ONLINE APPENDICES

Appendix A: Proofs

Characterization of Equilibrium

In Subsections 4.1.3 and 4.2.2, we stated and informally described the equivalence between the equilibrium consumption and the solution of the equilibrium program.

Lemma 2. *Consider either the model of forgetting to repay (Subsection 4.1.3) or the model of income shocks (Subsection 4.2.2). A set of state-dependent consumption* $\{\mathbf{C}_j\}_{j=1,\ldots,N}$ *and a set of acceptance and payment decisions is an equilibrium of the game if and only if:*

1. *At least two offers are accepted with positive probability,*
2. *All offers accepted with positive probability solve the equilibrium program, and*
3. *All offers that are not accepted give consumers a perceived utility lower than the solutions of the equilibrium program.*

Since firms get zero profits in equilibrium, when there are more than two firms, there exist equilibria in which some firms offer contracts that are never accepted. An equilibrium of the game is *essentially unique* if consumption in any contract accepted with positive probability is the same in all equilibria. An equilibrium of the game is *symmetric* if consumption in all contracts accepted with positive probability is the same: if $\mathbf{C}_j$ and $\mathbf{C}'_{j'}$ are accepted with positive probability, then $\mathbf{C}_j = \mathbf{C}'_{j'}$. The next lemma establishes existence, uniqueness, and symmetry of the equilibrium:

Lemma 3. *Consider either the model of forgetting to repay (Subsection 4.1.3) or the model of income shocks (Subsection 4.2.2). There exists an equilibrium. Moreover, the equilibrium is essentially unique and symmetric.*

Lemma 3 justifies the approach in the paper of omitting the index j from contracts that are accepted with positive probability.

Proof of Lemma 2

We establish the result for the model in which individuals forget to repay. The proof for the model with income shocks is analogous and therefore omitted.

The proof follows a standard Bertrand argument. Necessity:

1. First, suppose no offer is accepted in equilibrium. Then, a firm can get positive profits by offering full insurance conditional on remembering to pay and no insurance if the individual forgets to pay at a price slightly above actuarially fair. Since the perceived utility function is concave (consumers are risk averse), we can ensure that consumers buy the policy by taking prices to be close enough to actuarially fair. Next, suppose only one offer is accepted. If this offer yields strictly positive profits, another firm can profit by offering a policy with a slightly higher consumption, thereby attracting all customers. If the only offer that is accepted in equilibrium yields zero profits, there are two possibilities. If the policy gives a strictly positive perceived utility to consumers, the firm offering it can obtain strictly positive profits by charging a slightly higher price (say, at period 1). If, instead, the policy gives a zero perceived utility to consumers, because consumers are risk averse, the firm offering it can obtain strictly positive profits by shifting to a policy that offers full insurance conditional for those who remember to pay and no insurance for those that forget to pay. Therefore, at least two offers must be accepted with positive probability in equilibrium.

2. Because consumers put zero weight on forgetting to pay when they are choosing which policy to buy, they do not take into account any states that happen after they forget to pay. Therefore, policies must maximize the consumers' perceived utility (which attributes probability zero to forgetting to pay) subject to the incentive constraint.

Firms are willing to provide insurance policies as long as they obtain non-negative profits. If an offer with strictly positive profits is accepted in equilibrium, another firm can obtain a discrete gain by slightly undercutting the price of this policy. Moreover, if the policy does not maximize the consumer's perceived utility subject to the zero-profits constraint, another firm can offer a policy that yields a higher perceived utility and extract a positive profit.

3. If a consumer is accepting an offer with a lower perceived utility, either a policy that solves Program (2) is being rejected (which is not optimal for the consumer) or it is not being offered (which is not optimal for the firms).

To establish sufficiency, note that whenever these conditions are satisfied, any other offer by another firm must either not be accepted or yield non-positive profits.

Proof of Lemma 3

We establish the result for the model in which individuals forget to repay. The proof for the model with income shocks is analogous and therefore omitted.

