(t+1)} - R_{jt}}{R_{jt}} - \frac{R_{k(t+1)} - R_{kt}}{R_{kt}},\end{aligned}$$

where the first equality adds and subtracts $R_{jt}R_{kt}$ in the numerator and the second line factors $R_{jt}/R_{k(t+1)}$. To a first-order approximation, this is proportional to

$$\frac{1}{R_{jt}}\left(\frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{jt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right) - \frac{1}{R_{kt}}\left(\frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{jt}}\left[A_{j(t+1)} - A_{jt}\right] + \frac{\mathrm{d}R_{kt}}{\mathrm{d}A_{kt}}\left[A_{k(t+1)} - A_{kt}\right]\right),$$

with the derivatives evaluated at the time t allocation. Note that s_{jt} is included in A_{jt} when differentiating with respect to A_{jt}, which reflects that we will seek the allocation of scientists that holds R_{jt}/R_{kt} constant. Defining $H_j(R_{jt}, R_{kt}; s_{jt} = s)$ and $H_k(R_{jt}, R_{kt}; s_{jt} = s)$ as in the proof of Lemma E-6 and using the implicit function theorem, the previous expression becomes:

$$\begin{aligned}&\frac{1}{R_{jt}}\left(\frac{-\frac{\partial H_j}{\partial A_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_j}{\partial A_{kt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{\partial H_j}{\partial R_{kt}}\frac{\partial H_k}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ &- \frac{1}{R_{kt}}\left(\frac{-\frac{\partial H_k}{\partial A_{jt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{jt}}}{det(H)}\left[A_{j(t+1)} - A_{jt}\right] + \frac{-\frac{\partial H_k}{\partial A_{kt}}\frac{\partial H_j}{\partial R_{jt}} + \frac{\partial H_k}{\partial R_{jt}}\frac{\partial H_j}{\partial A_{kt}}}{det(H)}\left[A_{k(t+1)} - A_{kt}\right]\right) \\ \propto &\left[-\frac{\partial H_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial H_j}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_k}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_k}{\partial R_{jt}}\right] \\ &+ \left[\frac{\partial H_k}{\partial A_{jt}}s_{j(t+1)}A_{jt} + \frac{\partial H_k}{\partial A_{kt}}s_{k(t+1)}A_{kt}\right]\left[\frac{1}{R_{jt}}\frac{\partial H_j}{\partial R_{kt}} + \frac{1}{R_{kt}}\frac{\partial H_j}{\partial R_{jt}}\right],\end{aligned} \tag{E-18}$$

where the second expression factors $\eta\gamma/det(H)$, which is readily seen to be positive by altering the proof of Lemma E-5 to set the $\Sigma_{s,R}$ terms to zero. Differentiation and algebraic manipulations (including applying relationships from Lemma E-3) yield:

$$\begin{aligned}-\frac{\partial H_j}{\partial A_{jt}}s_{j(t+1)}A_{jt} - \frac{\partial H_j}{\partial A_{kt}}s_{k(t+1)}A_{kt} = &- H_j\left\{\frac{1}{\sigma} - \frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\right\}\Sigma_{E_{jt},X_{jt}}\Sigma_{X_{jt},A_{jt}}s_{j(t+1)} \\ &- H_j\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\Sigma_{E_{kt},X_{kt}}\Sigma_{X_{kt},A_{kt}}(1 - s_{j(t+1)}),\end{aligned}$$

$$\frac{\partial H_k}{\partial A_{jt}} s_{j(t+1)} A_{jt} + \frac{\partial H_k}{\partial A_{kt}} s_{k(t+1)} A_{kt} = H_k \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \right\} \Sigma_{E_{kt},X_{kt}} \Sigma_{X_{kt},A_{kt}} (1 - s_{j(t+1)})$$
$$+ H_k \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} \Sigma_{X_{jt},A_{jt}} s_{j(t+1)}.$$

Substitute these and expressions derived in the proof of Lemma E-6 into (E-18) and factor $\Sigma_{X,A} H_j H_k / [R_{jt} R_{kt}]$:

