E-Comment on Equilibrium Technology Diffusion, Trade, and Growth*

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Abstract

This comment aims to identify and correct an error in the solution to the firm's value function presented in Perla et al. (2021). The original paper omits an equilibrium condition, which leads to non-exporting firms failing to internalize the option value of becoming exporters. This error does not affect the theoretical results (Propositions 1-8 in Section IV and V of the original paper), since they are derived in a stationary environment in which this option value is zero. The quantitative results under the original paper's calibration are not significantly impacted. However, I demonstrate that the omission of the equilibrium condition can have substantial effects under alternative calibrations and hence caution is advised when applying a similar model in different settings.

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1 Introduction

This comment identifies and corrects a missing equilibrium condition in the computation of a firm's value function in Perla et al. (2021). The omission implies that non-exporting firms do not internalize the option value of becoming exporters. The following sections outline the economic environment, pinpoint the omitted condition, and provide the necessary correction to the solution. The theoretical results presented in Sections IV and V in the original paper are unaffected since they are derived from a nested model not subject to the error. To illustrate the quantitative significance of the error, the model is calibrated for two cases. First, the calibration is as in the original paper, and it turns out that the mischaracterization is not quantitatively important. Second, it is shown that for alternative calibrations, the omission of the equilibrium condition can be crucial.

2 The Environment

There are N symmetric countries. A firm of normalized productivity z receives flow profits as the sum of profits in their domestic market and all foreign markets, such that

$$\pi(z) = \pi_d(z) + (N-1)\pi_x(z).$$

Normalized productivity is the productivity level divided by the minimum in the economy. Domestic and foreign normalized profits are given below. The parameter $\pi_{\min}$ is an endogenous variable that for this note will be taken as given. Each firm is producing a differentiated variety with a constant elasticity of substitution $\sigma > 1$. For a firm to export they must pay an iceberg trading cost $d \geq 1$ for transporting goods abroad and a fixed cost to operate in each foreign market. The size of the fixed cost is governed by the parameter $\kappa > 0$.

$$\begin{aligned}
\pi_d(z) &= \overline{\pi}_{\min} z^{\sigma-1} \\
\pi_x(z) &= \max\left(\overline{\pi}_{\min} d^{1-\sigma} z^{\sigma-1} - \kappa, 0\right)
\end{aligned}$$

A firm exports if they garner positive profit from doing so. Thus, a firm exports if $\pi_x(z) > 0$. This is true when a firm's normalized productivity $z > \hat{z}$ where the threshold $\hat{z}$ is given by

$$\hat{z} = d\left(\frac{\kappa}{\overline{\pi}_{\min}}\right)^{\frac{1}{\sigma-1}}. \tag{1}$$

A firm's normalized productivity is ever evolving. First, the productivity in levels evolves according to a geometric Brownian motion with deterministic drift μ and variance υ^2. Second, the model features endogenous growth at rate g, such that if a firm's productivity remains stable their relative position in the distribution deteriorates. Perla et al. (2021) demonstrate that along the balanced growth path a firm's normalized Hamilton Jacobian Bellman equation can be represented as the second order differential equation below. The parameter r is the effective discount rate of a firm.

$$(r-g)v(z) = \pi(z) + \left(\mu + \frac{\upsilon^2}{2} - g\right) z v'(z) + \frac{\upsilon^2}{2} z^2 v''(z) \tag{2}$$

3 The Problem

The solution to the differential equation must satisfy four additional conditions. They are two value matching conditions and two smooth pasting conditions, evaluated at the minimum productivity $z = 1$, and the exporting threshold $z = \hat{z}$. Formally they are:

$$v(1) = \int_1^\infty v(z) f(z) dz - \xi \qquad \lim_{z\to\hat{z}^-} v(z) = \lim_{z\to\hat{z}^+} v(z) \qquad \text{(Value Matching)}$$

$$v'(1) = 0 \qquad \lim_{z\to\hat{z}^-} v'(z) = \lim_{z\to\hat{z}^+} v'(z) \qquad \text{(Smooth Pasting)}$$

