Corrigendum to “A Theory of Crowdfunding: A Mechanism Design Approach with Demand Uncertainty and Moral Hazard”

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1 Introduction

This corrigendum corrects a mistake in proving the optimality of a crowdfunding contract in Strausz (2017). In particular, the corrigendum proves that there is no loss in assuming that an optimal mechanism in Strausz (2017) does not leave any (information) rents to consumers *conditional on the project's cost structure* (I, c). As pointed out by Ellman and Hurkens (2017), Lemma 5 in Strausz (2017) falsely claims a stronger version: consumer rents are zero conditional on (I, c, l), ie, also conditional on the realization of a randomized contract realization l. The corrigendum corrects this mistake and thereby shows that it is inconsequential for the paper's results concerning the optimality of crowdfunding contracts.

This corrigendum is to replace subsection III.B and the associated proofs that appeared in the appendix. The numbering of equations and references in this corrigendum follow Strausz (2017).

B. Optimal Allocations and Mechanisms

A (possibly constrained) efficient mechanism $\breve{\Gamma} = \{(\breve{p}_l, \breve{t}_l, \breve{x}_l)\}_{l\in\mathcal{L}}$ maximizes $S^{\breve{\Gamma}}$ subject to constraints (21)–(29). In order to solve this maximization problem, we follow the usual approach in mechanism design to focus first on a relaxed maximization problem. In particular, we replace the entrepreneur's truthtelling constraint (27) by

$$\Pi^{\Gamma}(I, c) \geq \sum_{T\in\mathcal{T}^{\Gamma}(I',c')} P^{\Gamma}(T|I', c')\alpha T \;\forall(I, c, I', c'). \tag{30}$$

The constraint is weaker than (27), because its right-hand side is larger than the right-hand side of (30), whereas their left-hand sides are identical.[1]

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[1]Referring to (16), it considers only one element within the maximum operator and for $x_0(I', c', v) = 0$, constraint (22) implies $\Pi^{\gamma_l}(I', c'|I, c, v) = \sum_{i\in\mathcal{N}}[t^a_{li}(I', c', v) + t^p_{li}(I', c', v)] \geq 0$.

Formally, we say that Γ is *weakly feasible* if it satisfies constraints (21)–(26), and (28)-(30), and an output schedule $\breve{x} : \mathcal{K}\times\mathcal{V} \to \Delta\mathcal{X}$ is *weakly-implementable* if there exists a weakly feasible mechanism $\breve{\Gamma}$ that implements it. A weakly feasible mechanism $\breve{\Gamma}$ is *optimal* if it maximizes S^{Γ} over all weakly feasible mechanisms.

In the following, we derive an optimal weakly feasible mechanism $\breve{\Gamma}$ with the feature that it is also (strictly) feasible. Hence, it also represents a constrained efficient mechanism $\breve{\Gamma}$. In particular, we show that such a mechanism is a crowdfunding mechanism, i.e., there is a threshold function $T(I, c)$ so that all the deterministic mechanisms γ_l in $\breve{\Gamma}$ satisfy (10)–(13).

We first derive a series of lemmas that allow us to simplify the maximization problem. The first lemma establishes the relatively intuitive result that development-efficiency is a necessary feature of optimal weakly feasible mechanisms.

Lemma 2 *A weakly feasible mechanism $\breve{\Gamma} = \{(\breve{p}_l, \breve{t}_l, \breve{x}_l)\}_{l\in\mathcal{L}}$ is optimal only if each $\breve{x}_l$ is development-efficient.*

The next lemma validates the suggestion of the previous section that, in order to optimally control entrepreneurial moral hazard, a mechanism uses deferred payments and limits the entrepreneur's information. In particular, it shows that development-efficiency is a sufficient condition under which it is optimal to initially provide the entrepreneur only with the investment amount I and, hence, minimize the information which she gleans from receiving a recommendation to invest. The result is an illustration of Myerson's general observation that, accompanying a recommendation, mediators should give agents only the minimum information possible, as more information only makes it harder to satisfy incentive compatibility.

Lemma 3 *Suppose $\breve{\Gamma} = \{(\breve{p}_l, \breve{t}_l, \breve{x}_l)\}_{l\in\mathcal{L}}$ is weakly feasible and $\{\breve{x}_l\}_{l\in\mathcal{L}}$ are development-efficient. Then there are transfer schedules $\{\hat{t}_l\}_{l\in\mathcal{L}}$ such that* (21) *binds and the direct mechanism $\hat{\Gamma} = \{(\breve{p}_l, \hat{t}_l, \breve{x}_l)\}_{l\in\mathcal{L}}$ is weakly feasible and payoff equivalent, and* (22) *simplifies to*

$$\textstyle\sum_{i\in\mathcal{N}} t^p_{li}(I, c, v) \geq c \sum_{i\in\mathcal{N}} x_{li}(I, c, v) \; \forall (l, I, c, v) \in \mathcal{L}\times\mathcal{K}\times\mathcal{V}. \tag{31}$$

Because Lemma 2 shows that an optimal weakly feasible mechanism is development-efficient, there is no loss of generality in restricting attention to weakly feasible direct mechanisms that give the entrepreneur exactly the amount I if the entrepreneur is to develop the product.

Combining Lemmas 2 and 3 allows us to considerably simplify the optimization problem. Indeed, if the feasibility constraint (21) binds then $\mathcal{T}^{\Gamma}(I, c) = \{I\}$ so that the obedience constraint (26) has to hold only with regard to $T = I$. By defining, for an output schedule $x \in \mathbb{R}^{n+1}$, the set and probability

$$\mathcal{V}^x(I, c) \equiv \{v | x_0(I, c, v) = 1\} \text{ and } \pi^x(I, c) = \textstyle\sum_{v\in\mathcal{V}^x(I,c)} \pi(v),$$

the obedience constraint (26) simplifies to

$$\begin{aligned} &\textstyle\sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{x_l}(I,c)} \sum_{i\in\mathcal{N}} p_l \pi(v)(t^p_{li}(I, c, v) - c x_{li}(I, c, v)) \\ &\geq \textstyle\sum_{l\in\mathcal{L}} p_l \pi^{x_l}(I, c) \alpha I \; \forall (I, c) \in \mathcal{K}; \end{aligned} \tag{32}$$

and the relaxed truthfulness constraint (30) to

$$\Pi^{\Gamma}(I,c) \geq \pi^{\Gamma}(I',c')\alpha I' \; \forall (I,c,I',c') \in \mathcal{K}\times\mathcal{K}, \tag{33}$$

where $\pi^{\Gamma}(I,c) \equiv \sum_{l\in\mathcal{L}} p_l \pi^{x_l}(I,c)$.

Following the previous two lemmas, there is no loss of generality to focus on weakly feasible mechanisms $\check{\gamma} = (\check{t},\check{x})$ that satisfy (23), (24), (25), (28), (29), (31), (32), and (33), and (21) in equality. Given this observation, we next prove that optimal weakly feasible mechanisms do not produce a product for consumers who do not value them.

Lemma 4 *A weakly feasible mechanism $\check{\Gamma} = \{(\check{p}_l,\check{t}_l,\check{x}_l)\}_{l\in\mathcal{L}}$ is optimal only if it holds that*

$$x_{il}(I,c,0,v_{\text{-}i}) = 0 \; \forall (l,i,I,c,v_{\text{-}i}) \in \mathcal{L}\times\mathcal{N}\times\mathcal{K}\times\mathcal{V}_{\text{-}i}. \tag{34}$$

The result sounds intuitive, since it implies that an optimal weakly feasible mechanism does not display any form of artificial inefficiency. It is, however, not immediate because, in general, artificial inefficiencies may help to relax incentive constraints. The next lemma shows that it also implies that there is no loss of generality in assuming that an optimal weakly feasible mechanism leaves no rents to consumers.

Lemma 5 *Suppose a weakly feasible $\check{\Gamma} = \{(\check{p}_l,\check{t}_l,\check{x}_l)\}_{l\in\mathcal{L}}$ satisfies constraints* (9), (31)–(34), *and* (21) *in equality. Then there exists a weakly feasible mechanism $\hat{\Gamma} = \{(\check{p}_l,\hat{t}_l,\check{x}_l)\}_{l\in\mathcal{L}}$ which yields the same aggregate surplus $S^{\check{\Gamma}}$ and satisfies the additional constraints* (9), (31)– (34) *and the constraints* (21), (28), *and* (29) *in equality. Moreover if* (28) *and* (29) *are satisfied in equality, then* (34) *implies* (24) *and* (25).

The lemma provides the insight that optimal weakly feasible mechanisms extract all rents from consumers and assign them as revenues to the entrepreneur. The intuition as to why this rent extraction is optimal follows directly from the moral hazard problem: by giving all rents in the form of deferred payments to the entrepreneur, she has the least incentives to run with the money.