$$\left\{ - s_{j(t+1)} \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \right\} \Sigma_{E_{jt},X_{jt}} - (1 - s_{j(t+1)}) \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} \right\}$$
$$\left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{E_{kt},X_{kt}} \left[1 - \Sigma_{X,R} \right] \right\}$$
$$+ \left\{ (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \right\} \Sigma_{E_{kt},X_{kt}} + s_{j(t+1)} \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} \right\}$$
$$\left\{ -\frac{1}{\psi} - \frac{1}{\sigma} \Sigma_{E_{jt},X_{jt}} \left(1 - \Sigma_{X,R} \right) \right\}$$
$$+ \frac{1}{\epsilon} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}} \right]$$
$$\left\{ - s_{j(t+1)} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \right] \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} - (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \right\} \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} \right\}$$
$$- \frac{1}{\epsilon^2} \Sigma_{Y_t,E_{jt}} \Sigma_{Y_t,E_{kt}} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}} \right] \left\{ (1 - s_{j(t+1)}) \Sigma_{E_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{E_{jt},X_{jt}} \right\}$$
$$= s_{j(t+1)} \Sigma_{E_{jt},X_{jt}} \left\{ \frac{1}{\psi} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \right] \right.$$
$$\left. + \frac{1}{\sigma} \left(1 - \Sigma_{X,R} \right) \left[\frac{1}{\sigma} \Sigma_{E_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} \right] \right\}$$
$$- (1 - s_{j(t+1)}) \Sigma_{E_{kt},X_{kt}} \left\{ \frac{1}{\psi} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \right] \right.$$
$$\left. + \frac{1}{\sigma} \left(1 - \Sigma_{X,R} \right) \left[\frac{1}{\sigma} \Sigma_{E_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} \right] \right\}$$
$$+ \frac{1}{\epsilon} \left[1 - \Sigma_{X,R} \right] \left[\Sigma_{E_{kt},X_{kt}} - \Sigma_{E_{jt},X_{jt}} \right]$$
$$\left\{ - s_{j(t+1)} \left[\frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{kt}} \right] \Sigma_{Y_t,E_{jt}} \Sigma_{E_{jt},X_{jt}} - (1 - s_{j(t+1)}) \left\{ \frac{1}{\sigma} - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \right\} \Sigma_{Y_t,E_{kt}} \Sigma_{E_{kt},X_{kt}} \right.$$
$$\left. - \frac{1}{\epsilon} \Sigma_{Y_t,E_{jt}} \Sigma_{Y_t,E_{kt}} \left[(1 - s_{j(t+1)}) \Sigma_{E_{kt},X_{kt}} + s_{j(t+1)} \Sigma_{E_{jt},X_{jt}} \right] \right\}$$

$$
\begin{aligned}
=& s_{j(t+1)}\Sigma_{E_{jt},X_{jt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,E_{kt}}\right]\Sigma_{E_{kt},X_{kt}}\right\} \\
&-(1-s_{j(t+1)})\Sigma_{E_{kt},X_{kt}}\left\{\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]+\frac{1}{\sigma}\left(1-\Sigma_{X,R}\right)\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\Sigma_{Y_t,E_{jt}}\right]\Sigma_{E_{jt},X_{jt}}\right\} \\
&-s_{j(t+1)}\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,E_{jt}}\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}+(1-s_{j(t+1)})\frac{1}{\sigma}\frac{1}{\epsilon}\left[1-\Sigma_{X,R}\right]\Sigma_{Y_t,E_{kt}}\Sigma_{E_{kt},X_{kt}}\Sigma_{E_{jt},X_{jt}} \\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{E_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{E_{kt},X_{kt}}\right] \\
&+\frac{1}{\sigma^2}\left(1-\Sigma_{X,R}\right)\Sigma_{E_{kt},X_{kt}}\Sigma_{E_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)-\frac{1}{\sigma}\frac{1}{\epsilon}\left(1-\Sigma_{X,R}\right)\Sigma_{E_{jt},X_{jt}}\Sigma_{E_{kt},X_{kt}}\left(2s_{j(t+1)}-1\right) \\
=&\frac{1}{\psi}\left[\frac{1}{\sigma}-\frac{1}{\epsilon}\right]\left[s_{j(t+1)}\Sigma_{E_{jt},X_{jt}}-(1-s_{j(t+1)})\Sigma_{E_{kt},X_{kt}}\right]+\frac{1}{\sigma}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\left(1-\Sigma_{X,R}\right)\Sigma_{E_{kt},X_{kt}}\Sigma_{E_{jt},X_{jt}}\left(2s_{j(t+1)}-1\right)
\end{aligned}
$$

Substituting for $\Sigma_{X,R}$ and rearranging, we obtain

$$
\begin{aligned}
\frac{1}{\psi}\left(\frac{1}{\sigma}-\frac{1}{\epsilon}\right)\Bigg[& s_{j(t+1)}\Sigma_{E_{jt},X_{jt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{E_{kt},X_{kt}}\right) \\
&-(1-s_{j(t+1)})\Sigma_{E_{kt},X_{kt}}\left(1+\frac{\psi[1-\alpha]-\alpha}{\sigma(1-\alpha)+\alpha}\Sigma_{E_{jt},X_{jt}}\right)\Bigg].
\end{aligned} \tag{E-19}
$$

This expression is positive if and only if the term in brackets is positive. Define $\hat{s}_{t+1}$ as the $s_{j(t+1)}$ such that $R_{jt}/R_{kt} = R_{j(t+1)}/R_{k(t+1)}$. Then $\hat{s}_{t+1}$ is the root of the term in brackets. Solving for that root, we have:

$$
\hat{s}_{t+1} = \frac{\Sigma_{E_{kt},X_{kt}}C_{jt}}{\Sigma_{E_{jt},X_{jt}}C_{kt}+\Sigma_{E_{kt},X_{kt}}C_{jt}}, \tag{E-20}
$$

where $\Sigma_{w,z}$ is the elasticity of w with respect to z and where

$$
\begin{aligned}
C_{jt} &\triangleq 1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{E_{jt},X_{jt}} > 0, \\
C_{kt} &\triangleq 1+\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}\left[\psi-\frac{\alpha}{1-\alpha}\right]\Sigma_{E_{kt},X_{kt}} > 0.
\end{aligned}
$$