From Lemma 2, the equilibrium must solve the following program:

$$\max_{\left(c_{1,j},c_{2,j}^{R},c_{3,j}^{R,D},c_{3,j}^{R,A},c_{3,j}^{F,D},c_{3,j}^{F,A}\right)} u_A(c_{1,j})+u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A}) \tag{10}$$

subject to:

$$c_{1,j}+(1-l)\left[c_{2,j}^{R}+\alpha c_{3,j}^{R,D}+(1-\alpha)\,c_{3,j}^{R,A}\right]+l\left[I_2+\alpha c_{3,j}^{F,D}+(1-\alpha)\,c_{3,j}^{F,A}\right]\leq I_1+I_2+(1-\alpha)\,I_3. \tag{Zero profits}$$

$$u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq u_A(I_2)+\alpha u_D(c_{3,j}^{F,D})+(1-\alpha)\,u_A(c_{3,j}^{F,A}). \tag{IC}$$

$$u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq V, \tag{RP_1}$$

$$\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq U\left(I_3\right) \tag{RP_R}$$

and

$$\alpha u_D(c_{3,j}^{F,D})+(1-\alpha)\,u_A(c_{3,j}^{F,A})\geq U\left(I_3\right). \tag{RP_F}$$

By a duality argument, the solution of this program solves:

$$\min_{\left(c_{1,j},c_{2,j}^{R},c_{3,j}^{R,D},c_{3,j}^{R,A},c_{3,j}^{F,D},c_{3,j}^{F,A}\right)} c_{1,j}+(1-l)\left[c_{2,j}^{R}+\alpha c_{3,j}^{R,D}+(1-\alpha)\,c_{3,j}^{R,A}\right]+l\left[I_2+\alpha c_{3,j}^{F,D}+(1-\alpha)\,c_{3,j}^{F,A}\right]$$

subject to

$$u_A(c_{1,j})+u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq \bar{u},$$

$$u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq u_A(I_2)+\alpha u_D(c_{3,j}^{F,D})+(1-\alpha)\,u_A(c_{3,j}^{F,A}), \tag{IC}$$

$$u_A(c_{2,j}^{R})+\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq V, \tag{RP_1}$$

$$\alpha u_D(c_{3,j}^{R,D})+(1-\alpha)\,u_A(c_{3,j}^{R,A})\geq U\left(I_3\right) \tag{RP_R}$$

and

$$\alpha u_D(c_{3,j}^{F,D})+(1-\alpha)\,u_A(c_{3,j}^{F,A})\geq U\left(I_3\right). \tag{RP_F}$$

for some $\bar{u}\in\mathbb{R}$. Letting $v_A\equiv u_A^{-1}$ and $v_D\equiv u_D^{-1}$, we can rewrite this program in terms of utils rather than consumption:

$$\min_{\left(u_{1,j},u_{2,j}^{R},u_{3,j}^{R,D},u_{3,j}^{R,A},u_{3,j}^{F,D},u_{3,j}^{F,A}\right)} \begin{array}{l} v_A\left(u_{1,j}\right)+(1-l)\left[v_A\left(u_{2,j}^{R}\right)+\alpha v_D\left(u_{3,j}^{R,D}\right)+(1-\alpha)\,v_A\left(u_{3,j}^{R,A}\right)\right] \\ +l\left[I_2+\alpha v_D\left(u_{D,j}^{F}\right)+(1-\alpha)\,v_A\left(u_{A,j}^{F}\right)\right] \end{array}$$

subject to

$$u_{1,j}+u_{2,j}^{R}+\alpha u_{3,j}^{R,D}+(1-\alpha)\,u_{3,j}^{R,A}\geq \bar{u},$$

$$u_{2,j}^{R}+\alpha u_{3,j}^{R,D}+(1-\alpha)\,u_{3,j}^{R,A}\geq u_A(I_2)+\alpha u_{3,j}^{F,D}+(1-\alpha)\,u_{3,j}^{F,A}, \tag{IC}$$

$$u_{2,j}^{R}+\alpha u_{3,j}^{R,D}+(1-\alpha)\,u_{3,j}^{R,A}\geq V, \tag{RP_1}$$

$$\alpha u_{3,j}^{R,D}+(1-\alpha)\,u_{3,j}^{R,A}\geq U\left(I_3\right) \tag{RP_R}$$

and

$$\alpha u_{3,j}^{F,D} + (1-\alpha)\, u_{3,j}^{F,A} \geq U\left(I_3\right). \tag{RP_F}$$

It is straightforward to verify that the set of vectors satisfying the linear constraints is non-empty (in fact, we will construct one such vector below). Therefore, this program corresponds to the minimization of a continuous function in a non-empty and compact set, so a solution exists. Moreover, because the objective function is strictly convex and the feasibility set is convex, the solution is unique.