As we show in the next lemma, the rent extraction result implies that there is no conflict between maximizing the aggregate surplus and maximizing the entrepreneur's ex ante expected profits. In order to make this statement explicit, define for a mechanism $\Gamma = \{(p_l,\gamma_l)\}_{l\in\mathcal{L}} = \{(p_l,t_l,x_l)\}_{l\in\mathcal{L}}$ the entrepreneur's ex ante expected profits as

$$\Pi^{\Gamma} = \sum_{(I,c)\in\mathcal{K}} \rho(I,c)\Pi^{\Gamma}(I,c),$$

where $\Pi^{\Gamma}(I,c)$ represents the equilibrium profit in cost state (I,c):

$$\Pi^{\Gamma}(I,c) = \sum_{l\in\mathcal{L}} p_l \Pi^{\gamma_l}(I,c);$$

and the aggregate surplus in the cost state (I,c) as

$$S^{\Gamma}(I,c) = \sum_{l\in\mathcal{L}} p_l S^{x_l}(I,c).$$

Lemma 6 *It is without loss of generality to assume that an optimal weakly feasible mechanism* $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ *maximizes the entrepreneur's ex ante expected profits* Π^{Γ}*, and exhibits* $\Pi^{\check{\Gamma}}(I, c) = S^{\check{\Gamma}}(I, c)$ *for all* (I, c)*.*

To summarize, Lemmas 2 to 6 imply that, with respect to the optimal weakly feasible mechanism, it is without loss of generality to replace the constraints (21)–(29) with the following constraints:

$$\textstyle\sum_{i\in\mathcal{N}} t^a_{li}(I, c, v) = x_{l0}(I, c, v)I \;\forall(l, I, c, v); \tag{36}$$
$$\textstyle\sum_{i\in\mathcal{N}} t^p_{li}(I, c, v) \geq c\sum_{i\in\mathcal{N}} x_{li}(I, c, v) \;\forall(l, I, c, v); \tag{37}$$
$$\exists i \in \mathcal{N} : x_{li}(I, c, v) = 1 \Rightarrow x_{l0}(I, c, v) = 1 \;\forall(l, I, c, v); \tag{38}$$
$$U^{\Gamma}_i(I, c|1) = 0 \;\forall(i, I, c); \tag{39}$$
$$U^{\Gamma}_i(I, c|0) = 0 \;\forall(i, I, c); \tag{40}$$
$$\sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{x_l}(I,c)} \sum_{i\in\mathcal{N}} p_l\pi(v)(t^p_{li}(I, c, v) - cx_{li}(I, c, v)) \geq \pi^{\Gamma}(I, c)\alpha I \;\forall(I, c); \tag{41}$$
$$x_{li}(I, c, 0, v_{\text{-}i}) = 0 \;\forall(l, i, I, c, v_{\text{-}i}); \tag{42}$$
$$S^{\Gamma}(I, c) \geq \pi^{\Gamma}(I', c')\alpha I' \;\forall(I, c, I', c'). \tag{43}$$

Constraint (43) effectively represents the entrepreneur's incentive constraint (18). The insight that the mechanism leaves all rents to the entrepreneur in order to optimally deal with the entrepreneur's moral hazard problem, enables us to rewrite this constraint as depending only on output schedules and not on transfers.

Since the deterministic version of this constraint turns out to play a key role for implementability, we say that an output schedule $x \in \mathbb{R}^{n+1}$ is *affluent* if for all $(I, c) \in \mathcal{K}$ it holds:

$$S^x(I, c) \geq \Phi(x) \equiv \max_{(\tilde{I},\tilde{c})\in\mathcal{K}} \alpha\pi^x(\tilde{I}, \tilde{c})\tilde{I}. \tag{44}$$

We moreover denote by $(\bar{I}(x), \bar{c}(x))$ a maximizer of the right-hand side of (44). Note that for a deterministic mechanism $\Gamma = (1, \gamma_1) = (1, x_1, t_1)$, constraint (43) amounts to the requirement that x_1 is affluent. This leads to the following result.

Proposition 2 *The efficient output schedule* x^* *is implementable if and only if it is affluent. If implementable, a crowdfunding mechanism implements it and thereby maximizes both aggregate surplus and the entrepreneur's ex ante expected profits.*

The proposition identifies affluency as the crucial condition: it is both necessary and sufficient for the implementability of the efficient output schedule. The intuition behind this result is that the entrepreneur needs to receive a rent of at least $\Phi(x^*)$ to induce her to invest properly rather than employing the combined strategy of misreporting her cost structure and, subsequently, taking the money and running. Since the consumers ultimately pay this rent, the project then has to generate a surplus of at least $\Phi(x^*)$ so that the consumers' participation is still individual rational. The efficient output schedule x^*, however, only guarantees such a surplus if it is affluent.

More generally, we can interpret the required rent $\Phi(x)$ as the *agency costs* of implementing some output schedule x. To obtain more insights concerning the extent to which moral hazard and private cost information are responsible for these agency costs, note that if the entrepreneur cannot falsify her cost structure, the output schedule x induces the entrepreneur to invest if

$$S^x(I,c) \geq \Phi^m(x) \equiv \alpha \cdot \pi^x(I,c)I.$$

This suggests interpreting $\Phi^m(x)$ as the agency cost associated with moral hazard and the remaining part,

$$\Phi^i(x) \equiv \Phi(x) - \Phi^m(x) = \alpha \cdot \left[\pi^x(\bar{I}(x), \bar{c}(x))\bar{I}(x) - \pi^x(I,c)I\right] \geq 0,$$

as the agency cost associated with private information about the cost structure.

The proposition further shows that if there is no moral hazard problem ($\alpha = 0$), the efficient output schedule is implementable even if the entrepreneur has private information about the cost structure. In this case, agency costs $\Phi^m(x)$ and $\Phi^i(x)$ are both zero. Hence, private cost information alone does not lead to distortions in crowdfunding. This observation formalizes the insight of Section 2 that entrepreneurial moral hazard is a first-order problem in crowdfunding while private cost information is of second order.

It also demonstrates that the presence of private cost information does not alter the intuition behind the inefficiency result of Proposition 1. Effectively, the existence of a tension between the entrepreneur's budget constraint and the moral hazard problem remains solely responsible for the inefficiencies, and prevents the implementability of the efficient output.

Yet, even though private cost information by itself cannot lead to an inefficiency, it does, however, intensify the moral hazard problem. This is because with private cost information, consumers have to grant enough rents to prevent the *double* deviation of the entrepreneur combining lies about the cost structure with the intent to take the money and run. In the extreme, this multiplier effect destroys all potential benefits from crowdfunding. In particular, if there is a cost structure (I,c) in $\mathcal{K}$ for which $S^{x^*}(I,c) = 0$, then an affluent output schedule necessarily exhibits $\pi^x(\tilde{I},\tilde{c}) = 0$ for all $(\tilde{I},\tilde{c}) \in \mathcal{K}$. This means that crowdfunding is ineffective: for any demand realization and any cost structure, implementability implies $x_0 = 0$.

We next address the question of which constrained efficient output schedule is optimal when the efficient output schedule is not affluent. Note that affluency is a necessary condition for an implementable output schedule x. Hence, an intuitive approach toward finding the constrained efficient output level is to start with the efficient output x^* and adapt it to make it affluent. Because the efficient output x^* maximizes $S^x(\cdot)$ and, hence, the left-hand side of (44), such an adaptation requires a change in x that lowers its right-hand side. That is, the output schedule should decrease $\pi^x(\cdot)$. Effectively, this means lowering the likelihood that the entrepreneur will receive a recommendation to invest when reporting the cost structure $(\bar{I}(x), \bar{c}(x))$. Intuitively, this change reduces the profitability of the double deviation to misreport the cost structure as $(\bar{I}(x), \bar{c}(x))$ and subsequently take the money and run.

The required adaptation of x^* implies a downward distortion of the output schedule: the constrained efficient mechanism has to recommend the entrepreneur not to invest for some demand revelations that yield a positive surplus. Hence, lowering π^x comes at the cost of underinvestment. These costs are minimized when the mechanism makes the inefficient recommendation not to invest for those demand realizations that yield the least surplus. In terms of crowdfunding, this means that the crowdfunding target T is raised above the efficient one, as the demand realizations closest to target yield the least.

The reasoning to adapt x^* toward some affluent output schedule suggests that also the constrained efficient mechanism is a crowdfunding mechanism, but with an inefficiently high target T. Since the adaptation away from x^* comes at a cost, the crowdfunding target should be raised such that the affluency constraint (44) is just met. Due to the discreteness of the problem, this is generally not possible with deterministic output schedules. As a consequence, we cannot exclude the possibility that the optimal mechanism is stochastic and displays the minor form of randomness in that it randomizes between two crowdfunding schemes such that the affluency constraint is satisfied with equality.

In two steps, we formally confirm that the heuristic arguments presented above are correct. In a first lemma, we show that optimal weakly feasible mechanisms necessarily exhibit a single cut-off T for each cost structure (I, c). This implies that crowdfunding mechanisms implement them. Proposition 3 then shows that these weakly feasible mechanisms are actually (strictly) feasible.

Lemma 7 *A weakly feasible mechanism* $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l \in \mathcal{L}}$ *that satisfies* (36)–(43) *is optimal only if for each* $(I, c) \in \mathcal{K}$ *there exists some* $T \in \mathcal{N}$ *such that for all* $(l, i, v) \in \mathcal{L} \times \mathcal{N} \times \mathcal{V}$ *it holds:*

$$\check{x}_{l0}(I, c, v) = \begin{cases} 1 & \textit{if } n(v) > T, \\ 0 & \textit{if } n(v) < T; \end{cases} \quad \textit{and} \quad \check{x}_{li}(I, c, v) = \begin{cases} v_i & \textit{if } n(v) > T, \\ 0 & \textit{if } n(v) < T. \end{cases} \tag{45}$$

The next proposition shows that any output schedule that satisfies (45), is actually implementable by a (strictly) feasible mechanism that, in addition to (36)–(43), also satisfies properties (11)–(13).