Thus,

$$
\left\{\hat{s}_{t+1} \geq \frac{1}{2}\right\} \Leftrightarrow \left\{\Sigma_{E_{kt},X_{kt}} \geq \Sigma_{E_{jt},X_{jt}}\right\},
$$

where the right-hand side is evaluated at $\hat{s}_{t+1}$. Using the explicit expressions for the elasticities, for intermediate-good production, and for X_{jt} and X_{kt} (see equation (E-2)), we have:

$$\begin{aligned}\Sigma_{E_{kt},X_{kt}} \geq& \Sigma_{E_{jt},X_{jt}}\\ \Leftrightarrow 0 \leq& \frac{(1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}E_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}E_{kt}^{\frac{\sigma-1}{\sigma}}}{E_{kt}^{\frac{\sigma-1}{\sigma}}E_{jt}^{\frac{\sigma-1}{\sigma}}} \qquad \text{(E-21)}\\ \Leftrightarrow 0 \leq& X_{kt}^{\frac{\sigma-1}{\sigma}}E_{jt}^{\frac{\sigma-1}{\sigma}} - X_{jt}^{\frac{\sigma-1}{\sigma}}E_{kt}^{\frac{\sigma-1}{\sigma}}\\ \Leftrightarrow 0 \leq& \kappa R_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} + (1-\kappa)X_{jt}^{\frac{\sigma-1}{\sigma}}X_{kt}^{\frac{\sigma-1}{\sigma}} - \kappa R_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}} - (1-\kappa)X_{kt}^{\frac{\sigma-1}{\sigma}}X_{jt}^{\frac{\sigma-1}{\sigma}}\\ \Leftrightarrow 1 \leq& \left(\frac{R_{jt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{kt}}{\Psi_k}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{kt}}{A_{kt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{kt}}{R_{kt}\left[\frac{1-\kappa}{\kappa}\left(\frac{R_{jt}}{\Psi_j}\right)^{1/\psi}\right]^{\frac{\alpha\,\sigma}{\sigma(1-\alpha)+\alpha}}\left[\frac{R_{jt}}{A_{jt}}\right]^{\frac{\alpha}{\sigma(1-\alpha)+\alpha}}A_{jt}}\right)^{\frac{\sigma-1}{\sigma}}\\ \Leftrightarrow 1 \leq& \left[\left(\frac{\Psi_j}{\Psi_k}\right)^{\frac{\alpha\,\sigma/\psi}{\sigma(1-\alpha)+\alpha}}\left(\frac{R_{jt}}{R_{kt}}\right)^{\frac{\sigma(1-\alpha-\alpha/\psi)}{\sigma(1-\alpha)+\alpha}}\left(\frac{A_{kt}}{A_{jt}}\right)^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}}\right]^{\frac{\sigma-1}{\sigma}}\\ \Leftrightarrow 1 \leq& \left(\frac{\Psi_j}{\Psi_k}\right)^{\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]}\left(\frac{1+\eta\gamma s_{jt}}{1+\eta\gamma s_{kt}}\right)^{-\chi\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]}\left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{\chi(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]}, \qquad \text{(E-22)}\end{aligned}$$

where the final line substitutes for R_{jt}/R_{kt} from equation (12) (which must hold for $\hat{s}_{t+1}$ interior) and where

$$\chi \triangleq \frac{\sigma-1}{[\sigma(1-\alpha)+\alpha][1+\sigma/\psi]} < 0 \text{ iff } \sigma < 1.$$

The right-hand side of inequality (E-22) is increasing in s_{jt} if and only if $\sigma < 1$. Therefore, if $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the strict version of the inequality does not hold at $s_{jt} = 0.5$, and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if the inequality holds at $s_{jt} = 0.5$. If $\sigma < 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \geq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

and if $\sigma > 1$, then $\hat{s}_{t+1} \geq 0.5$ if and only if

$$\frac{A_{j(t-1)}}{A_{k(t-1)}} \leq \left[\frac{\Psi_j}{\Psi_k}\right]^{\theta},$$

where

$$\theta \triangleq \frac{-\frac{1}{\psi}[\alpha+\sigma(1-\alpha)]}{(1-\alpha)[(1-\sigma)(1-\alpha-\alpha/\psi)-(1+\sigma/\psi)]} = \frac{1}{(1-\alpha)(1+\psi)} > 0.$$

□

E.3 Proof of Proposition E-1

The tâtonnement adjustment process generates, to constants of proportionality, the following system for finding the equilibrium within period t:

$$\begin{aligned}\dot{R}_{jt} =& h\Big(G_j(R_{jt}, R_{kt}) - 1\Big),\\ \dot{R}_{kt} =& h\Big(G_k(R_{jt}, R_{kt}) - 1\Big),\end{aligned}$$

where dots indicate time derivatives (with the fictional time for finding an equilibrium here flowing within a period t), $h(0) = 0$, and $h'(\cdot) > 0$. The system's steady state occurs at the equilibrium values, which I denote with stars. Linearizing around the steady state, we have

$$\begin{bmatrix}\dot{R}_{jt}\\ \dot{R}_{kt}\end{bmatrix} \approx h'(0)\begin{bmatrix}\frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_j(R_{jt},R_{kt})}{\partial R_{kt}}\\ \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{jt}} & \frac{\partial G_k(R_{jt},R_{kt})}{\partial R_{kt}}\end{bmatrix}\begin{bmatrix}R_{jt} - R^*_{jt}\\ R_{kt} - R^*_{kt}\end{bmatrix} = h'(0)\, G\begin{bmatrix}R_{jt} - R^*_{jt}\\ R_{kt} - R^*_{kt}\end{bmatrix},$$

where G is the 2×2 matrix of derivatives, each evaluated at (R^*_{jt}, R^*_{kt}). Lemma E-4 implies that the trace of G is strictly negative, in which case at least one of the two eigenvalues must be strictly negative. Lemma E-5 shows that $det(G) > 0$, which means that both eigenvalues must have the same sign. Therefore both eigenvalues are strictly negative. The linearized system is therefore globally asymptotically stable, and, by Lyapunov's Theorem of the First Approximation, the full nonlinear system is locally asymptotically stable around the equilibrium.

