Online Appendix
"The Inverse Product Differentiation Logit Model"

Mogens Fosgerau* Julien Monardo† André de Palma‡

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Abstract

We present simulations investigating some properties of the inverse product differentiation logit (IPDL) model.

Notation We use italics for scalar variables and real-valued functions, boldface for vectors, matrices and vector-valued functions, and calligraphic for sets. $\mathbb{R}_+$ is the set of nonnegative real numbers, $\mathbb{R}_{++}$ is the set of positive real numbers, and $\mathbb{R}_{++}^{J+1} = (0,\infty)^{J+1}$. As default, vectors are column vectors: $\mathbf{s} = (s_0, \ldots, s_J)^\intercal \in \mathbb{R}^{J+1}$.

$\Delta_J \subset \mathbb{R}^{J+1}$ is the unit simplex: $\Delta_J = \left\{\mathbf{s} \in [0,\infty)^{J+1} : \sum_{j\in\mathcal{J}} s_j = 1\right\}$, and $\Delta_J^\circ = \left\{\mathbf{s} \in (0,\infty)^{J+1} : \sum_{j\in\mathcal{J}} s_j = 1\right\}$ is its relative interior.

Let $\mathbf{G} = (G_0, \ldots, G_J) : \mathbb{R}^{J+1} \to \mathbb{R}^{J+1}$ be a vector function composed of functions $G_j : \mathbb{R}^{J+1} \to \mathbb{R}$. The matrix $\mathbf{J}_{\mathbf{G}}^{\mathbf{s}}(\bar{\mathbf{s}}) \in \mathbb{R}^{(J+1)\times(J+1)}$ with entries $(i+1, j+1)$ given by $\frac{\partial G_i(\bar{\mathbf{s}})}{\partial s_j}$ denotes the Jacobian matrix of $\mathbf{G}$ with respect to $\mathbf{s}$ at point $\bar{\mathbf{s}}$.

A univariate function $\mathbb{R} \to \mathbb{R}$ applied to a vector is a coordinate-wise application of the function, e.g., $\ln(\mathbf{s}) = (\ln(s_0), \ldots, \ln(s_J))$. $|\tilde{\mathbf{s}}| = \sum_{j\in\mathcal{J}} |\tilde{s}_j|$ denotes the 1-norm of vector $\tilde{\mathbf{s}}$.

*University of Copenhagen; mogens.fosgerau@econ.ku.dk.
†University of Bristol; julien.monardo@bristol.ac.uk.
‡CY Cergy Paris Université; andre.de-palma@cyu.fr.

The IPDL Model Recall first that $d(j)$ is the set of products that are grouped with product j according to grouping characteristic d and that $s_{d(j)} = \sum_{k \in d(j)} s_k$ denotes the market share of group $d\,(j)$. To ease exposition, we omit notation for parameters $\boldsymbol{\theta}_2$ and markets t.

In the IPDL model, the matrix of derivatives of the market share function $\boldsymbol{\sigma}$ with respect to prices $\mathbf{p}$, $\mathbf{J}^{\mathbf{p}}_{\boldsymbol{\sigma}}(\boldsymbol{\delta})$, has entries $\partial\sigma_i(\boldsymbol{\delta})/\partial p_j$ equal to

$$\mathbf{J}^{\mathbf{p}}_{\boldsymbol{\sigma}}(\boldsymbol{\delta}) = -\alpha \left(\left[\mathbf{J}^{\mathbf{s}}_{\ln \mathbf{G}}(\mathbf{s})\right]^{-1} - \mathbf{s}\mathbf{s}^{\intercal} \right), \tag{1}$$

with $\mathbf{s} = \boldsymbol{\sigma}(\boldsymbol{\delta})$ and where $\mathbf{J}^{\mathbf{s}}_{\ln \mathbf{G}}(\mathbf{s})$ has entries given by

$$\frac{\partial \ln G_i(\mathbf{s})}{\partial s_j} = \begin{cases} \dfrac{1 - \sum_{d=1}^{D} \mu_d}{s_i} + \sum_{d=1}^{D} \dfrac{\mu_d}{s_{d(i)}} & \text{if } i = j > 0 \\ \sum_{d=1}^{D} \dfrac{\mu_d}{s_{d(i)}} \mathbf{1}\{j \in d(i)\} & \text{if } i \neq j,\ i > 0,\ j > 0 \\ \dfrac{1}{s_0} & \text{if } i = j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We cannot obtain closed-form formulae for the entries of the matrix of price derivatives of market share, and in turn, for the diversion ratios between products. We therefore use simulations to better understand the substitution patterns of the IPDL model. We focus on the diversion ratios.