Proposition 3 *If the efficient output* x^* *is not affluent, the optimal allocation is constrained efficient. A crowdfunding mechanism implements it and thereby also maximizes the entrepreneur's ex ante expected profits.*

A Appendix

Proof of Lemma 2 Consider a weakly feasible mechanism $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l \in \mathcal{L}}$ with some $\check{x}_l$ that is not development-efficient. That is, $\check{\Gamma}$ satisfies (21)–(26), and (28)-(30), and there exists a combination $(\tilde{I}, \tilde{c}, \bar{v})$ such that $\check{x}_{l0}(\tilde{I}, \tilde{c}, \bar{v}) = 1$ and $\check{x}_{li}(\tilde{I}, \tilde{c}, \bar{v}) = 0$ for all $i \in \mathcal{N}$. Lowering $\check{x}_{l0}(\tilde{I}, \tilde{c}, \bar{v})$ to zero raises the objective S^{Γ} by $p_l \rho(\tilde{I}, \tilde{c}) \pi(\bar{v}) \tilde{I}$. We show that this change yields a weakly feasible Γ', and as a result $\check{\Gamma}$ is not optimal. To show that Γ' is weakly feasible, we show that it satisfies (21)–(26), and (28)-(30), given that $\check{\Gamma}$ satisfies these constraints. Note first that the change does not affect any of the constraints (24), (25), (28), and (29), while it affects (21) and (22) only for $(l, \tilde{I}, \tilde{c}, \bar{v})$ by lowering the right-hand side by $\tilde{I}$. Hence, these constraints remain satisfied. Note further that because $\check{x}_{li}(\tilde{I}, \tilde{c}, \bar{v}) = 0$ for all $i \in \mathcal{N}$, (23) is vacuous for $(l, \tilde{I}, \tilde{c}, \bar{v})$ so that the change does not affect it. Moreover, the change only affects (26) for $(\tilde{I}, \tilde{c}, \bar{v})$ by raising the left-hand side and, hence, it remains satisfied. Finally, the change also keeps (30) satisfied, because it raises $\Pi^{\check{\Gamma}}(I, c)$, i.e., the left-hand side, while it lowers $P^{\check{\Gamma}}(T|\tilde{I}, \tilde{c})$, i.e., the right-hand side. Q.E.D.

Proof of Lemma 3 Fix a weakly feasible $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ with $\check{x}_1, \dots, \check{x}_L$ development-efficient. Define for each (l, I, c, v):

$$K_l(I,c,v) \equiv \sum_{i\in\mathcal{N}} \check{t}^a_{li}(I,c,v) - I\check{x}_{l0}(I,c,v).$$

Since $\check{\Gamma}$ is weakly feasible, (21) implies that $K_l(I,c,v) \geq 0$ for all (l,I,c,v). For any (l,I,c,v), let $n_l(I,c,v) \equiv \sum_{i\in\mathcal{N}} \check{x}_{li}(I,c,v)$ represent the total number of consumers with $x_i = 1$. For any (l,I,c,v) with $\check{x}_{l0}(I,c,v) = 0$, define $\hat{t}^a_{li}(I,c,v) \equiv 0$ and $\hat{t}^p_{li}(I,c,v) \equiv \check{t}^a_{li}(I,c,v) + \check{t}^p_{li}(I,c,v)$. Similarly, for $\check{x}_{l0}(I,c,v) = 1$ define $\hat{t}^a_{li}(I,c,v) \equiv \check{t}^a_{li}(I,c,v) - \check{x}_{li}(I,c,v)K_l(I,c,v)/n_l(I,c,v)$ and $\hat{t}^p_{li}(I,c,v) \equiv \check{t}^p_{li}(I,c,v) + \check{x}_{li}(I,c,v)\ K_l(I,c,v)/n_l(I,c,v)$. Since $\check{\Gamma}$ is weakly feasible and $\check{x}_l$ is development-efficient, it holds $n_l(I,c,v) > 0$ if and only if $\check{x}_{l0}(I,c,v) = 1$. Hence, the transformed transfer schedule $\hat{t}$ is well defined.

By construction, $\sum_{i\in\mathcal{N}} \hat{t}^a_{li}(I,c,v) = 0$ for any (l,I,c,v) with $\check{x}_{l0}(I,c,v) = 0$, and $\sum_{i\in\mathcal{N}} \hat{t}^a_{li}(I,c,v) = \sum_{i\in\mathcal{N}} \check{t}^a_{li}(I,c,v) - \check{x}_{li}(I,c,v)K_l(I,c,v)/n_l(I,c,v) = \sum_{i\in\mathcal{N}} \check{t}^a_{li}(I,c,v) - K_l(I,c,v) = I$ for any (l,I,c,v) with $\check{x}_{l0}(I,c,v) = 1$. Hence, $(\hat{t}, \check{x}_l)$ satisfies (21) in equality. We show that, because $\check{\Gamma}$ is weakly feasible, $\hat{\Gamma} = \{(\check{p}_l, \hat{t}_l, \check{x}_l)\}$ is weakly feasible. To see this, note first that—because $\hat{t}^a_{li}(I,c,v) + \hat{t}^p_{li}(I,c,v) = \check{t}^a_{li}(I,c,v) + \check{t}^p_{li}(I,c,v)$ for all (l,I,c,v)—the change from $\check{\Gamma}$ to $\hat{\Gamma}$ leaves all constraints (22)–(25) and (28)–(29) unaffected. We therefore only have to check that $\hat{\Gamma}$ remains to satisfy (26) and (30).

In order to show that $\hat{\Gamma}$ satisfies (26), first note that, by construction of $\hat{t}_l$, for all (l,I,c) we have

$$v \in \mathcal{V}^{\hat{\gamma}_l}(I|I,c) \Leftrightarrow \exists T \in \mathcal{T}^{\check{\Gamma}}(I,c)\colon v \in \mathcal{V}^{\check{\gamma}_l}(T|I,c).$$

Hence, for all (l,I,c) we have

$$\{(v,l)|v \in \mathcal{V}^{\hat{\gamma}_l}(I|I,c)\} = \{(v,l)|\exists T \in \mathcal{T}^{\check{\Gamma}}(I,c)\colon v \in \mathcal{V}^{\check{\gamma}_l}(T|I,c)\}, \tag{A7}$$

which for all (l,I,c) implies

$$\sum_{v\in\mathcal{V}^{\hat{\gamma}_l}(I|I,c)} \pi(v) = \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I,c)} \sum_{v\in\mathcal{V}^{\check{\gamma}_l}(T|I,c)} \pi(v).$$

Multiplying by p_l, summing over l, and rearranging terms yields

$$\begin{aligned} P^{\hat{\Gamma}}(I|I,c) &= \sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{\hat{\gamma}_l}(I|I,c)} p_l\pi(v) = \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I,c)} \sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{\check{\gamma}_l}(T|I,c)} p_l\pi(v) \\ &= \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I,c)} P^{\check{\Gamma}}(T|I,c). \end{aligned} \tag{A8}$$

Note that, by definition of Π^{Γ}_o,

$$P^{\check{\Gamma}}(T|I,c)\Pi^{\check{\Gamma}}_o(T|I,c,I,c) = \sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{\check{\gamma}_l}(T|I,c)} p_l\pi(v)\Pi^{\check{\gamma}_l}(I,c|I,c,v).$$

Because $\check{\Gamma}$ satisfies (26), a multiplication of (26) by $P^{\check{\Gamma}}(T|I,c)$ yields

$$\sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{\check{\gamma}_l}(T|I,c)} p_l\pi(v)\Pi^{\check{\gamma}_l}(I,c|I,c,v) \geq P^{\check{\Gamma}}(T|I,c)\alpha T\ \forall T \in \mathcal{T}^{\check{\Gamma}}(I,c).$$

Summing over $T \in \mathcal{T}^{\check{\Gamma}}(I,c)$ and noting that $T \geq I$ yields after an exchange of sums,

$$\sum_{l\in\mathcal{L}} \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I,c)} \sum_{v\in\mathcal{V}^{\check{\gamma}_l}(T|I,c)} p_l\pi(v)\Pi^{\check{\gamma}_l}(I,c|I,c,v) \geq \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I,c)} P^{\check{\Gamma}}(T|I,c)\alpha I.$$

Using (A7), $\Pi^{\hat{\gamma}_l}(I,c|I,c,v) = \Pi^{\check{\gamma}_l}(I,c|I,c,v)$, and (A8) yields

$$\sum_{l\in\mathcal{L}} \sum_{v\in\mathcal{V}^{\hat{\gamma}_l}(I|I,c)} p_l\pi(v)\Pi^{\hat{\gamma}_l}(I,c|I,c,v) \geq P^{\hat{\Gamma}}(I|I,c)\alpha I.$$

Dividing both sides by $P^{\hat{\Gamma}}(I|I,c)$ shows that $\hat{\Gamma}$ satisfies (26), since $\mathcal{T}^{\hat{\Gamma}}(I,c) = \{I\}$.