E.4 Proof of Corollary E-2

Now treat the versions of equation (A-4) corresponding to each resource as functions of R_{jt}, R_{kt}, and s_{jt} (recognizing that $s_{kt} = 1 - s_{jt}$):

$$\begin{aligned}1 &= \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{E_{jt}(R_{jt}, s_{jt})}\right]^{1/\epsilon}\left[\frac{E_{jt}(R_{jt}, s_{jt})}{R_{jt}}\right]^{1/\sigma}\left[\frac{R_{jt}}{\Psi_j}\right]^{-1/\psi} &&\triangleq \hat{G}_j(R_{jt}, R_{kt}; s_{jt}),\\ 1 &= \kappa\,(1-\nu_j) A_Y^{\frac{\epsilon-1}{\epsilon}}\left[\frac{Y_t(R_{jt}, R_{kt}, s_{jt})}{E_{kt}(R_{kt}, s_{jt})}\right]^{1/\epsilon}\left[\frac{E_{kt}(R_{kt}, s_{jt})}{R_{kt}}\right]^{1/\sigma}\left[\frac{R_{kt}}{\Psi_k}\right]^{-1/\psi} &&\triangleq \hat{G}_k(R_{jt}, R_{kt}; s_{jt}).\end{aligned}$$

This system of equations implicitly defines R_{jt} and R_{kt} as functions of the parameter s_{jt}. Define the matrix $\hat{G}$ analogously to the matrix G. Using the implicit function theorem, we have

$$\frac{\partial R_{jt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} \quad\text{and}\quad \frac{\partial R_{kt}}{\partial s_{jt}} = \frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})}.$$

Interpreting equation (12) as implicitly defining s_{jt} as a function of R_{jt} and R_{kt}, we have:

$$\frac{\partial s_{jt}}{\partial R_{jt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}} \text{ and } \frac{\partial s_{jt}}{\partial R_{kt}} = -\frac{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}}{\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}},$$

and thus

$$\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \text{ and } \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}} = -\frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}.$$

Using these expressions, consider how the right-hand side of equation (E-1) changes in s_{jt}:

$$\begin{aligned}
\frac{\mathrm{d}[\Pi_{jt}/\Pi_{kt}]}{\mathrm{d}s_{jt}} =& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{jt}}\frac{\partial R_{jt}}{\partial s_{jt}} + \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial R_{kt}}\frac{\partial R_{kt}}{\partial s_{jt}} \\
=& \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}} \\
&- \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{-\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}}{det(\hat{G})} - \frac{\partial[\Pi_{jt}/\Pi_{kt}]}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{-\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}}{det(\hat{G})} \\
\propto& -\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} \\
&- \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}} - \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial \hat{G}_j}{\partial R_{jt}} + \frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}}\frac{\partial \hat{G}_j}{\partial s_{jt}} \\
=& -\left(\frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial R_{kt}} + \frac{\partial \hat{G}_j}{\partial R_{jt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\right) \\
&+ \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial R_{jt}} + \frac{\partial \hat{G}_j}{\partial R_{kt}}\frac{\partial \hat{G}_k}{\partial s_{jt}}\frac{\partial s_{jt}}{\partial R_{jt}} \\
=& -det(G).
\end{aligned}$$

The third expression factored $det(\hat{G})$, which is positive by the proof of Proposition E-1 for a corner solution in s_{jt}, and it also factored $\partial[\Pi_{jt}/\Pi_{kt}]/\partial s_{jt}$, which is negative. The final equality recognizes that the only difference between the equations with a hat and the equations without a hat are that the equations without a hat allow s_{jt} to vary with R_{jt} and R_{kt}. Lemma E-5 showed that $det(G) > 0$. Thus the right-hand side of equation (E-1) strictly decreases in s_{jt}.

E.5 Proof of Lemma 1

Under the given assumption that $\nu = 0.5$ and $\Psi_j = \Psi_k$, we have $R_{jt} = R_{kt}$ when $A_{j(t-1)} = A_{k(t-1)}$ and $s_{jt} = 0.5$. Therefore, it is easy to see that $\Pi_{jt}/\Pi_{kt} = 1$ at $s_{jt} = 0.5$ when $A_{j(t-1)} = A_{k(t-1)}$. By Lemma E-7, increasing $A_{j(t-1)}$ increases Π_{jt}/Π_{kt} if either $\sigma > 1$ or σ is not too much smaller than 1. In those cases, Corollary E-2 gives us that $A_{j(t-1)} > A_{k(t-1)}$ implies $s^*_{jt} > 0.5$. The lemma follows from observing that $A_{j(t-1)} > A_{k(t-1)}$ and $\Psi_j = \Psi_k$ imply that $A_{j(t-1)}/A_{k(t-1)} > (\Psi_j/\Psi_k)^{1/[(1-\alpha)(1+\psi)]}$.