The value matching condition at the infimum determines the value of $\pi_{\min}$ in equilibrium. For the purpose of this note, the first value matching condition is ignored and it is assumed that the value ξ is such that it is satisfied. Perla et al. (2021) propose solving the differential equation (2) through the method of undetermined coefficients. The value function is segmented between exporting and non-exporting firms such that

$$v(z) = \begin{cases} v_d(z) & \text{if } z \leq \hat{z} \\ v_d(z) & \text{if } z \geq \hat{z}. \end{cases} \tag{3}$$

The candidate solution they propose takes the form below, with undetermined coefficients a, v and b.

$$v_d(z) = a\overline{\pi}_{\min}\left(z^{\sigma-1} + \frac{\sigma-1}{\mathrm{v}}z^{-\mathrm{v}}\right)$$
$$v_x(z) = a\overline{\pi}_{\min}\left(\left(1+(N-1)d^{1-\sigma}\right)z^{\sigma-1} + \frac{\sigma-1}{\mathrm{v}}z^{-\mathrm{v}} + (N-1)\frac{1}{a(r-g)}\left(bz^{-\mathrm{v}} - \frac{\kappa}{\overline{\pi}_{\min}}\right)\right)$$

Taking this guess and equating coefficients yields the following expressions for a, b and v.

$$a = \frac{1}{r-g-(\sigma-1)(\mu-g+(\sigma-1)\upsilon^2/2)}$$
$$b = (1-a(r-g))\, d^{1-\sigma}\hat{z}^{\mathrm{v}+\sigma-1}$$
$$\mathrm{v} = \frac{\mu-g}{\upsilon^2} + \sqrt{\left(\frac{g-\mu}{\upsilon^2}\right)^2 + \frac{r-g}{\upsilon^2/2}}$$

Hence, the expression for the value function can be written as

$$v_d(z) = a\overline{\pi}_{\min}\left(z^{\sigma-1} + \frac{\sigma-1}{\mathrm{v}}z^{-\mathrm{v}}\right) \tag{4}$$
$$v_x(z) = a\overline{\pi}_{\min}\left(\left(1+(N-1)d^{1-\sigma}\right)z^{\sigma-1} + \frac{\sigma-1}{\mathrm{v}}z^{-\mathrm{v}} + (N-1)\frac{1}{a(r-g)}\left(bz^{-\mathrm{v}} - \frac{\kappa}{\overline{\pi}_{\min}}\right)\right) \tag{5}$$

The paper states that "*[b]y construction, the form of these guesses ensures that value matching and smooth pasting are satisfied, both at the adoption threshold* ($z = 1$) *and the exporter threshold* ($z = \hat{z}$)." However, there is nothing implied by either the initial guess nor the solution that guarantees the interior smooth pasting condition does indeed hold at the exporter threshold ($z = \hat{z}$). To demonstrate the discontinuity in the value function, in Figure 1 the functions $v_d(z)$ and $v_x(z)$ and their first derivatives are plotted. Parameters are calibrated to coincide with the parameterization in Perla et al. (2021), see Appendix A.1 for details.

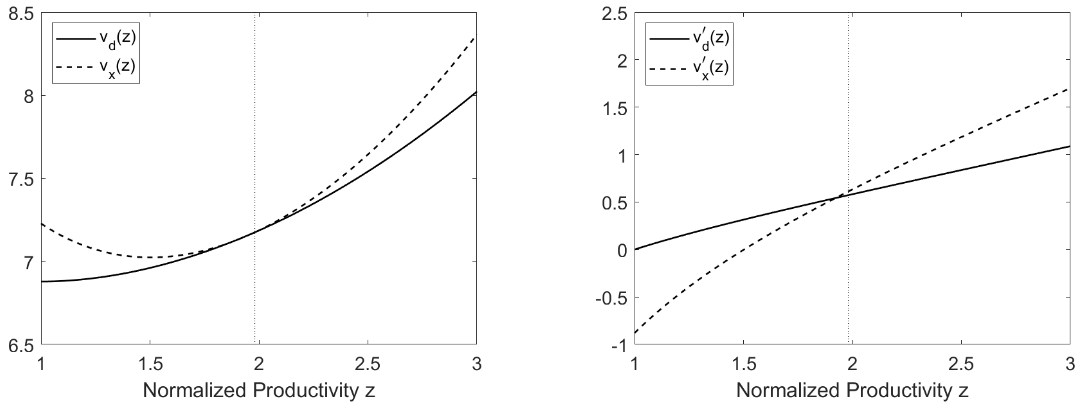
Inspection of Figure 1 highlights how close the smooth pasting condition at the threshold appears. Looking at just the first panel, without magnifying around the threshold $\hat{z}$, one could easily observe the value function $v_x(z)$ being tangential to $v_d(z)$ at $z = \hat{z}$. Once one plots the derivative of the functions it becomes apparent that the gradients coincide with a single crossing at a value of $z < \hat{z}$. Equation (6) computes the differences in the derivative of the two functions at $\hat{z}$ evaluated at the parameter values in Appendix A.1. Since this value is not zero, the second smooth pasting condition does not hold in this solution and the value function $v(z)$ thus exhibits a discontinuity.