Moreover, since $\check{\Gamma}$ satisfies (30) and, for any $T \in \mathcal{T}^{\check{\Gamma}}(I,c)$, we have $T \geq I$ and $\mathcal{T}^{\hat{\Gamma}}(I,c) = \{I\}$, it follows for all $(I,c,I',c') \in \mathcal{K}\times\mathcal{K}$ that, by (A8),

$$\begin{aligned}\Pi^{\hat{\Gamma}}(I,c) = \Pi^{\check{\Gamma}}(I,c) &\geq \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I',c')} P^{\check{\Gamma}}(T|I',c')\alpha T \\ &\geq \sum_{T\in\mathcal{T}^{\check{\Gamma}}(I',c')} P^{\check{\Gamma}}(T|I',c')\alpha I' = P^{\hat{\Gamma}}(I'|I',c')\alpha I',\end{aligned}$$

which shows that $\hat{\Gamma}$ satisfies (30).

We conclude that $\hat{\Gamma}$ is weakly feasible. Because for all (l,I,c,v) we have $\hat{x}_{l0}(I,c,v) = \check{x}_{l0}(I,c,v)$, $\hat{x}_{li}(I,c,v) = \check{x}_{li}(I,c,v)$, and $\hat{t}^a_{li}(I,c,v) + \hat{t}^p_{li}(I,c,v) = \check{t}^a_{li}(I,c,v) + \check{t}^p_{li}(I,c,v)$, $\hat{\Gamma}$ is payoff equivalent to $\check{\Gamma}$. Finally, because (21) holds in equality for $\hat{\Gamma}$, (22) reduces to (31). Q.E.D.

Proof of Lemma 4 To see that any maximizer $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l\}_{l\in\mathcal{L}}$ of S^{Γ} subject to the constraints (23), (24), (25), (28), (29), (31), (32), and (33), and (21) in equality, exhibits (34), suppose to the contrary that it is violated for some $(I,c,0,v_{\text{-}i}) \in \mathcal{K}\times\mathcal{V}$, i.e., for some l, we have $\check{x}_{li}(I,c,0,v_{\text{-}i}) = 1$. But then lowering it to 0 and lowering $\check{t}^p_{li}(I,c,0,v_{\text{-}i})$ by c raises the objective by $\check{p}_l\rho(I,c)\pi(0,v_{\text{-}i})c$ so that $\check{\Gamma}$ is not optimal if the changed mechanism respects all the constraints. To see that it does so, first note that the change does not affect (21) and (23) and (29). The combined reduction in $\check{x}_i(I,c,0,v_{\text{-}i})$ and $\check{t}^p_i(I,c,0,v_{\text{-}i})$ also implies that (31) and (32) remain satisfied, while also $\Pi(\gamma(I',c',v)|I,c)$ remains unaffected for any $(I,c,I',c') \in K^2$. Hence, $\Pi^{\gamma}(I,c)$ remains unaffected and, therefore (33) remains satisfied. The change further relaxes (24) and (28), since it raises the left-hand side. Finally, the change also keeps (25) satisfied, because it does not affect its left-hand side, while it lowers the right-hand side by $\check{p}_l\rho(I,c)\pi_i(v_{\text{-}i})(1-c)$. Q.E.D.

Proof of Lemma 5 We first prove that if $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ is weakly feasible and satisfies constraints (9), (31)–(34), and (21) in equality, then we find $\hat{t}_l = (\hat{t}^a_l, \hat{t}^p_l)$ with $\hat{t}^a_l$ identical to $\check{t}^a_l$ such that the mechanism $\hat{\Gamma} = \{(\check{p}_l, \hat{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ is weakly feasible, satisfies constraints (9), (31)–(34), and (21) in equality, yields the same surplus, and exhibits (28) binding, ie. $U^{\hat{\Gamma}}_i(I,c|0) = 0$ for any (i,I,c). Define $\hat{t}^p_l$ as follows: $\hat{t}^p_{li}(I,c,0,v_{\text{-}i}) = \check{t}^p_{li}(I,c,0,v_{\text{-}i}) + U^{\check{\Gamma}}_i(I,c|0)$ and $\hat{t}^p_{li}(I,c,1,v_{\text{-}i}) = \check{t}^p_{li}(I,c,1,v_{\text{-}i}) + U^{\check{\Gamma}}_i(I,c|0)$. By construction the mechanism $\hat{\Gamma}$ exhibits $U^{\hat{\Gamma}}_i(I,c|0) = 0$. Because $\hat{\Gamma}$ and $\check{\Gamma}$ exhibit the same output schedules, they generate the same surplus: $S^{\hat{\Gamma}} = S^{\check{\Gamma}}$. We next show that, because $\check{\Gamma}$ satisfies (9), (23), (24), (25), (28), (29) (31)–(34), and (21) in equality, so does the constructed $\hat{\Gamma}$. To see this, note first that the change from $\check{\Gamma}$ to $\hat{\Gamma}$ affects only the transfers $t^p_i(\cdot)$ so that (9), (21), (23), and (34) remain unaffected and, therefore, satisfied for $\hat{\Gamma}$. Because $\check{t}_i^p(I,c,0,v_{\text{-}i})$ and $\check{t}_i^p(I,c,1,v_{\text{-}i})$ are changed by the same amount, the change lowers the left- and right-hand side of (24) and (25) also by the same amount so that they remain satisfied. By construction $\hat{\Gamma}$ satisfies (28), while (29) follows from $U_i^{\hat{\Gamma}}(1|1) \geq U_i^{\hat{\Gamma}}(0|1) \geq U_i^{\hat{\Gamma}}(0|0) = \sum_{(I,c)\in\mathcal{K}} \rho(I,c) U_i^{\hat{\Gamma}}(I,c|0) = 0$, where the first inequality uses (25), the second inequality uses $v_1 = 1 > v_0 = 0$, and the final equality uses (28). Finally note that, because $\check{\Gamma}$ satisfies (28), it holds $U_i^{\check{\Gamma}}(I,c|0) \geq 0$ so that the change from $\check{\Gamma}$ to $\hat{\Gamma}$ only raises the transfers, i.e., $\hat{t}_i^p(I,c,v) \geq \check{t}_i^p(I,c,v)$. Hence, the constraints (31), (32), and (33) are relaxed so that $\hat{\Gamma}$ remains to satisfy them.

To see that for a weakly feasible $\check{\Gamma}$ that satisfies (9), (23), (24), (25), (28), (29) (31)–(34), and (21) in equality, we can also find such a weakly feasible $\bar{\Gamma}$ for which, in addition, both (28) and (29) bind, note first that we already established that given $\check{\Gamma}$, there exists such a mechanism $\hat{\Gamma}$ for which (28) binds, yielding the same aggregate surplus as $\check{\Gamma}$. It therefore remains to show that given this $\hat{\Gamma}$ there exists such a $\bar{\Gamma}$ for which, in addition, (29) is binding, ie. $U_i^{\bar{\Gamma}}(I,c,|1) = 0$ for all (i,I,c), and which yields the same surplus. To construct such a $\bar{\Gamma}$, take $\bar{\Gamma}$ as identical to $\hat{\Gamma}$ except that $\bar{t}_{li}^p(I,c,1,v_{\text{-}i}) = \hat{t}_{li}^p(I,c,1,v_{\text{-}i}) + U_i^{\hat{\Gamma}}(I,c,|1)$. By construction, $\bar{\Gamma}$ satisfies (29). Note moreover that, because $\hat{\Gamma}$ satisfies (29), it holds $U_i^{\hat{\Gamma}}(I,c|1) \geq 0$ so that $\bar{\Gamma}$ differs from $\hat{\Gamma}$ only in that the ex post transfers t^p are higher. Hence, if $\hat{\Gamma}$ satisfies (9), (23), (31)–(34), and (21) in equality, then also $\bar{\Gamma}$.

Finally, note that (34) implies $U_i^{\bar{\Gamma}}(0|1) = U_i^{\bar{\Gamma}}(0|0)$. Together with (28) and (29) binding, this implies $U_i^{\bar{\Gamma}}(1|1) = 0$ and $U_i^{\bar{\Gamma}}(0|1) = U_i^{\bar{\Gamma}}(0|0) = 0$, and therefore (25) is satisfied (in equality). Moreover, (24) follows from $U_i^{\bar{\Gamma}}(0|0) = 0 = U_i^{\bar{\Gamma}}(1|1) \geq U_i^{\bar{\Gamma}}(1|0)$. Hence, (34), with (28) and (29) binding, implies (24) and (25). Q.E.D.