E.6 Proof of Proposition 2

To start, let Assumption 1 hold. From Lemma E-8, $\hat{s}_{t+1} < 0.5$. Therefore $s_{jt_0} > \hat{s}_{t+1}$. Assume that $s_{j(t_0+1)} < s_{jt_0}$. From equation (12), $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)}$ increases in A_{jt_0}/A_{kt_0} for any given $s_{j(t_0+1)}$ if $\sigma > 1$. Therefore, for the equilibrium to have $s_{j(t_0+1)} < s_{jt_0}$, it must be true that $R_{jt_0}/R_{kt_0} > R_{j(t_0+1)}/R_{k(t_0+1)}$ and thus $s_{j(t_0+1)} < \hat{s}_{t_0+1}$. From Corollary E-2 and $s_{jt_0} > \hat{s}_{t_0+1}$, it must be true that $\Pi_{jt_0}/\Pi_{kt_0} > 1$ when evaluated at $\hat{s}_{t_0+1}$. Because $R_{jt_0}/R_{kt_0} = R_{j(t_0+1)}/R_{k(t_0+1)}$ if $s_{j(t_0+1)} = \hat{s}_{t_0+1}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$ by $s_{jt_0} > 0.5$, it therefore must be true that $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > 1$ when evaluated at $\hat{s}_{t_0+1}$. By Corollary E-2, it then must be true that $s_{j(t_0+1)} > \hat{s}_{t_0+1}$. We have a contradiction. It must be true that $s_{j(t_0+1)} \geq s_{jt_0}$.

Because $s_{j(t_0+1)} \geq s_{jt_0} > 0.5 > \hat{s}_{t+1}$, it follows that $R_{jt_0}/R_{kt_0} \leq R_{j(t_0+1)}/R_{k(t_0+1)}$ and $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Therefore Assumption 1 still holds at time $t_0 + 1$. Proceeding by induction, sector j's shares of research and resource use increase forever: resource j is locked-in from time t_0 if $\sigma > 1$ and Assumption 1 holds at time t_0. We have established the first part of the proposition.

Now consider the remaining parts of the proposition, no longer imposing Assumption 1. We know that $\Pi^*_{jt}/\Pi^*_{kt} = 1$ when $s^*_{jt} \in (0, 1)$. Assume that $s^*_{jt} \in (0.5, 1)$. By Lemma E-7, $\Pi_{j(t+1)}/\Pi_{k(t+1)} > 1$ when evaluated at s^*_{jt}. Therefore, by Corollary E-2, $s^*_{j(t+1)} > s^*_{jt}$. Analogous arguments apply when $s^*_{jt} \in (0, 0.5)$. We have established the second part of the proposition.

By the foregoing, the only possible steady states are at $s^*_{jt} = 0.5$, $s^*_{jt} = 0$, and $s^*_{jt} = 1$. We just saw that a steady state at $s^*_{jt} = 0.5$ cannot be stable (should it even exist). When $s^*_{jt} = 1$, only $A_{j(t-1)}$ changes over time, increasing by $\eta\gamma A_{j(t-1)}$ at each time t. By Lemma E-7, $\Pi_{j(t_0+1)}/\Pi_{k(t_0+1)} > \Pi_{jt_0}/\Pi_{kt_0}$ if $s_{j(t_0+1)} \geq s_{jt_0}$. If $s_{jt_0} = 1$, then $\Pi_{jt_0} > \Pi_{kt_0}$, in which case $\Pi_{j(t_0+1)} > \Pi_{k(t_0+1)}$ if $s_{j(t_0+1)} = s_{jt_0}$. It is then an equilibrium for s^*_{jt} to equal 1 for all $t \geq t_0$. An analogous proof covers the case where $s^*_{jt} = 0$.

E.7 Proof of Proposition 3

First consider whether a corner allocation can persist indefinitely. If $s^*_{jt} = 1$ for all $t \geq t_0$, then $A_{j(t-1)} \to \infty$ as $t \to \infty$ and, by Lemma E-6, R_{jt}/R_{kt} goes to a constant. In that case, from equation (12), Π_{jt}/Π_{kt} goes to zero for all s_{jt}. But Π_{jt}/Π_{kt} cannot be zero if $s^*_{jt} = 1$ because $s^*_{jt} = 1$ implies that $\Pi_{jt}/\Pi_{kt} \geq 1$. We have contradicted the assumption that $s^*_{jt} = 1$ for all $t \geq t_0$. Analogous arguments show that it cannot be true that $s^*_{kt} = 1$ for all $t \geq t_0$. It therefore must be true that, for all t_0, there exists some $t > t_0$ such that $s^*_{jt} \in (0, 1)$.