Proof of Lemma 1

We start with the renegotiation proofness constraint in period 1:

$$u_A(c_2^R) + \alpha u_D(c_D^{3,R}) + (1-\alpha)\, u_A(c_A^{3,R}) \geq V, \tag{RP_1}$$

where the outside option V solves the subprogram:

$$V \equiv \max_{\tilde{c}_2^R, \tilde{c}_D^{3,R}, \tilde{c}_A^{3,R}} u_A(\tilde{c}_2^R) + \alpha u_D(\tilde{c}_3^{R,D}) + (1-\alpha)\, u_A(\tilde{c}_3^{R,A})$$

subject to

$$(1-l)\left[\tilde{c}_2^R + \alpha \tilde{c}_3^{R,D} + (1-\alpha)\, \tilde{c}_3^{R,A}\right] + l\left[I_2 + \alpha \tilde{c}_3^{F,D} + (1-\alpha)\, \tilde{c}_3^{F,A}\right] \leq I_2 + I_3 \tag{Zero profits}$$

$$\alpha u_D(\tilde{c}_3^{R,D}) + (1-\alpha)\, u_A(\tilde{c}_3^{R,A}) \geq U\left(I_3\right) \tag{RP_R}$$

$$\alpha u_D(\tilde{c}_3^{F,D}) + (1-\alpha)\, u_A(\tilde{c}_3^{F,A}) \geq U\left(I_3\right) \tag{RP_F}$$

Note that (RP_F) must bind in the program above (otherwise, it would be possible to increase the objective by transferring consumption from $\tilde{c}_3^{F,D}$ or $\tilde{c}_3^{F,A}$ to $\tilde{c}_3^{R,D}$ or $\tilde{c}_3^{R,A}$).

There must be perfect consumption smoothing after forgetting to pay—i.e., $u_D'(\tilde{c}_3^{F,D}) = u_A'(\tilde{c}_3^{F,A})$—, which minimizes the cost of providing utility $U(I_3)$ in constraint (RP_F). There must also be perfect consumption smoothing after remembering to pay, which, not only minimizes the cost of providing utility $U(I_3)$ in (RP_S) but also maximizes the objective function.

Let $c_3^R \equiv \alpha c_3^{R,D} + (1-\alpha)\, c_3^{R,A}$ denote the average consumption in period 3 when the individual remembers to pay (which he believes will happens with probability 1). Using the previous observations, we can rewrite the outside option as

$$V \equiv \max_{\tilde{c}_2^R, \tilde{c}_3^R} u_A(\tilde{c}_2^R) + U(\tilde{c}_3^R) \tag{11}$$

subject to

$$\tilde{c}_2^R + \tilde{c}_3^R \leq I_2 + I_3 \tag{12}$$

$$\tilde{c}_3^R \geq I_3 \tag{13}$$

where (12) is the zero profits condition and (13) is the renegotiation proofness constraint (RP_R).

Recall that the equilibrium program is:

$$\max_{\left(c_1, c_2^R, c_3^{R,D}, c_3^{R,A}, c_3^{F,D}, c_3^{F,A}\right)} u_A(c_1) + u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A})$$

subject to (*Zero Profits*), (RP_R), (RP_F), (RP_1), and (IC).

As in the renegotiation program considered above, (RP_F) must bind. Otherwise, it would be possible to relax both (IC) and (*Zero Profits*) by reducing $c_3^{F,D}$ or $c_3^{F,A}$, while not affecting any other constraint or the objective.