Proof of Lemma 6 Following Lemma 5, we may assume without loss of generality that an optimal weakly feasible mechanism $\check{\Gamma}$ satisfies (28) and (29) in equality. That is, for all (i,I,c) it holds $U_i^{\check{\Gamma}}(I,c|0) = 0$ and $U_i^{\check{\Gamma}}(I,c|1) = 0$. It then follows for any (I,c) that

$$\begin{aligned}
\Pi^{\check{\Gamma}}(I,c) &= \textstyle\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l[\check{t}_{li}^a(I,c,v) + \check{t}_{li}^p(I,c,v)] \\
&\quad - \textstyle\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{li}(I,c,v)c - \sum_{v\in\mathcal{V}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{0l}(I,c,v)I \\
&= \textstyle\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{li}(I,c,v)v_i \\
&\quad - \textstyle\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{li}(I,c,v)c - \sum_{v\in\mathcal{V}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{0l}(I,c,v)I = S^{\check{\Gamma}}(I,c);
\end{aligned}$$

where the second equality follows from

$$\begin{aligned}
&\textstyle\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l[\check{t}_{li}^a(I,c,v) + \check{t}_{li}^p(I,c,v)] \\
&\quad = \textstyle\sum_{i\in\mathcal{N}}\sum_{(v_i,v_{\text{-}i})\in\mathcal{V}}\pi(v_i,v_{\text{-}i})\{\sum_{l\in\mathcal{L}}\check{p}_l[\check{t}_{li}^a(I,c,v) + \check{t}_{li}^p(I,c,v)]\} \\
&\quad = \textstyle\sum_{i\in\mathcal{N}}\Big[\sum_{(0,v_{\text{-}i})\in\mathcal{V}}\pi(0,v_{\text{-}i})\left\{\sum_{l\in\mathcal{L}}\check{p}_l[\check{t}_{li}^a(I,c,0,v_{-i}) + \check{t}_{li}^p(I,c,0,v_{-i})]\right\} \\
&\qquad\quad + \textstyle\sum_{(1,v_{\text{-}i})\in\mathcal{V}}\pi(1,v_{\text{-}i})\left\{\sum_{l\in\mathcal{L}}\check{p}_l[\check{t}_{li}^a(I,c,1,v_{-i}) + \check{t}_{li}^p(I,c,1,v_{-i})]\right\}\Big] \\
&\quad = \textstyle\sum_{i\in\mathcal{N}}\big[\pi_i(0)\sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}}\pi_i(v_{\text{-}i})\left\{\sum_{l\in\mathcal{L}}\check{p}_l[\check{t}_{li}^a(I,c,0,v_{-i}) + \check{t}_{li}^p(I,c,0,v_{-i})]\right\} \\
&\qquad\quad + \pi_i(1)\textstyle\sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}}\pi_i(v_{\text{-}i})\left\{\sum_{l\in\mathcal{L}}\check{p}_l[\check{t}_{li}^a(I,c,1,v_{-i}) + \check{t}_{li}^p(I,c,1,v_{-i})]\right\}\big] \\
&\quad = \textstyle\sum_{i\in\mathcal{N}}\Big[\pi_i(0)\Big\{\sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}}\sum_{l\in\mathcal{L}}\pi_i(v_{\text{-}i})\check{p}_l\check{x}_{li}(I,c,0,v_{-i})\cdot 0 - U_i^{\check{\Gamma}}(I,c|0)\Big\}
\end{aligned}$$

$$+\pi_i(1)\left\{\sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}}\sum_{l\in\mathcal{L}}\pi_i(v_{\text{-}i})\check{p}_l\check{x}_{li}(I,c,1,v_{-i})\cdot 1-U_i^{\check{\Gamma}}(I,c|0)\right\}\Big]$$
$$=\sum_{i\in\mathcal{N}}\Big[\sum_{(0,v_{\text{-}i})\in\mathcal{V}}\pi(0,v_{\text{-}i})\sum_{l\in\mathcal{L}}\check{p}_l\check{x}_{li}(I,c,0,v_{-i})\cdot 0$$
$$+\sum_{(1,v_{\text{-}i})\in\mathcal{V}}\pi(1,v_{\text{-}i})\sum_{l\in\mathcal{L}}\check{p}_l\check{x}_{li}(I,c,1,v_{-i})\cdot 1\Big]$$
$$=\sum_{i\in\mathcal{N}}\sum_{(v_i,v_{\text{-}i})\in\mathcal{V}}\pi(v_i,v_{\text{-}i})\sum_{l\in\mathcal{L}}\check{p}_l\check{x}_{li}(I,c,v_i,v_{-i})v_i$$
$$=\sum_{v\in\mathcal{V}}\sum_{i\in\mathcal{N}}\sum_{l\in\mathcal{L}}\pi(v)\check{p}_l\check{x}_{li}(I,c,v)v_i.$$

Q.E.D.

Proof of Proposition 2 If the efficient output schedule x^* is implementable, then the optimal feasible mechanism $\check{\Gamma}$ must implement it, because, by definition, no other output schedule yields a larger surplus. Moreover, the proof of Proposition 1 already noted that, because x^* is deterministic, it is implementable if and only if there exists a transfer schedule $\check{t}$ such that the deterministic mechanism $\check{\Gamma}=(1,\check{\gamma})=(1,\check{t},x^*)$ is feasible.

Note that for deterministic mechanisms, constraint (43) simplifies to

$$S^{x^*}(I,c)\geq\pi^{x^*}(I',c')\alpha I'\ \forall(I,c,I',c')\in\mathcal{K}\times\mathcal{K}.$$

It is therefore immediate that affluency is a necessary condition for the implementability of x^* by a weakly feasible mechanism $\check{\Gamma}$ and, hence, also for the implementability by a (fully) feasible mechanism $\check{\Gamma}$.

It remains to prove that affluency is also a sufficient condition for the implementability of x^*. We will do so constructively and, under the assumption that x^* is affluent, construct an explicit crowdfunding mechanism that implements it.

Because x^* is development-efficient, it holds $n(v)=\sum_{i\in\mathcal{N}}v_i>0$ for any $x_0^*(I,c,v)=1$ so that defining $\check{t}=(\check{t}^a,\check{t}^p)$ as

$$(\check{t}_i^a(I,c,v),\check{t}_i^p(I,c,v))\equiv\begin{cases}(v_iI/n(v),v_i[1-I/n(v)]) & \text{if } x_0^*(I,c,v)=1,\\ (0,0) & \text{otherwise,}\end{cases}$$

yields a well-defined $\check{t}$. For $T(I,c)=I/(1-c)$, the output schedule x^* and transfers $\check{t}$ satisfy (10)–(13) and the deterministic mechanism $\check{\Gamma}=(1,\check{\gamma})=(1,\check{t},x^*)$ is, therefore, a crowdfunding mechanism.

As we next show, given that x^* is affluent, the crowdfunding mechanism $\check{\Gamma}$ satisfies constraints (36)–(43) so that it is weakly feasible and, moreover, (27) so that it is also feasible.

To see (36), note for $x_0^*(I,c,v)=0$, it follows $\sum_{i\in\mathcal{N}}\check{t}_i^a(I,c,v)=0=x_0^*(I,c,v)I$. Moreover, because x^* is development-efficient it follows for $x_0^*(I,c,v)=1$ that

$$\sum_{i\in\mathcal{N}}\check{t}_i^a(I,c,v)=\sum_{i\in\mathcal{N}}v_iI/n(v)=\Big[\sum_{i\in\mathcal{N}}v_i\Big]I/\sum_j v_j=I=x_0^*(I,c,v)I.$$

Note that (38) holds, because x^* is development-feasible. To see (39) and (40), note that, because x^* is development-efficient,

$$U_i^{\check{\gamma}}(v_i|I,c,v_i)=\sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}}\pi_i(v_{\text{-}i})[v_ix_i^*(I,c,1)-\check{t}_i^a(I,c,1)-\check{t}_i^p(I,c,1)]=0.$$

To see (37), note that, since x^* is development-efficient, for $x_0^*(I,c,v)=0$ we have $\sum_{i\in\mathcal{N}} \check{t}_i^p(I,c,v) = 0 = \sum_{i\in\mathcal{N}} x_i^*(I,c,v)c$. Moreover, because x^* is development-efficient and $x_0^*(I,c,v)=1$ implies $n(v)\geq I/(1-c)$, for $x_0^*(I,c,v)=1$ it follows that

$$\textstyle\sum_{i\in\mathcal{N}}\check{t}_i^p(I,c,v) = \sum_{i\in\mathcal{N}} v_i[1-I/n(v)] = n(v)-I \geq cn(v) = c\sum_{i\in\mathcal{N}} x_i^*(I,c,v).$$

To see (39), note

$$\begin{aligned}
U_i^{\check{\Gamma}}(I,c|1) &= U_i^{\check{\gamma}}(1|I,c,1)\\
&= \sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}} \pi_i(v_{\text{-}i}) U_i(\check{\gamma}(I,c,1,v_{\text{-}i})|1)\\
&= \sum_{v_{\text{-}i}\in\mathcal{V}_{\text{-}i}} \pi_i(v_{\text{-}i})[x_i^*(I,c,1,v_{\text{-}i}) - \check{t}_i^a(I,c,1,v_{\text{-}i}) - \check{t}_i^p(I,c,1,v_{\text{-}i})]\\
&= \sum_{v_{\text{-}i}:x_0^*(I,c,1,v_{\text{-}i})=1} \pi_i(v_{\text{-}i}) x_i^*(I,c,1,v_{\text{-}i})[1-1] = 0,
\end{aligned}$$

while (40) follows directly from the observation that $\check{t}_i^a(I,c,0,v_{-i}) = \check{t}_i^p(I,c,0,v_{-i}) = 0$.