Because a corner research allocation cannot persist indefinitely, A_{jt} and A_{kt} both become arbitrarily large as t becomes large. From equations (A-2), (7), and (2), we have

$$X_{jt} = \left\{ \left[\left(\frac{R_{jt}}{\Psi_j} \right)^{1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} \left[\frac{R_{jt}}{A_{jt}} \right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha}} \right\}^{\frac{\alpha}{1-\alpha}} A_{jt}$$
$$= \left[\Psi_j^{-1/\psi} \frac{1-\kappa}{\kappa} \right]^{\frac{\sigma\alpha}{\sigma(1-\alpha)+\alpha}} A_{jt}^{\frac{\sigma(1-\alpha)}{\sigma(1-\alpha)+\alpha}} R_{jt}^{\frac{\alpha(1+\sigma/\psi)}{\sigma(1-\alpha)+\alpha}}.$$

X_{jt} and X_{kt} thus also become arbitrarily large as t becomes large. This in turn implies that $E_{jt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{jt}$ and $E_{kt} \to \kappa^{\frac{\sigma}{\sigma-1}} R_{kt}$ as t becomes large. From equation (13), we have:

$$\left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}+\frac{1}{\psi}} \to \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \left[\frac{R_{jt}}{R_{kt}} \right]^{\frac{1}{\sigma}-\frac{1}{\epsilon}}$$

as t becomes large. Therefore, as $t \to \infty$,

$$\frac{R_{jt}}{R_{kt}} \to \left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}}. \tag{E-23}$$

Define $\Omega_t \triangleq A_{jt}/A_{kt}$, so that

$$\Omega_t = \frac{1+\eta\gamma s_{jt}}{1+\eta\gamma(1-s_{jt})} \Omega_{t-1}. \tag{E-24}$$

Because a corner allocation cannot persist indefinitely, $\Pi^*_{jt}/\Pi^*_{kt} = 1$ for some t sufficiently large. Using this and equation (E-23) in equation (12), we have:

$$\frac{1+\eta\gamma s^*_{jt}}{1+\eta\gamma(1-s^*_{jt})} = \Omega_{t-1}^{-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Therefore, from equation (E-24),

$$\Omega_t = \Omega_{t-1}^{1-(1-\sigma)(1-\alpha)} \left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi}.$$

Define $\tilde{\Omega}_t \triangleq \ln[\Omega_t]$. We then have:

$$\tilde{\Omega}_t = [1-(1-\sigma)(1-\alpha)]\tilde{\Omega}_{t-1} + \ln \left[\left(\left\{ \frac{\nu}{1-\nu} \left[\frac{\Psi_j}{\Psi_k} \right]^{1/\psi} \right\}^{\frac{\epsilon\psi}{\epsilon+\psi}} \right)^{1+\sigma/\psi} \left[\frac{\Psi_j}{\Psi_k} \right]^{-\sigma/\psi} \right].$$

This is a linear difference equation. For $\sigma < 1$, the coefficient on $\tilde{\Omega}_{t-1}$ is strictly between 0 and 1. The linear difference equation is therefore stable. The system approaches a steady state in $\tilde{\Omega}_t$ and therefore in Ω_t. From equation (E-24), any steady state in Ω_t must have $s^*_{jt} = 0.5$. Therefore as $t \to \infty$, $s^*_{jt} \to 0.5$. We have established the first result.

Equation (E-23) implies that if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $R^*_{jt} = R^*_{kt}$. Further, if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $R^*_{jt} > R^*_{kt}$. Now substitute into equation (12) and use $s_{jt} = 0.5$:

$$
\begin{aligned}
\frac{\Pi_{jt}}{\Pi_{kt}} \to & \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\left\{\frac{\nu}{1-\nu}\left[\frac{\Psi_j}{\Psi_k}\right]^{1/\psi}\right\}^{\frac{\epsilon\psi}{\epsilon+\psi}}\right)^{\frac{1+\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{-\sigma/\psi}{\sigma+\alpha(1-\sigma)}} \\
= & \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{\frac{-(1-\sigma)(1-\alpha)}{\sigma+\alpha(1-\sigma)}} \left(\frac{\nu_j}{1-\nu_j}\right)^{\frac{\sigma+\psi}{\sigma+\alpha(1-\sigma)}\frac{\epsilon}{\epsilon+\psi}} \left(\frac{\Psi_j}{\Psi_k}\right)^{\frac{\epsilon-\sigma}{\sigma+\alpha(1-\sigma)}\frac{1}{\epsilon+\psi}},
\end{aligned}
$$

and this must equal 1 because $s^*_{jt} = 0.5$. Therefore, if $\nu_j = 0.5$ and $\Psi_j = \Psi_k$ then $A_{jt} = A_{kt}$, and if $\nu_j \geq 0.5$ and $\Psi_j \geq \Psi_k$ with at least one inequality being strict, then $A_{jt} > A_{kt}$. We have established the second and third results.