$$\frac{\partial v_x(\hat{z})}{\partial z} - \frac{\partial v_d(\hat{z})}{\partial z} = \overline{\pi}_{\min}^{\frac{1}{\sigma-1}}(N-1)\frac{1}{d}\kappa^{\frac{\sigma-2}{\sigma-1}}\left(a(\sigma-1+\upsilon) - \frac{\upsilon}{r-g}\right) = 0.0389 \tag{6}$$

4 The Solution

In order to resolve the smooth pasting condition that does not hold in the solution in Perla et al. (2021) one needs an extra degree of freedom in the undetermined coefficients. Thus to solve the value function,

Figure 1: Value Functions and their Derivatives



The Figure plots the functions given in equation (4) and (5) and their first derivative evaluated at the parameters shown in Appenix A.1. The dashed vertical line is at the exporter threshold ($z = \hat{z}$).

the candidate solution takes the form of equations (7) and (8), where the undetermined coefficients are $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathrm{v}_1, \mathrm{v}_2)$

$$v_d(z) = \mathcal{B}_1 z^{\sigma-1} + \mathcal{B}_2 z^{-\mathrm{v}_1} + \mathcal{B}_3 z^{-\mathrm{v}_2} \tag{7}$$

$$v_x(z) = \mathcal{C}_0 + \mathcal{C}_1 z^{\sigma-1} + \mathcal{C}_2 z^{-\mathrm{v}_1} \tag{8}$$

To compute the undetermined coefficients the same approach as Perla et al. (2021) is implemented. In addition to matching coefficients it is also ensured that the value matching condition at the exporter threshold and smooth pasting conditions at the the minimum and exporter threshold are satisfied. Formally that is that,

$$v_d(\hat{z}) = v_x(\hat{z}) \quad , \quad v_d'(1) = 0 \quad \text{and} \quad v_d'(\hat{z}) = v_x'(\hat{z}).$$

The solution for the undetermined coefficients in (7) and (8) are given in equations (9) through (16).

$$\mathrm{v}_1 = \frac{-(g-\mu) + \sqrt{(g-\mu)^2 + 2\upsilon^2(r-g)}}{\upsilon^2} \tag{9}$$

$$\mathrm{v}_2 = \frac{-(g-\mu) - \sqrt{(g-\mu)^2 + 2\upsilon^2(r-g)}}{\upsilon^2} \tag{10}$$

$$\mathcal{C}_0 = -(N-1)\frac{\kappa}{r-g} \tag{11}$$

$$\mathcal{B}_1 = \overline{\pi}_{\min}\left(r - g - (\sigma-1)\left(\mu + (\sigma-1)\frac{\upsilon^2}{2} - g\right)\right)^{-1} \tag{12}$$

$$\mathcal{C}_1 = \overline{\pi}_{\min}\left(1 + (N-1)d^{1-\sigma}\right)\left(r - g - (\sigma-1)\left(\mu + (\sigma-1)\frac{\upsilon^2}{2} - g\right)\right)^{-1} \tag{13}$$