Note also that solution must provide full insurance against mortality conditional on forgetting to pay: $u_D'(c_3^{F,D}) = u_A'(c_3^{F,A})$ with $\alpha c_3^{F,D} + (1-\alpha)\, c_3^{F,A} = I_3$. This follows from the fact that the solution must minimize the cost of providing utility $U(I_3)$ in case of forgetting (so as to satisfy RP_F), and this is obtained by providing full insurance. Substituting these conditions in the equilibrium program, we obtain:

$$\max_{\mathbf{c}} u_A(c_1) + u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A})$$

subject to

$$\frac{c_1}{1-l} + c_2^R + \alpha c_3^{R,D} + (1-\alpha)\, c_3^{R,A} \leq \frac{I_1}{1-l} + I_2 + I_3 \tag{Zero profits}$$

$$\alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq U(I_3) \tag{RP_R}$$

$$u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq V \tag{RP_1}$$

$$u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq u_A(I_2) + U(I_3) \tag{IC}$$

By the same argument as in the renegotiation program, there must be full insurance against mortality conditionally on remembering to pay:

$$u_D'(c_3^{R,D}) = u_A'(c_3^{R,A}).$$

Then, (RP_R) becomes

$$\alpha c_3^{R,D} + (1-\alpha)\, c_3^{R,A} \geq I_3.$$

Letting, $c_3^R \equiv \alpha c_3^{R,D} + (1-\alpha)\, c_3^{R,A}$, we can rewrite the program as

$$\max_{c_1, c_2^R, c_3^R} u_A(c_1) + u_A(c_2^R) + U(c_3^R) \tag{14}$$

subject to

$$\frac{c_1}{1-l} + c_2^R + c_3^R \leq \frac{I_1}{1-l} + I_2 + I_3 \tag{Zero profits}$$

$$c_3^R \geq I_3 \tag{ RP_R}$$

$$u_A(c_2^R) + U(c_3^R) \geq V \tag{RP_1}$$

$$u_A(c_2^R) + U(c_3^R) \geq u_A(I_2) + U(I_3) \tag{IC}$$

We claim that any allocation satisfying (RP_1) also satisfies (IC), so that (IC) can be omitted from this program. To see this, recall that V solves Program (11)-(13). But, since $\tilde{c}_2^R = I_2$ and $\tilde{c}_3^R = I_3$ satisfies the constraints (12) and (13), revealed preference gives

$$V \geq u_A(I_2) + U(I_3),$$

showing that (RP_1) is tighter than (IC).

Let $\{\hat{c}_1, \hat{c}_2^R, \hat{c}_3^R\}$ be a solution of the program above. We claim that we must have

$$\left(\hat{c}_2^R, \hat{c}_3^R\right) = \arg\max_{c_2^R, c_3^R} u_A(c_2^R) + U(c_3^R)$$

subject to

$$c_3^R \geq I_3 \tag{ RP_R}$$

and

$$c_2^R + c_3^R = \hat{c}_2^R + \hat{c}_3^R.$$

First note that this program corresponds to the maximization of a strictly concave function in a convex set, so the solution must be unique. Suppose $\left(\hat{c}_2^R, \hat{c}_3^R\right)$ is not the solution, so there exists (c_2^R, c_3^R) satisfying the constraints which has

$$u_A(c_2^R) + U(c_3^R) > u_A(\hat{c}_2^R) + U(\hat{c}_3^R).$$

But this implies that $\{\hat{c}_1, c_2^R, c_3^R\}$ is also feasible---i.e., it satisfies the (Zero Profits), (RP_R), (RP_1), and (IC)---and attains a higher objective, contradicting the optimality of $\{\hat{c}_1, \hat{c}_2^R, \hat{c}_3^R\}$.