Finally, as t becomes large along a path with $s^*_{jt} = 0.5$, using previous results in equation (A-4) yields:

$$
\begin{aligned}
\left[\frac{R_{jt}}{\Psi_j}\right]^{1/\psi} \to & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{E_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\frac{R_{jt}}{E_{jt}}\right]^{-1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{Y_t}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{\kappa^{\frac{\sigma}{\sigma-1}} R_{jt}}{A_Y E_{jt}\left(\nu_j + (1-\nu_j)\left(\frac{E_{kt}}{E_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \kappa\, \nu_j A_Y^{\frac{\epsilon-1}{\epsilon}} \left[\frac{1}{A_Y\left(\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}}\right]^{-1/\epsilon} \left[\kappa^{\frac{\sigma}{\sigma-1}}\right]^{1/\sigma} \\
= & \nu_j \kappa^{\frac{\sigma}{\sigma-1}} A_Y \left[\nu_j + (1-\nu_j)\left(\frac{R_{kt}}{R_{jt}}\right)^{\frac{\epsilon-1}{\epsilon}}\right]^{\frac{1}{\epsilon-1}}.
\end{aligned}
\tag{E-25}
$$

From equation (E-23), R^*_{jt}/R^*_{kt} becomes constant as t becomes large. Then from (E-25), R^*_{jt} approaches a constant. An analogous derivation establishes that R^*_{kt} approaches a constant. We have established the final result.

E.8 Proof of Proposition 4

Let time $w \geq t_0$ be the first time after t_0 at which sector j's share of resource use begins decreasing, so that $R_{jx}/R_{kx} \leq R_{j(x+1)}/R_{k(x+1)}$ for all $x \in [t_0, w-1]$ and $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$, which in turn requires $s_{jx} \geq \hat{s}_x$ for all $x \in [t_0+1, w]$ and $s_{j(w+1)} < \hat{s}_{w+1}$. Note that $s_{jt_0} > 0.5$ implies that $A_{jt_0}/A_{kt_0} > A_{j(t_0-1)}/A_{k(t_0-1)}$. Assume that sector j's share of research begins declining sometime after its share of resource use does, so that $s_{jx} \leq s_{j(x+1)}$ for all $x \in [t_0, w]$. Then we have $A_{jx}/A_{kx} > A_{j(x-1)}/A_{k(x-1)}$ for all $x \in [t_0, w+1]$, and thus $A_{jx}/A_{kx} > [\Psi_j/\Psi_k]^\theta$ for all $x \in [t_0, w+1]$. Using this with Lemma E-8 and $\sigma < 1$ then implies $\hat{s}_{x+1} \geq 0.5$ for all $x \in [t_0, w+2]$. Combining this with the requirement that $s_{jw} \geq \hat{s}_w$, we have $s_{jw} \geq 0.5$. From equation (12) and $\sigma < 1$, we then have $s_{j(w+1)} \geq s_{jw}$ only if $R_{jw}/R_{kw} \leq R_{j(w+1)}/R_{k(w+1)}$. But that contradicts the definition of w, which required $R_{jw}/R_{kw} > R_{j(w+1)}/R_{k(w+1)}$. Sector j's share of research must have begun declining no later than time w. We have shown that a transition in resource use occurs only after a transition in research.

We now have two possibilities. We will see that the first one implies that $s_{jx} \geq 0.5$ at all times $x \in [t+1, w]$ and the second one generates a contradiction.

First, we could have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ at all times $x \in [t_0+1, w]$. Then by Lemma E-8, $\hat{s}_x \geq 0.5$ at all times $x \in [t_0+1, w]$. The definition of time w then requires $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$.

Second, we could have $A_{j(x-2)}/A_{k(x-2)} < [\Psi_j/\Psi_k]^\theta$ at some time $x \in [t_0+1, w]$. In order for this to happen, it must be true that $s_{jx} < 0.5$ at some times $x \in [t_0+2, w]$.[18] Let z be the first time at which $s_{jx} < 0.5$. $A_{j(t_0-1)}/A_{k(t_0-1)} > [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ for all $x \in [t_0, z-1]$ imply that $A_{j(z-2)}/A_{k(z-2)} > [\Psi_j/\Psi_k]^\theta$, which implies by Lemma E-8 and $\sigma < 1$ that $\hat{s}_z \geq 0.5$. So we have $s_{jz} < \hat{s}_z$, which means that $R_{j(z-1)}/R_{k(z-1)} > R_{jz}/R_{kz}$. But this contradicts the definition of time w as the first time at which sector j's share of resource use begins decreasing.

Therefore, we must have $A_{j(x-2)}/A_{k(x-2)} \geq [\Psi_j/\Psi_k]^\theta$ and $s_{jx} \geq 0.5$ at all times $x \in [t_0+1, w]$. Observe that $s_{jx} \geq 0.5$ at all times $x \in [t_0, w]$ implies $A_{jx}/A_{kx} \geq A_{j(x-1)}/A_{k(x-1)}$ at all times $x \in [t_0, w]$. We have shown that a transition in technology happens only after a transition in resource use. We have established the first part of the proposition.

Now consider the first time $z > t_0$ at which $R_{jz} < R_{kz}$. Assume that $\Psi_j \geq \Psi_k$ and that $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Assumption 1, $\Psi_j \geq \Psi_k$, and $s_{jx} \geq 0.5$ imply $A_{jx} \geq A_{kx}$ for $x \in [t_0, z]$. Using $\sigma < 1$, we see that $A_{j(z-1)} \geq A_{k(z-1)}$, $\Psi_j \geq \Psi_k$, and $R_{jz} < R_{kz}$ imply that the right-hand side of equation (E-1) is < 1 when evaluated at $s_{jz} = 0.5$. So by Corollary E-2, time z equilibrium scientists must be less than 0.5. But $s_{jz} < 0.5$ contradicts $s_{jx} \geq 0.5$ for $x \in [t_0, z]$. Therefore, if $\Psi_j \geq \Psi_k$, then there must be some time $x \in [t_0, z]$ at which $s_{jx} < 0.5$. We have shown that if $\Psi_j \geq \Psi_k$, then sector k must begin dominating research before it begins dominating resource use. We have established the second part of the proposition.