To see (41), note that since x^* is development efficient and affluent, we have

$$\begin{aligned}
&\sum_{v\in\mathcal{V}^{x^*}(I,c)} \sum_{i\in\mathcal{N}} \pi(v)(\check{t}_i^p(I,c,v) - cx_i^*(I,c,v))\\
&= \sum_{v\in\mathcal{V}^{x^*}(I,c)} \pi(v)\Big[\sum_{i:x_i^*(I,c,v)=1}(1-I/n(v)-c)\Big] = \sum_{v\in\mathcal{V}^{x^*}(I,c)} \pi(v)[n(v)(1-c)-I]\\
&= \sum_{v\in\mathcal{V}} \pi(v)\Big[\sum_{i\in\mathcal{N}}^{n}(v_i-c)x_i^*(I,c,v) - Ix_0^*(I,c,v)\Big] = S^{x^*}(I,c) \geq \pi^{x^*}(I,c)\alpha I.
\end{aligned}$$

Finally, (43) follows because x^* is affluent and x^* satisfies (42) by definition. Since x^* is efficient, it also satisfies (42). Hence, $\check{\gamma}$ satisfies all constraints (36)–(43) and, therefore, is weakly feasible.

To see (27), note that, because $x_0^*(I',c',v)=0$ implies $\Pi^{\check{\gamma}}(I',c'|I,c,v)=0$, (27) holds if

$$\Pi^{\check{\Gamma}}(I,c) \geq \pi^{x^*}(I',c')\max\{\Pi_o^{\check{\Gamma}}(T|I,c,I',c'),\alpha I'\}.$$

That is, it holds if

$$\Pi^{\check{\Gamma}}(I,c) \geq \pi^{x^*}(I',c')\Pi_o^{\check{\Gamma}}(I'|I,c,I',c') \text{ and } \Pi^{\check{\Gamma}}(I,c) \geq \pi^{x^*}(I',c')\alpha I'.$$

The latter follows, since, by Lemma 6, $\Pi^{\check{\Gamma}}(I,c) = S^{x^*}(I,c)$ and x^* is affluent. To see also the former inequality, note, because $x_0^*(I',c',v)=0$ implies $\Pi^{\check{\gamma}}(I',c'|I,c,v)=0$, we have

$$\begin{aligned}
\pi^{x^*}(I',c')\Pi_o^{\check{\Gamma}}(I'|I,c,I',c') &= \pi^{x^*}(I',c')\sum_{v\in\mathcal{V}} \eta^{\check{\Gamma}}(v,1|I',I',c')\Pi^{\check{\gamma}}(I',c'|I,c,v)\\
&= \sum_{v\in\mathcal{V}^{x^*}(I',c')} \pi(v)\Pi^{\check{\gamma}}(I',c'|I,c,v) = \sum_{v\in\mathcal{V}} \pi(v)\Pi^{\check{\gamma}}(I',c'|I,c,v)\\
&= \sum_{v\in\mathcal{V}} \pi(v)\Pi(\check{\gamma}(I',c',v)|I,c)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{v\in\mathcal{V}} \pi(v)\Big\{ \sum_{i\in\mathcal{N}} [\check{t}_i^a(I',c',v) + \check{t}_i^p(I',c',v) - x_i^*(I',c',v)c] - I x_0^*(I',c',v)\Big\} \\
&= \sum_{v\in\mathcal{V}} \pi(v)\{x_0^*(I',c',v)[n(v)(1-c) - I]\} \\
&\le \sum_{v\in\mathcal{V}} \pi(v)\{x_0^*(I,c,v)[n(v)(1-c) - I]\} = S^{\check{\gamma}}(I,c) = \Pi^{\check{\gamma}}(I,c) = \Pi^{\check{\Gamma}}(I,c),
\end{aligned}
$$

where the inequality follows because x^* is efficient. Q.E.D.

Proof of Lemma 7 The proof consists of 3 steps. We first prove that, for an optimal $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ satisfying (36)–(43), for all $(l,I,c,v) \in \mathcal{L}\times\mathcal{K}\times\mathcal{V}$, it holds

$$\check{x}_{l0}(I,c,v) = 1 \Rightarrow \check{x}_{li}(I,c,v) = v_i. \tag{A9}$$

Second, we prove that if $\check{\Gamma}$ is optimal, then for each $(l,I,c) \in \mathcal{L}\times\mathcal{K}$ there exists a $T \in \mathcal{N}$ such that (10) holds. In a final step, we prove that T is independent of l so that for each $(I,c) \in \mathcal{K}$ there exists a $T \in \mathcal{N}$ such that (10) holds for any $l \in \mathcal{L}$.

Step 1: Consider a $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ that satisfies (36)–(43), but for which condition (A9) is not satisfied. Hence, it holds that for some $(l,I,c,v) \in \mathcal{L}\times K\times\mathcal{V}$ that $\check{x}_{l0}(I,c,v) = 1$ but $\check{x}_{li}(I,c,v) \neq v_i \in \{0,1\}$. Constraint (42) then implies $v_i = 1$ so that $\check{x}_{li}(I,c,v) = 0$. It then follows that by raising $\check{x}_{li}(I,c,v)$ to 1, the objective $S^{\check{\Gamma}}$ is increased by $\check{p}_l\rho(I,c)\pi(v)(1-c)$. By accompanying the raise in $\check{x}_{li}(I,c,v)$ by a raise in $\check{t}_{li}^p(I,c,v)$ of 1 a changed mechanism obtains that remains to respect all constraints (36)–(43). It is therefore also weakly feasible, and hence $\check{\Gamma}$ is not optimal.

Step 2: Next we show that if $\check{\Gamma}$ is optimal then i) $\check{x}_{l0}(I,c,\hat{v}) = 1$ implies $\check{x}_{l0}(I,c,\bar{v}) = 1$ for any $\bar{v}$ such that $n(\bar{v}) > n(\hat{v})$, and ii) $\check{x}_{l0}(I,c,\hat{v}) = 0$ implies $\check{x}_{l0}(I,c,\bar{v}) = 0$ for any $n(\bar{v}) < n(\hat{v})$. From this it then directly follows that, for any $(l,I,c) \in \mathcal{L}\times\mathcal{K}$, there is a $T \in \mathcal{N}$ such that, for all $v \in \mathcal{V}$, it holds $x_{0l}(I,c,v) = 1$ if $n(v) > T$ and $x_{0l}(I,c,v) = 0$ if $n(v) < T$.

To see i) and ii), assume to the contrary that one of the two conditions does not hold, meaning there exists an $(\bar{l},\tilde{I},\tilde{c}) \in \mathcal{L}\times\mathcal{K}$ and $\bar{v},\hat{v} \in \mathcal{V}$ with $n(\bar{v}) < n(\hat{v})$ such that $\check{x}_{\bar{l}0}(\tilde{I},\tilde{c},\bar{v}) = 1$ and $\check{x}_{\bar{l}0}(\tilde{I},\tilde{c},\hat{v}) = 0$. Since $n(\bar{v}) < n(\hat{v})$ there exists a bijection $j : \mathcal{N} \to \mathcal{N}$ such that $\bar{v}_i = 1$ implies $\hat{v}_{j(i)} = 1$. To show that $\check{\Gamma}$ is not optimal, we distinguish three cases: 1. $\pi(\bar{v}) = \pi(\hat{v})$; 2. $\pi(\bar{v}) < \pi(\hat{v})$, and 3. $\pi(\bar{v}) > \pi(\hat{v})$.

Case 1: Adapt the mechanism $\check{\Gamma}$ to the mechanism $\hat{\Gamma}$ by only replacing $\check{\gamma}_{\bar{l}}$ by the mechanism $\hat{\gamma} = (\hat{t},\hat{x})$, which is identical to $\check{\gamma}_{\bar{l}}$ for all $(I,c,v) \in \mathcal{K}\times\mathcal{V}$ except for $(\tilde{I},\tilde{c},\bar{v})$ and $(\tilde{I},\tilde{c},\hat{v})$. Hence, for all $(I,c,v) \in (\mathcal{K}\times\mathcal{V})\backslash\{(\tilde{I},\tilde{c},\bar{v}),(\tilde{I},\tilde{c},\hat{v})\}$, it holds $\hat{t}(I,c,v) = \check{t}_{\bar{l}}(I,c,v) \in \mathbb{R}^{2n}$ and $\hat{x}(I,c,v) = \check{x}_{\bar{l}}(I,c,v) \in \{0,1\}^{n+1}$. For all $i \in \mathcal{N}$, let $\hat{x}_0(\tilde{I},\tilde{c},\bar{v}) = \hat{x}_i(\tilde{I},\tilde{c},\bar{v}) = 0$, $\hat{t}_i^a(\tilde{I},\tilde{c},\bar{v}) = \check{t}_{\bar{l}i}^a(\tilde{I},\tilde{c},\bar{v}) - \check{x}_{\bar{l}i}(\tilde{I},\tilde{c},\bar{v})\tilde{I}/n(\bar{v})$, and $\hat{t}_i^p(\tilde{I},\tilde{c},\bar{v}) = \check{t}_{\bar{l}i}^p(\tilde{I},\tilde{c},\bar{v}) - \check{x}_{\bar{l}i}(\tilde{I},\tilde{c},\bar{v})[1-\tilde{I}/n(\bar{v})]$. Moreover, for all $i \in \mathcal{N}$, let $\hat{x}_0(\tilde{I},\tilde{c},\hat{v}) = 1$ and $\hat{x}_{j(i)}(\tilde{I},\tilde{c},\hat{v}) = \check{x}_{\bar{l}i}(\tilde{I},\tilde{c},\bar{v})$, $\hat{t}_{j(i)}^a(\tilde{I},\tilde{c},\hat{v}) = \check{t}_{\bar{l}j(i)}^a(\tilde{I},\tilde{c},\hat{v}) + \hat{x}_{j(i)}(\tilde{I},\tilde{c},\hat{v})\tilde{I}/n(\bar{v})$, and $\hat{t}_{j(i)}^p(\tilde{I},\tilde{c},\hat{v}) = \check{t}_{\bar{l}j(i)}^p(\tilde{I},\tilde{c},\hat{v}) + \hat{x}_{j(i)}(\tilde{I},\tilde{c},\hat{v})[1-\tilde{I}/n(\bar{v})]$. Because $\pi(\bar{v}) = \pi(\hat{v})$, it holds $\pi^{\check{x}_{\bar{l}}}(\tilde{I},\tilde{c}) = \pi^{\hat{x}}(\tilde{I},\tilde{c})$ and, therefore, $\pi^{\check{\Gamma}}(\tilde{I},\tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I},\tilde{c})$.