Let the indirect utility of future consumption be:

$$W(C) \equiv \max u_A(c_2^R) + U(c_3^R)$$

subject to

$$c_3^R \geq I_3 \tag{ RP_R}$$

and

$$c_2^R + c_3^R = C.$$

Note that $V = W(I_2 + I_3)$, so RP2 can be rewritten as

$$W(C) \geq W(I_2 + I_3),$$

which, by the monotonicity of W, is equivalent to

$$C \geq I_2 + I_3. \tag{RP_1}$$

By zero profits (which must bind at the optimum), we can rewrite this constraint as

$$c_1 \leq I_1. \tag{RP_2}$$

Therefore, the equilibrium program can be rewritten as

$$\max_{c_1, C} u_A(c_1) + W(C)$$

subject to

$$\frac{c_1}{1-l} + C = \frac{I_1}{1-l} + I_2 + I_3 \tag{Zero profits}$$

$$c_1 \leq I_1. \tag{RP_1}$$

Proof of Proposition 1

The equilibrium contract solves:

$$\max_{c_1, c_2^R, c_3^R} u_A(c_1) + u_A(c_2^R) + U(c_3^R)$$

subject to

$$\frac{c_1}{1-l} + c_2^R + c_3^R \leq \frac{I_1}{1-l} + I_2 + I_3 \tag{Zero profits}$$

$$c_1 \leq I_1 \tag{RP_1}$$

$$c_3^R \geq I_3 \tag{RP_R}$$

The necessary FOCs for the solution to entail $c_1 = I_1$ are:

$$(1-l)\, u_A'(I_1) \geq u_A'(c_2^R) \geq U'(c_3^R), \tag{15}$$

with $u_A'(c_2^R) = U'(c_3^R)$ if $c_3^R > I_3$, as well as the zero profits condition (which must bind):

$$c_2^R + c_3^R = I_2 + I_3. \tag{16}$$

By zero profits, we have:

$$c_3^R \geq I_3 \iff c_2^R \leq I_2 \iff u_A'(c_2^R) \geq u_A'(I_2).$$

Substituting in (15), we find that the following inequality is necessary for the solution to entail $c_1 = I_1$:

$$(1-l)\,u_A'(I_1) \geq u_A'(I_2).$$

Proof of Claim made in Footnote 30

Let $\mathscr{W}(s) \equiv U(s+I_2+I_3)$ denote the utility from saving s dollars (in future value) to period 3. By the auxiliary program, the following liquidity constraint must hold: $s \geq 0$. The corollary below determines how the equilibrium consumption in the first period changes with the probability of forgetting to pay l:

Corollary 1. *c_1 is weakly decreasing (increasing) in the probability of forgetting to pay l if the elasticity of $\mathscr{W}$ is greater (smaller) than 1: $-\frac{\mathscr{W}'(s)}{s\mathscr{W}''(s)} \geq (\leq) 1$.*

Proof. There are two possible cases. If (RP_1) does not bind, c_1 is implicitly determined by the following Euler equation:

$$u_A'(c_1) = \frac{1}{1-l} W'\left(\frac{I_1-c_1}{1-l} + I_2 + (1-\alpha)\,I_3\right).$$

Using the Implicit Function Theorem, we find that $\frac{dc_1}{dl} < 0$ if and only if

$$-\frac{\left(\frac{I_1-c_1}{1-l}\right) W''\left(\frac{I_1-c_1}{1-l} + I_2 + (1-\alpha)\,I_3\right)}{W'\left(\frac{I_1-c_1}{1-l} + I_2 + (1-\alpha)\,I_3\right)} < 1.$$

If (RP_1) binds, we must have $c_1 = I_1$, which is constant in l. □

Recall that the lapse fee equals $I_1 - c_1$. Therefore, the previous corollary implies that the lapse fee is weakly increasing (decreasing) in the probability of forgetting to pay l if the elasticity of $\mathscr{W}$ is greater (smaller) than 1. Since, in equilibrium, a lapse happens whenever a consumer forgets to pay, if individuals only differ with respect to their probability of forgetting to pay, the model predicts a positive (negative) relationship between lapses and lapse fees if if the elasticity of $\mathscr{W}$ is greater (smaller) than 1.

Forgetfulness Model with Sophistication (Subsection 4.1.4)

We now formally consider the model described in Subsection 4.1.4, in which consumers have rational expectations about their likelihood of forgetting to pay the fee. As before, forgetting to pay the fee corresponds to consuming the entire income in period 2: $c_2^F = I_2$.