Finally, let $\nu_j = \nu_k$ and $\Psi_j = \Psi_k$. By Proposition 3, $A_{jt} = A_{kt}$ in the steady-state

[18] Recall that $s_{jt} \geq 0.5$ and $s_{j(t+1)} \geq s_{jt}$ imply $s_{j(t+1)} \geq 0.5$.

research allocation. But Assumption 1 ensures that $A_{jt_0} > A_{kt_0}$. Thus there exists $t_1 > t_0$ such that $s_{jt_1} < 0.5$. By the foregoing parts of this proposition, a transition in research, a transition in resource use, and a transition in technology must happen between t_0 and t_1. We have established the third part of the proposition.

E.9 Intermediate steps for Cobb-Douglas special case

Substituting the Cobb-Douglas forms, equation (13) becomes

$$\left[\frac{R_{jt}}{R_{kt}}\right]^{\frac{\psi+1}{\psi}-\kappa\frac{\epsilon-1}{\epsilon}} = \frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{1/\psi}\left[\frac{X_{jt}}{X_{kt}}\right]^{(1-\kappa)\frac{\epsilon-1}{\epsilon}}.$$

Substituting equation (A-1) into equation (A-2) and then using equation (2), we have:

$$X_{jt} = \left[\frac{1-\kappa}{\kappa} R_{jt}^{\frac{\psi+1}{\psi}} \Psi_j^{-1/\psi}\right]^{\alpha} A_{jt}^{1-\alpha}.$$

We then have equation (15).

E.10 Intermediate steps for Leontief special case

From equation (A-2) and $R_{jt} = X_{jt}$,

$$p_{jXt} = \left(\frac{R_{jt}}{A_{jt}}\right)^{\frac{1-\alpha}{\alpha}}.$$

Equation (17) follows from equation (6).

From equation (A-2),

$$p_{jXt}X_{jt} = X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

And from equation (2),

$$p_{jRt}R_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}}.$$

Intermediate good producers' zero-profit condition is

$$p_{jt}E_{jt} = \Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + X_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}.$$

Substituting for p_{jt} from the final good producers' first-order condition and then setting $X_{jt} = R_{jt}$ and $E_{jt} = R_{jt}$, we have:

$$\nu_j Y_t^{1/\epsilon} = A_Y^{\frac{1-\epsilon}{\epsilon}} R_{jt}^{\frac{1-\epsilon}{\epsilon}} \left[\Psi_j^{-1/\psi} R_{jt}^{\frac{1+\psi}{\psi}} + R_{jt}^{1/\alpha} A_{jt}^{-\frac{1-\alpha}{\alpha}}\right].$$

Using $\psi = \alpha/(1-\alpha)$, we have:

$$\nu_j Y_t^{1/\epsilon} = A_Y^{\frac{1-\epsilon}{\epsilon}} R_{jt}^{\frac{1-\epsilon}{\epsilon}+\frac{1}{\alpha}} \left[\Psi_j^{-\frac{1-\alpha}{\alpha}} + A_{jt}^{-\frac{1-\alpha}{\alpha}}\right].$$

An analogous result holds for sector k. Equation (16) follows.

Now consider the steady-state research allocation. For $s \in (0,1)$, $A_{j(t-1)}$ and $A_{k(t-1)}$ become arbitrarily large as t increases. From equations (16) and (17), we have:

$$\lim_{t\to\infty} \frac{\Pi_{jt}}{\Pi_{kt}} \to \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-\frac{1-\alpha}{\alpha}} \left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{-\frac{1}{\alpha}} \left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}. \tag{E-26}$$

At an equilibrium with $s \in (0,1)$, $\Pi_{jt} = \Pi_{kt}$. Then, for t sufficiently large,

$$\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{\frac{1}{\alpha}} = \left(\frac{A_{j(t-1)}}{A_{k(t-1)}}\right)^{-\frac{1-\alpha}{\alpha}} \left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}.$$

At a steady state, $A_{j(t-1)} = (1+\eta\gamma s)^{\Delta} A_{j(t-1-\Delta)}$ and $A_{k(t-1)} = (1+\eta\gamma(1-s))^{\Delta} A_{k(t-1-\Delta)}$. Therefore the following must hold for all $\Delta \geq 0$:

$$\left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{\frac{1}{\alpha}} = \left(\frac{1+\eta\gamma s}{1+\eta\gamma(1-s)}\right)^{-\Delta\frac{1-\alpha}{\alpha}} \left(\frac{A_{j(t-1-\Delta)}}{A_{k(t-1-\Delta)}}\right)^{-\frac{1-\alpha}{\alpha}} \left(\frac{\nu_j}{\nu_k}\left[\frac{\Psi_j}{\Psi_k}\right]^{\frac{1-\alpha}{\alpha}}\right)^{\frac{\epsilon}{\alpha+(1-\alpha)\epsilon}}.$$

This implies equation (18).

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