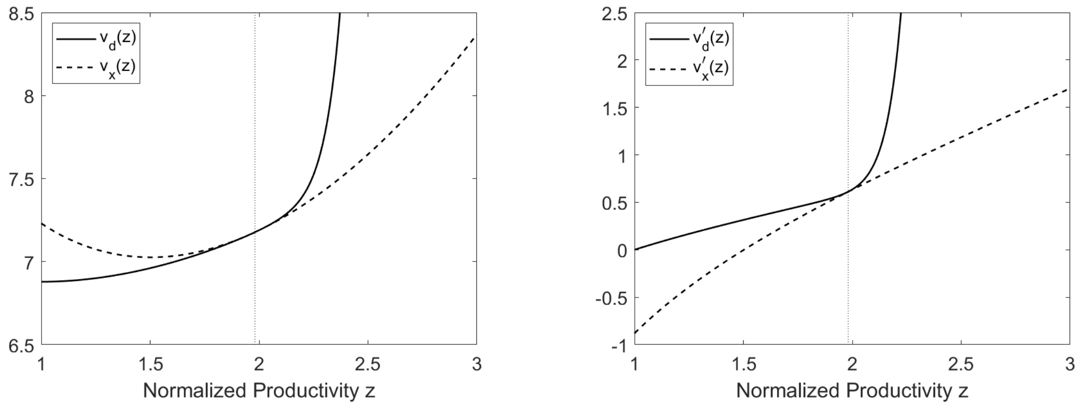
$$\mathcal{B}_2 = \frac{\mathrm{v}_2}{\mathrm{v}_2 - \mathrm{v}_1}\mathcal{C}_0 \hat{z}^{\mathrm{v}_2} + \frac{\mathrm{v}_2(\mathrm{v}_1 + \sigma - 1)}{\mathrm{v}_1(\mathrm{v}_2 - \mathrm{v}_1)}(\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\sigma + \mathrm{v}_2 - 1} + (\sigma - 1)\left(\frac{1}{\mathrm{v}_1}\right)\mathcal{B}_1 \tag{14}$$

$$\mathcal{C}_2 = \frac{1}{\mathrm{v}_1}(\sigma-1)\left((\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\sigma+\mathrm{v}_1-1} + \mathcal{B}_1 \hat{z}^{\mathrm{v}_1 - \mathrm{v}_2}\right) + \mathcal{B}_2\left(1 - \hat{z}^{\mathrm{v}_1 - \mathrm{v}_2}\right) \tag{15}$$

$$\mathcal{B}_3 = \frac{\mathrm{v}_1}{\mathrm{v}_1 - \mathrm{v}_2}\mathcal{C}_0 \hat{z}^{\mathrm{v}_2} + \frac{(\mathrm{v}_1 + \sigma - 1)}{(\mathrm{v}_1 - \mathrm{v}_2)}(\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\sigma + \mathrm{v}_2 - 1} \tag{16}$$

Figure 2 plots the value functions described in equations (7) and (8) and their first derivatives. The value function of exporting firms looks extremely similar to that in Figure 1. The value of domestic firms appears quite different, particularly in the irrelevant region $z > \hat{z}$, where the value function becomes extremely steep. Unlike Figure 1, under this solution both smooth pasting conditions hold, $v_d'(1) = 0$, and $v_d'(\hat{z}) = v_x'(\hat{z})$.

Figure 2: Value Functions and their Derivatives



The Figure plots the functions given in equation (7) and (8) and their first derivative evaluated at the parameters shown in Appendix A.1. The dashed vertical line is at the exporter threshold ($z = \hat{z}$).

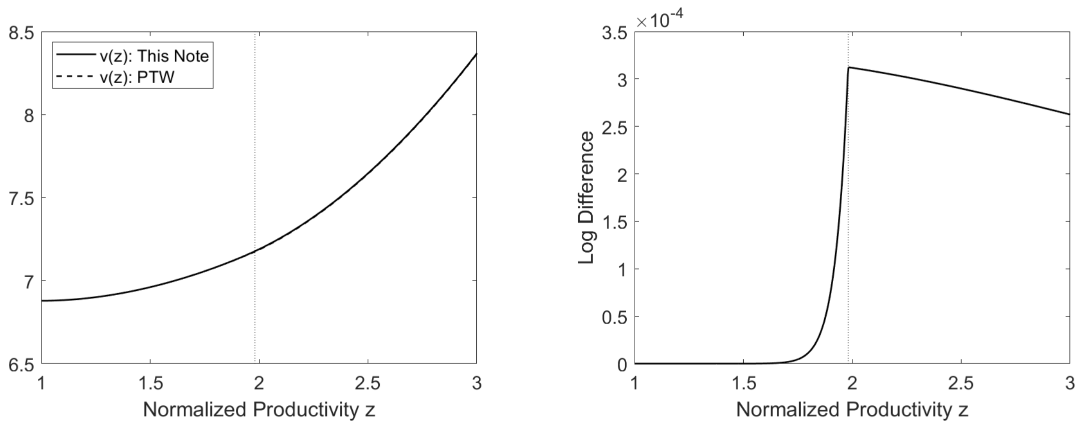
Another useful comparison is the case where the productivity of a firm remains fixed, which occurs when there is no deterministic drift ($\mu = 0$) or volatility ($\upsilon = 0$) in the geometric Brownian motion. All propositions in Perla et al. (2021) rely on this simplified nested model as an illustrative example. Appendix A.2 derives the coefficients from equations (9)–(16) under this specification. Comparing the coefficients in the appendix with those from the previous section confirms that the two models are equivalent under the stationary restriction. The economic intuition behind this equivalence is straightforward. The original paper neglects to include the option value of a non-exporting firm exporting in the future. A firm exports if its productivity z exceeds the threshold $\hat{z}$. In this model, productivity evolves exogenously, but in this setting that evolution is eliminated, preventing firms from crossing the threshold.