We start by specifying the renegotiation proofness constraints, which, as before, arise from the fact that the consumer is allowed to drop the policy at the end of periods 1 and 2.

Renegotiation Proofness Constraints at $t = 2$

At the end period 2, the individual either remembered (R) or forgot (F) to make the payment. At that point, beliefs are the same as in the model consider previously, so the renegotiation proofness constraints remain the same:

$$\alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq U\,(I_3) \tag{RP_R}$$

$$\alpha u_D(c_3^{F,D}) + (1-\alpha)\, u_A(c_3^{F,A}) \geq U\,(I_3) \tag{RP_F}$$

Renegotiation Proofness Constraints at $t = 1$

The renegotiation proofness constraints in period 1 are different from before because the consumer now has rational expectations about the probability of forgetting to pay, and therefore takes the correct probability of forgetting to pay into account. The consumer's outside option at the end of period 1 is:

$$V \equiv \max_{\mathbf{c}}\,(1-l)\left[u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A})\right] + l\left[u_A\,(I_2) + \alpha u_D\left(c_3^{F,D}\right) + (1-\alpha)\, u_A\left(c_3^{F,A}\right)\right]$$

subject to

$$(1-l)\left[c_2^R + \alpha c_3^{R,D} + (1-\alpha)\, c_3^{R,A}\right] + l\left[I_2 + \alpha c_3^{F,D} + (1-\alpha)\, c_3^{F,A}\right] \leq I_2 + I_3 \tag{Zero profits}$$

$$\alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq U\,(I_3) \tag{RP_R}$$

$$\alpha u_D(c_3^{F,D}) + (1-\alpha)\, u_A(c_3^{F,A}) \geq U\,(I_3) \tag{RP_F}$$

$$u_A(c_2^R) + \alpha u_D(c_3^{R,D}) + (1-\alpha)\, u_A(c_3^{R,A}) \geq u_A(I_2) + \alpha u_D(c_3^{F,D}) + (1-\alpha)\, u_A(c_3^{F,A}) \tag{IC}$$

It is straightforward to see that the solution of this program must provide full insurance in period 3:

$$u_D'(c_3^{j,D}) = u_A'(c_3^{j,A}) \quad j = R, F.$$

Therefore, the outside option can be rewritten as

$$V \equiv \max_{c}\,(1-l)\left[u_A(c_2^R) + U\left(C_3^R\right)\right] + l\left[u_A\,(I_2) + U\left(c_3^F\right)\right]$$

subject to

$$(1-l)\left(c_2^R + c_3^R\right) + l\left(I_2 + c_3^F\right) \leq I_2 + I_3 \tag{Zero profits}$$

$$u_A(c_2^R) + U\left(C_3^R\right) \geq u_A(I_2) + U\left(c_3^F\right) \tag{IC}$$

$$c_3^j \geq I_3 \quad j = R, F \tag{RP_j}$$

Equilibrium Program

The equilibrium program is:

$$\max_{\mathbf{c}} u_A(c_1)+(1-l)\left[u_A(c_2^R)+\alpha u_D(c_3^{R,D})+(1-\alpha)\,u_A(c_3^{R,A})\right]+l\left[u_A\left(I_2\right)+\alpha u_D\left(c_3^{F,D}\right)+(1-\alpha)\,u_A\left(c_3^{F,A}\right)\right]$$

subject to

$$c_1+(1-l)\left[c_2^R+\alpha c_3^{R,D}+(1-\alpha)\,c_3^{R,A}\right]+l\left[I_2+\alpha c_3^{F,D}+(1-\alpha)\,c_3^{F,A}\right]\leq I_1+I_2+I_3 \qquad \text{(Zero profits)}$$

$$\alpha u_D(c_3^{R,D})+(1-\alpha)\,u_A(c_3^{R,A})\geq U\left(I_3\right) \qquad (RP_R)$$

$$\alpha u_D(c_3^{F,D})+(1-\alpha)\,u_A(c_3^{F,A})\geq U\left(I_3\right) \qquad (RP_F)$$