The diversion ratio from product j to product k is the fraction of consumers who leave product j to switch to product k following a price increase of product j. It is given in percentage terms by $100(\partial\sigma_k(\boldsymbol{\delta}_t)/\partial p_{jt})/|(\partial\sigma_j(\boldsymbol{\delta}_t)/\partial p_{jt}))|$.

Simulated Data We simulate markets with 45 products and an outside good. For this, we first simulate:

- 20 different grouping structures according to 3 grouping characteristics, with 3 groups per characteristic. We obtain a grouping structure by simulating a 20×3 matrix of random numbers following a generalized Bernoulli distribution.

- 20 different vectors of grouping parameters $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_3)$. We obtain a vector of $\boldsymbol{\mu}$ by simulating a 4-vector of uniformly distributed random numbers, where the first element is μ_0, then normalizing so that $\boldsymbol{\mu} \in \Delta_3^{\circ}$. This normalization ensures that we simulate markets with very low and very high values for μ_0.

- 20 different vectors of market shares $\mathbf{s} = (s_0, \ldots, s_{45})$. We obtain a vector of market shares by simulating a 46-vector of uniformly distributed random numbers, where the first element is s_0, then by normalizing the vector of market shares of products so that $\mathbf{s} \in \Delta_{45}^{\circ}$. This normalization ensures that we simulate markets with very low and very high values for s_0.

Then, we combine the grouping structures, the grouping parameters and the market shares to form $8,000$ markets. Table 1 gives summary statistics.

Table 1: Summary Statistics on the Simulated Data

Variable	Mean	Min	Max
s_0	0.400	0.022	0.999
s	0.013	0.000	0.042
μ_0	0.624	0.019	0.950
μ_1	0.102	0.002	0.448
μ_2	0.113	0.012	0.318
μ_3	0.162	0.018	0.544

Grouping Structures Table 2 shows the distribution of the diversion ratios between products according to the number of common groups.

Diversion ratios can be either negative (complementarity in demand) or positive (substitutability in demand). Products of the same type are always substitutes in demand. Otherwise, products can be either substitutes or complements in demand. Products are more likely to be complements in demand as they become more different.

Table 2: Diversion Ratios according to the Number of Common Groups

# Common groups	Median	Mean	Complements
0 (None)	0.287	0.329	29.77%
1	1.156	1.520	4.35%
2	2.112	2.765	0.00%
3 (All)	2.780	3.904	0.00%

Notes: Column "Complements" gives the percentage of negative diversion ratios according to the number of common groups. E.g., 52.29% of the pairs of products sharing zero group are complements in demand.

Grouping Parameters Table 3 shows the distribution of diversion ratios according to the proximity of products into the characteristics space used to form product types, as measured by $\mu_{jk} = \sum_{d=1}^{3} \mu_d \mathbf{1}\{j \in d(k)\}$ for two products j and k.

As the parameter μ_{jk} becomes larger, we observe that the diversion ratios increase in values, and that the share of complements in demand decreases. This is because higher μ_d means that products of the same group according to grouping characteristic d become more similar.

Table 3: Percentage of Complements in Demand according to the Value of μ_{jk}

μ_{jk}	Median	Mean	Complements
$[0, 0.1[$	0.681	-2.872	16.89%
$[0.1, 0.2[$	1.373	1.657	1.68%
$[0.2, 0.3[$	1.849	2.196	3.21%
$[0.3, 0.4[$	2.166	2.615	3.39%
$[0.4, 0.5[$	2.822	3.246	8.21%
$[0.5, 0.6[$	3.841	4.554	0.00%
$[0.6, 0.7[$	5.350	6.533	0.00%
$[0.7, 0.8[$	6.036	6.854	0.00%
$[0.8, 0.9[$	7.770	8.253	0.00%
$[0.9, 1[$	11.79	12.76	0.00%

Notes: Column "Complements" gives the percentage of negative diversion ratios according to the number of common groups. E.g., 52.29% of the pairs of products sharing zero group are complements in demand.

Summary In the IPDL model,

1. (Grouping structure) Products of the same type are always substitutes in demand. Products of different types may be substitutes or complements in demand, depending on the degree of closeness between products as measured by the value of the parameters μ_d and by the closeness of the products into the characteristics space used to form product types. The closer two products are, the more likely they are to be substitutes in demand.

2. (Grouping parameters) The size of the diversion ratios depends on the degree of closeness. The closer two products are, the higher is their diversion ratio.