5 Economic Implications

Equations (7) and (8) are more convoluted than the solution in Perla et al. (2021), given by (4) and (5). The additional complexity comes through an option value of non-exporting firms. A firm not exporting today, could conceivably grow in productivity through volatility in the geometric Brownian motion. In equation (7), this is captured by the coefficient $\mathcal{B}_3$ and exponent $-\mathrm{v}_2$. Indirectly, this will also impact the value of exporting firms, since they understand that after stopping exporting there is a chance that they can export again. Since there is no $\hat{z}$ in equation (4), it is clear that this option value is not captured without taking into account the smooth pasting condition at the exporting threshold.

The left hand panel of Figure 3 plots the solutions to the value function from this note and compares with Perla et al. (2021). The two are indistinguishable when plotted in levels. The right hand panel computes the log difference between the value function here and in Perla et al. (2021). The figure implies that for a firm at the exporting threshold, the associated value is 0.03% under-computed in Perla et al. (2021). The differences in the value function are largest for firms most affected by the option value omitted. They are: firms currently exporting and non-exporting firms close to the exporting threshold. It should be stressed that these differences are negligible and will have very little impact on the quantitative results in Perla et al. (2021). However, in an alternative calibration the different value function could indeed be consequential. In this setting, the option value of a currently non-exporting firm to commence exporting is minimal as it is an extremely unlikely event. Recall, the productivity z is normalized. In levels, the exporting threshold is growing at a rate g, and a firm's productivity on average deteriorates since the drift $\mu < 0$. If one calibrates a similar model with larger volatility υ^2, and positive drift then a non-exporting firm to commence exporting increases in likelihood and omitting the interior smooth pasting condition can be consequential quantitatively. To demonstrate this point Figure 4 compares the two value functions in an alternative calibration where half of growth is driven by exogenous growth in firms $\mu = g/2 = 0.395$, and the volatility parameter $\upsilon = 0.4382$ is twice the value in Appendix A.1. In this hypothetical scenario, the likelihood of moving up the productivity

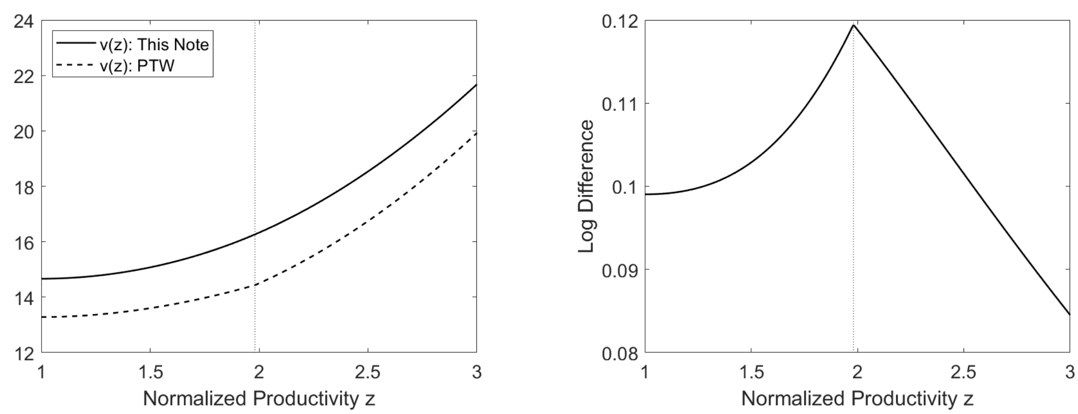
Figure 3: Value Function Comparison



The left hand panel plots the value function, equation (3) in Perla et al. (2021) and this note. The right hand panel is the log difference between the two.