Case 2: Consider the mechanism $\hat{\Gamma} = \{(\hat{p}_l, \check{\gamma}_l)\}_{l\in\{0,\dots,L\}}$, which, in addition to the same collection of deterministic mechanisms $\check{\gamma}_l$ as $\check{\Gamma}$ but with $\check{\gamma}_{\bar{l}}$ exchanged by the deterministic mechanism $\hat{\gamma}$

as defined in Case 1, also contains the deterministic mechanism $\check{\gamma}_0 = (\check{t}_0, \check{x}_0)$. This deterministic mechanism is identical to $\check{\gamma}_{\bar{l}}$ for all $(I, c, v) \in \mathcal{K}\times\mathcal{V}$ except for $(\tilde{I}, \tilde{c}, \bar{v})$. Hence, for all $(I, c, v) \in \mathcal{K}\times\mathcal{V}\backslash\{(\tilde{I}, \tilde{c}, \bar{v})\}$, let $\check{t}_0(I, c, v) = \check{t}_{\bar{l}}(I, c, v) \in \mathbb{R}^{2n}$ and $\check{x}_0(I, c, v) = \check{x}_{\bar{l}}(I, c, v) \in \{0, 1\}^{n+1}$. For all $i \in \mathcal{N}$, let $\check{x}_{00}(\tilde{I}, \tilde{c}, \bar{v}) = \check{x}_{0i}(\tilde{I}, \tilde{c}, \bar{v}) = 0$, $\check{t}^a_{0i}(\tilde{I}, \tilde{c}, \bar{v}) = \check{t}^a_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v}) - \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})\tilde{I}/n(\bar{v})$, and $\check{t}^p_{0i}(\tilde{I}, \tilde{c}, \bar{v}) = \check{t}^p_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v}) - \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})[1 - \tilde{I}/n(\bar{v})]$. For $\hat{\Gamma}$ we further set $\hat{p}_l = \check{p}_l$ for all $l \in \mathcal{L}\backslash\{\bar{l}\}$, $\hat{p}_{\bar{l}} = \check{p}_{\bar{l}}\pi(\bar{v})/\pi(\hat{v}) < \check{p}_{\bar{l}}$ and $\hat{p}_0 = \check{p}_{\bar{l}}[\pi(\hat{v}) - \pi(\bar{v})]/\pi(\hat{v}) \in (0, 1)$. Hence, $\sum_{l=0}^{L} \hat{p}_l = 1$. Note that $\pi^{\check{\Gamma}}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \check{p}_l \pi^{\check{x}_l}(\tilde{I}, \tilde{c}) = \sum_{l\in\{0,\ldots,L\}} \hat{p}_l \pi^{\hat{x}_l}(\tilde{I}, \tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I}, \tilde{c})$.

Case 3: Consider the mechanism $\hat{\Gamma} = \{(\hat{p}_l, \check{\gamma}_l)\}_{l\in\{0,\ldots,L\}}$, which, in addition to the same collection of deterministic mechanisms $\check{\gamma}_l$ as $\check{\Gamma}$ but with $\check{\gamma}_{\bar{l}}$ exchanged by the deterministic mechanism $\hat{\gamma}$ as defined in Case 1, also contains the deterministic mechanism $\check{\gamma}_0 = (\check{t}_0, \check{x}_0)$. This deterministic mechanism is identical to $\check{\gamma}_{\bar{l}}$ for all $(I, c, v) \in \mathcal{K}\times\mathcal{V}$ except for $(\tilde{I}, \tilde{c}, \hat{v})$. Hence, for all $(I, c, v) \in (\mathcal{K}\times\mathcal{V})\backslash\{(\tilde{I}, \tilde{c}, \hat{v})\}$, let $\check{t}_0(I, c, v) = \check{t}_{\bar{l}}(I, c, v) \in \mathbb{R}^{2n}$ and $\check{x}_0(I, c, v) = \check{x}_{\bar{l}}(I, c, v) \in \{0, 1\}^{n+1}$. For all $i \in \mathcal{N}$, let $\check{x}_{00}(\tilde{I}, \tilde{c}, \hat{v}) = 1$ and $\check{x}_{0j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})$, $\check{t}^a_{0j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{t}^a_{\bar{l}j(i)}(\tilde{I}, \tilde{c}, \hat{v}) + \check{x}_{0j(i)}(\tilde{I}, \tilde{c}, \hat{v})\tilde{I}/n(\bar{v})$, and $\check{t}^p_{0j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{t}^p_{\bar{l}j(i)}(\tilde{I}, \tilde{c}, \hat{v}) + \check{x}_{0j(i)}(\tilde{I}, \tilde{c}, \hat{v})[1 - \tilde{I}/n(\bar{v})]$. For $\hat{\Gamma}$, we further set $\hat{p}_l = \check{p}_l$ for all $l \in \mathcal{L}\backslash\{\bar{l}\}$, $\hat{p}_{\bar{l}} = \check{p}_{\bar{l}}\pi(\hat{v})/\pi(\bar{v}) < \check{p}_{\bar{l}}$ and $\hat{p}_0 = \check{p}_{\bar{l}}[\pi(\bar{v}) - \pi(\hat{v})]/\pi(\bar{v}) \in (0, 1)$. Hence, $\sum_{l=0}^{L} \hat{p}_l = 1$. Note that $\pi^{\check{\Gamma}}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \check{p}_l \pi^{\check{x}_l}(\tilde{I}, \tilde{c}) = \sum_{l=0}^{L} \hat{p}_l \pi^{\hat{x}_l}(\tilde{I}, \tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I}, \tilde{c})$.

In all 3 cases, we obtain an adapted mechanism $\hat{\Gamma}$ that satisfies (36)–(43), but, because $\sum_{i\in\mathcal{N}} x_i(\tilde{I}, \tilde{c}, \hat{v}) = n(\bar{v}) < n(\hat{v})$, it does not satisfy (A9). According to step 1, the mechanism $\hat{\Gamma}$ is not optimal. Since $S^{\check{\Gamma}} = S^{\hat{\Gamma}}$, this means that also $\check{\Gamma}$ is not optimal.

Step 3: Due to step 2, if $\check{\Gamma}$ is optimal, then, for any $(l, I, c) \in \mathcal{L}\times\mathcal{K}$, there exists an integer $T_l(I, c) \in \mathcal{N}$ such that if $x_{l0}(I, c, v_1) \neq x_{l0}(I, c, v_2)$ and $n(v_1) = n(v_2)$, then $n(v_1) = n(v_2) = T_l(I, c)$. Moreover, $T_l(I, c)$ is a cutoff in the sense that $x_{l0}(I, c, v) = 0$ for all $v \in \mathcal{V}$ such that $n(v) < T_l(I, c)$, and $x_{l0}(I, c, v) = 1$ for all $v \in \mathcal{V}$ such that $n(v) > T_l(I, c)$.

We next show that for an optimal $\check{\Gamma}$ there is a cutoff $T_l(I, c)$ that is independent of l. That is, we show that if $x_{\bar{l}0}(\tilde{I}, \tilde{c}, \bar{v}_1) \neq x_{\bar{l}0}(\tilde{I}, \tilde{c}, \bar{v}_2)$, $n(\bar{v}_1) = n(\bar{v}_2) = n(\bar{v})$, $x_{\hat{l}0}(\tilde{I}, \tilde{c}, \hat{v}_1) \neq x_{\hat{l}0}(\tilde{I}, \tilde{c}, \hat{v}_2)$ and $n(\hat{v}_1) = n(\hat{v}_2) = n(\hat{v})$, then $n(\bar{v}) = n(\hat{v})$. By step 2 it then follows that $T(\tilde{I}, \tilde{c}) = n(\bar{v}) = n(\hat{v})$ is such an l-independent cutoff.