$$(1-l)\left[u_A(c_2^R)+\alpha u_D(c_D^{3,R})+(1-\alpha)\,u_A(c_A^{3,R})\right]+l\left[u_A\left(I_2\right)+\alpha u_D\left(c_3^{F,D}\right)+(1-\alpha)\,u_A\left(c_3^{F,A}\right)\right]\geq V \qquad (RP_1)$$

$$u_A(c_2^R)+\alpha u_D(c_3^{R,D})+(1-\alpha)\,u_A(c_3^{R,A})\geq u_A(I_2)+\alpha u_D(c_3^{F,D})+(1-\alpha)\,u_A(c_3^{F,A}) \qquad \text{(IC)}$$

As before, it is straightforward to show that the period-1 renegotiation proofness constraints ($RP_j,\ j=R,F$) can be substituted by liquidity constraints:

$$\alpha c_3^{j,D}+(1-\alpha)\,c_3^{j,A}\geq I_3,\quad j=R,F.$$

Using the function U, we can rewrite the equilibrium program as

$$\max_{\mathbf{c}} u_A(c_1)+(1-l)\left[u_A(c_2^R)+U\left(c_3^R\right)\right]+l\left[u_A\left(I_2\right)+U\left(c_3^D\right)\right]$$

subject to

$$c_1+(1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)\leq I_1+I_2+I_3 \qquad \text{(Zero profits)}$$

$$c_3^R\geq I_3 \qquad (RP_R)$$

$$c_3^F\geq I_3 \qquad (RP_F)$$

$$u_A(c_2^R)+U\left(c_3^R\right)\geq u_A(I_2)+U\left(c_3^F\right) \qquad \text{(IC)}$$

$$(1-l)\left[u_A(c_2^R)+U\left(c_3^R\right)\right]+l\left[u_A\left(I_2\right)+U\left(c_3^F\right)\right]\geq V \qquad (RP_1)$$

Let $\mathscr{V}(C)$ denote the highest continuation utility that can be obtained at period 2 with a total expected consumption of C:

$$\mathscr{V}(C)\equiv\max_{c}\,(1-l)\left[u_A(c_2^R)+U\left(C_3^R\right)\right]+l\left[u_A\left(I_2\right)+U\left(c_3^F\right)\right]$$

subject to

$$(1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)\leq C$$

$$c_3^j\geq I_3 \quad j=R,F$$

$$u_A(c_2^R)+U\left(c_3^R\right)\geq u_A(I_2)+U\left(c_3^F\right) \tag{IC}$$

Note that $V=\mathscr{V}(I_2+I_3)$. It is straightforward to show that the solution of the equilibrium program must solve this continuation program for $C=(1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)$. Thus, ($RP_1$) can be written as

$$\mathscr{V}\left((1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)\right)\geq \mathscr{V}\left(I_2+I_3\right),$$

which, since $\mathscr{V}$ is strictly increasing, can be further simplified to:

$$(1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)\geq I_2+I_3.$$

Then, using the zero profit constraint, which must bind at the solution, it follows that RP_2 is equivalent to the following liquidity constraint:

$$c_1\leq I_1.$$

The equilibrium program can thus be written as:

$$\max_{\mathbf{c}} u_A(c_1)+(1-l)\left[u_A(c_2^R)+U\left(c_3^R\right)\right]+l\left[u_A\left(I_2\right)+U\left(c_3^F\right)\right]$$

subject to

$$c_1+(1-l)\left(c_2^R+c_3^R\right)+l\left(I_2+c_3^F\right)\leq I_1+I_2+I_3 \tag{Zero profits}$$

$$u_A(c_2^R)+U\left(c_3^R\right)\geq u_A(I_2)+U\left(c_3^F\right) \tag{IC}$$

$$c_3^R\geq I_3 \tag{RP_R}$$

$$c_3^F\geq I_3 \tag{RP_F}$$

$$c_1\leq I_1 \tag{RP_1}$$

There are two possibilities depending on whether IC binds.