ladder is significantly increased. Consequently, there are large differences between the two value functions, peaking at the exporting threshold, the value function in Perla et al. (2021) would understate the value by approximately 12%.

Figure 4: Value Function Comparison: Alternative Calibration



The left hand panel plots the value function, equation (3) in Perla et al. (2021) and this note. The calibration is purely hypothetical and follows Appendix A.1 with two modifications. The drift parameter $\mu = g/2$, and the value of $\upsilon = 0.4382$. The right hand panel is the log difference between the two value functions.

6 Conclusion

In conclusion, this comment identifies an error in the solution to the firm's value function in Perla et al. (2021), specifically an equilibrium condition that affects non-exporting firms' consideration of the option value in beginning to export. Although this oversight does not significantly alter the results of the original calibration, its importance becomes pronounced when firms experience large individual shocks to their productivity and when their individual growth rates closely align with the aggregate growth rate. Under these circumstances, the missing equilibrium condition can substantially influence the model's outcomes, highlighting the need for careful calibration in such scenarios. Thus, while the main quantitative conclusions of the original paper remain intact, researchers should exercise caution and consider the potential impact of this omission in alternative economic environments.

A Appendix

A.1 Calibration

All of the parameters below are taken from Table 1 of Perla et al. (2021) with the exception of the growth rate g and the share of exporters which are taken from their empirical counterparts in Table 2. The parameters are as follows:

- Drift of GBM process $\mu = -0.031$
- Variance of GBM process $\upsilon^2 = 0.048$
- Variety elasticity of substitution, $\sigma = 3.17$
- Number of countries, $N = 10$
- Iceberg trade cost, $d = 3.02$
- Export fixed cost, $\kappa = 0.104$
- Growth rate of the economy $g = 0.79$
- Effective discount rate $r = \rho + g + \delta = 0.0203 + 0.79 + 0.02 = 0.8303$
- Exporting threshold, $\hat{z} = (\text{Share of Exporting Firms})^{-1/\theta} = 0.033^{-1/4.99} = 1.981$
- Minimum productivity $\pi_{\min} = \kappa \left(\frac{\hat{z}}{d}\right)^{1-\sigma} = 0.2597$

A.2 Stationary Productivity

The following are derived by setting the drift $\mu = 0$, taking the limit as volatility $\upsilon \to 0$, and substituting the expression for the exporter threshold as $\hat{z} = d\left(\frac{\kappa}{\overline{\pi}_{\min}}\right)^{\frac{1}{\sigma-1}}$.

$$
\begin{aligned}
\mathrm{v}_1 &= \frac{r-g}{g} \\
\mathrm{v}_2 &= -\frac{r-g}{g} \\
\mathcal{C}_0 &= -(N-1)\frac{\kappa}{r-g} \\
\mathcal{B}_1 &= \frac{\overline{\pi}_{\min}}{r-(2-\sigma)g} \\
\mathcal{C}_1 &= \frac{\overline{\pi}_{\min}}{r-(2-\sigma)g}\left(1+(N-1)d^{1-\sigma}\right) \\
\mathcal{B}_2 &= \overline{\pi}_{\min}\cdot\frac{(\sigma-1)g}{(r-g)(r-(2-\sigma)g)} \\
\mathcal{C}_2 &= \overline{\pi}_{\min}\cdot\frac{(\sigma-1)g}{(r-g)(r-(2-\sigma)g)}\left((N-1)\frac{\kappa}{\overline{\pi}_{\min}}\hat{z}^{\mathrm{v}_1}+1\right) \\
\mathcal{B}_3 &= 0
\end{aligned}
$$

References

Perla, J., C. Tonetti and M. E. Waugh, “Equilibrium Technology Diffusion, Trade, and Growth,” *American Economic Review* 111(1) (2021), 73–128.