To see this, suppose to the contrary that $n(\bar{v}) \neq n(\hat{v})$ and, without of loss of generality, assume $n(\bar{v}) < n(\hat{v})$. This implies a bijection $j : \mathcal{N} \to \mathcal{N}$ such that $\bar{v}_i = 1$ implies $\hat{v}_{j(i)} = 1$. By step 1, optimality of $\check{\Gamma}$ implies $\check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v}) = v_i$, and $\check{x}_{\hat{l}0}(\tilde{I}, \tilde{c}, \hat{v}) = 0$ implies $\check{x}_{\hat{l}i}(\tilde{I}, \tilde{c}, \hat{v}) = 0$.

Consider the (deterministic) direct mechanism $\check{\gamma}_{\bar{l}'}$ that is identical to $\check{\gamma}_{\bar{l}}$ except for $(\tilde{I}, \tilde{c}, \bar{v})$ in that $\check{x}_{\bar{l}'0}(\tilde{I}, \tilde{c}, \bar{v}) = 0$ and, for all $i \in \mathcal{N}$, it holds $\check{x}_{\bar{l}'i}(\tilde{I}, \tilde{c}, \bar{v}) = 0$, $\check{t}^a_{\bar{l}'i}(\tilde{I}, \tilde{c}, \bar{v}) = \check{t}^a_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v}) - \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})\tilde{I}/n(\bar{v})$, and $\check{t}^p_{\bar{l}'i}(\tilde{I}, \tilde{c}, \bar{v}) = \check{t}^p_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v}) - \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})[1 - \tilde{I}/n(\bar{v})]$.

Consider the (deterministic) direct mechanism $\check{\gamma}_{\hat{l}'}$ which is identical to $\check{\gamma}_{\hat{l}}$ except for $(\tilde{I}, \tilde{c}, \hat{v})$ in that $\check{x}_{\hat{l}'0}(\tilde{I}, \tilde{c}, \hat{v}) = 1$ and, for all $i \in \mathcal{N}$, it holds $\check{x}_{\hat{l}'j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{x}_{\bar{l}i}(\tilde{I}, \tilde{c}, \bar{v})$, $\check{t}^a_{\hat{l}'j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{t}^a_{\hat{l}j(i)}(\tilde{I}, \tilde{c}, \hat{v}) + \check{x}_{\bar{l}'j(i)}(\tilde{I}, \tilde{c}, \bar{v})\tilde{I}/n(\bar{v})$, and, similarly, $\check{t}^p_{\hat{l}'j(i)}(\tilde{I}, \tilde{c}, \hat{v}) = \check{t}^p_{\hat{l}j(i)}(\tilde{I}, \tilde{c}, \hat{v}) + \check{x}_{\bar{l}'j(i)}(\tilde{I}, \tilde{c}, \bar{v})[1 - \tilde{I}/n(\bar{v})]$.

Once more, we distinguish three cases: 1. $\pi(\bar{v}) = \pi(\hat{v})$; 2. $\pi(\bar{v}) < \pi(\hat{v})$, and 3. $\pi(\bar{v}) > \pi(\hat{v})$.

Case 1: We adapt the mechanism $\check{\Gamma}$ to $\hat{\Gamma}$ by exchanging $\check{\gamma}_{\bar{l}}$ by $\check{\gamma}_{\bar{l}'}$ and $\check{\gamma}_{\hat{l}}$ by $\check{\gamma}_{\hat{l}'}$. It then follows that, because $\pi(\bar{v}) = \pi(\hat{v})$, we have $\pi^{\check{\Gamma}}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \pi^{\check{x}_l}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \pi^{\hat{x}_l}(\tilde{I}, \tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I}, \tilde{c})$.

Case 2: We adapt the mechanism $\check{\Gamma}$ to $\hat{\Gamma}$ by exchanging $\check{\gamma}_{\bar{l}}$ by $\check{\gamma}_{\bar{l}'}$ and $\check{\gamma}_{\hat{l}}$ by $\check{\gamma}_{\hat{l}'}$. In addition, we add to the collection $\hat{\Gamma}$ the mechanism $\check{\gamma}_0 = (\check{t}_0, \check{x}_0)$ as defined in Case 2 above. For $\hat{\Gamma}$ we further set

$\hat{p}_l = \check{p}_l$ for all $l \in \mathcal{L}\backslash\{\hat{l}\}$, $\hat{p}_{\hat{l}} = \check{p}_{\hat{l}}\pi(\bar{v})/\pi(\hat{v}) < \check{p}_{\hat{l}}$ and $\hat{p}_0 = \check{p}_{\hat{l}}[\pi(\hat{v}) - \pi(\bar{v})]/\pi(\hat{v}) \in (0,1)$. Hence, $\sum_{l=0}^{L} \hat{p}_l = 1$. Note that $\pi^{\check{\Gamma}}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \check{p}_l \pi^{\check{x}_l}(\tilde{I}, \tilde{c}) = \sum_{l\in\{0,\ldots,L\}} \hat{p}_l \pi^{\hat{x}_l}(\tilde{I}, \tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I}, \tilde{c})$.

Case 3: We adapt the mechanism $\check{\Gamma}$ to $\hat{\Gamma}$ by exchanging $\check{\gamma}_{\bar{l}}$ by $\check{\gamma}_{\bar{l}'}$ and $\check{\gamma}_{\hat{l}}$ by $\check{\gamma}_{\hat{l}'}$. In addition, we add to the collection $\hat{\Gamma}$ the mechanism $\check{\gamma}_0 = (\check{t}_0, \check{x}_0)$ as defined in Case 2 above. For $\hat{\Gamma}$ we further set $\hat{p}_l = \check{p}_l$ for all $l \in \mathcal{L}\backslash\{\hat{l}\}$, $\hat{p}_{\hat{l}} = \check{p}_{\hat{l}}\pi(\bar{v})/\pi(\hat{v}) < \check{p}_{\hat{l}}$ and $\hat{p}_0 = \check{p}_{\hat{l}}[\pi(\hat{v}) - \pi(\bar{v})]/\pi(\hat{v}) \in (0,1)$. Hence, $\sum_{l=0}^{L} \hat{p}_l = 1$. Note that $\pi^{\check{\Gamma}}(\tilde{I}, \tilde{c}) = \sum_{l\in\mathcal{L}} \check{p}_l \pi^{\check{x}_l}(\tilde{I}, \tilde{c}) = \sum_{l\in\{0,\ldots,L\}} \hat{p}_l \pi^{\hat{x}_l}(\tilde{I}, \tilde{c}) = \pi^{\hat{\Gamma}}(\tilde{I}, \tilde{c})$.

In all 3 cases, we obtain an adapted mechanism $\hat{\Gamma}$ that satisfies (36)–(43), but, because $\sum_{i\in\mathcal{N}} x_{\hat{l}i}(\tilde{I}, \tilde{c}, \hat{v}) = n(\bar{v}) < n(\hat{v})$, it does not satisfy (A9). According to Lemma 7, the mechanism $\hat{\Gamma}$ is not optimal. Since $S^{\check{\Gamma}} = S^{\hat{\Gamma}}$, also $\check{\Gamma}$ is not optimal. Q.E.D.

Proof of Proposition 3 By Lemmas 2–6, we can assume that the optimal weakly feasible mechanism $\check{\Gamma} = \{(\check{p}_l, \check{t}_l, \check{x}_l)\}_{l\in\mathcal{L}}$ satisfies (36)–(43). By Lemma 7, we can moreover assume that for an optimal weakly feasible mechanism, there is a function $T : \mathcal{K} \to \mathbb{N}$ that satisfies (10). Lemma 7 implies that for any $(l, i, I, c, v) \in \mathcal{L}\times\mathcal{N}\times\mathcal{K}\times\mathcal{V}$ such that $n(v) = T(I, c)$, we have $(\check{x}_0(I, c, v), \check{x}_{li}(I, c, v)) = (0, 0)$ or $(\check{x}_0(I, c, v), \check{x}_{li}(I, c, v)) = (1, v_i)$. Hence, the optimal weakly feasible mechanism specifies a unique output schedule $x(I, c, v) \in \{0, 1\}^{n+1}$ for any (I, c, v) such that $n(v) \neq T(I, c)$, and it mixes between at most two output schedules when $n(v) = T(I, c)$.

With these observations, the proposition then follows by noting that we can complete any collection $\{(\hat{p}_l, \hat{x}_l)\}_{l\in\mathcal{L}}$ that satisfies the above conditions by a transfers schedule $\{\hat{t}_l\}_{l\in\mathcal{L}}$ as defined by (11)–(13). The resulting mechanism $\hat{\Gamma} = \{(\hat{p}_l, \hat{t}_l, \hat{x}_l)\}_{l\in\mathcal{L}}$ then satisfies (36)–(43) and the constraints (27). It is therefore not only weakly feasible but also (strictly) feasible. We conclude that any constrained efficient allocation is implementable by a crowdfunding mechanism and maximizes the entrepreneur's ex ante profits. Q.E.D.

REFERENCES

Ellman, Matthew and Sjaak Hurkens. (2017). "A Theory of Crowdfunding — A Mechanism Design Approach with Demand Uncertainty and Moral Hazard: Comment." Unpublished.

Strausz, Roland. (2017). "A Theory of Crowdfunding: A Mechanism Design Approach with Demand Uncertainty and Moral Hazard." *American Economic Review* 107(6): 1430-1476.