Case 1: IC does not bind

If the IC does not bind, the solution must have full insurance against the risk of forgetting to make the payment $c_3^R=c_3^D$. Substituting back in the IC, we find that it holds if and only if the equilibrium contract has a negative premium at period 2, so the individual does not have an incentive to forget to "pay": $c_2^R\geq I_2$. Then, the consumer collects a payment from the insurance company in period 2, and forgetting to do so does not affect future consumption.

This is not realistic for three reasons. First, because the premium is negative, "forgetting to pay" actually corresponds to forgetting to collect a benefit. Insurance policies do not typically make payments before they expire. Second, forgetting to pay does not cause the policyholder to lapse (the consumer either always lapses or he never lapses). Third, because either everyone lapses or no one lapses, policies are not lapse-based, meaning that there are no cross-subsidies between lapsers and non-lapsers.

Case 2: IC binds

Now, consider the case where the IC binds. The necessary FOCs are:

$$u_A'(c_1) = \lambda + \mu_{RP_2}$$

$$u_A'(c_2^R)\left(1+\frac{\mu_{IC}}{1-l}\right) = \lambda$$

$$U'(c_3^R)\left(1+\frac{\mu_{IC}}{1-l}\right) + \frac{\mu_{RP_R}}{1-l} = \lambda$$

$$U'(c_3^F)\left(1-\frac{\mu_{IC}}{l}\right) + \frac{\mu_{RP_F}}{l} = \lambda$$

We claim that the solution must entail $c_3^F \leq c_3^R$. Suppose $c_3^F > c_3^R$. Then, we must have $\mu_{RP_F} = 0$, so that the previous conditions give:

$$U'(c_3^F)\left(1-\frac{\mu_{IC}}{l}\right) = U'(c_3^R)\left(1+\frac{\mu_{IC}}{1-l}\right) + \frac{\mu_{RP_R}}{1-l}.$$

Rearranging, we obtain

$$U'(c_3^F) \geq U'(c_3^F)\left(1-\frac{\mu_{IC}}{l}\right) \geq U'(c_3^R)\left(1+\frac{\mu_{IC}}{1-l}\right) \geq U'(c_3^R),$$

which, by the concavity of U, gives $c_3^F \leq c_3^R$, a contradiction.

Since $c_3^F \leq c_3^R$, we can omit (RP_R) from the program (i.e., policies cannot lapse only for individuals who remember to pay the premium). Removing RP_R from the program, we find that there must be perfect smoothing between periods 2 and 3 for those that remember to pay:

$$u_A'(c_2^R) = U'\left(c_3^R\right). \tag{17}$$

By the binding IC constraint, we must have

$$u_A(c_2^R) + U\left(c_3^R\right) = u_A(I_2) + U\left(c_3^F\right). \tag{18}$$

Then, since $c_3^F \leq c_3^R$, it must be the case that $c_2^R \leq I_2$.

By 18, c_2^R and c_3^R provide the same utility as I_2 and c_3^F more efficiently---since, by 17 there is perfect smoothing between c_2^R and c_3^R. Therefore, c_2^R and c_3^R must be cheaper than I_2 and c_3^F:

$$c_2^R + c_3^R \leq I_2 + c_3^F.$$

Substituting this inequality in the (binding) zero profits condition, we obtain:

$$c_1 + I_2 + c_3^F \geq I_1 + I_2 + I_3 \geq c_1 + c_2^R + c_3^R, \tag{19}$$

and both inequalities are strict if $c_3^F < c_3^R$. In words: firms make (weakly) positive profits on consumers who remember to pay and lose money on those who forget. That is, cross-subsidies go in the opposite direction from the one observed in practice.

Suppose there are lapses in period 2. Since $c_3^F \leq c_3^R$, the policyholder must lapse after forgetting to pay:

$$c_3^F = I_3.$$

Substituting in (19), we obtain

$$c_1 \geq I_1,$$

which, by (RP_1), gives $c_1 = I_1$. Therefore, the firm makes zero profits on those who forget to pay, so (by zero profits) it must also make zero profits on those who remember to pay.

To summarize, either there are no lapses after forgetting to pay, or there is no cross-subsidy from lapsers to non-lapsers. And, when there are no lapses, cross-subsidies go in the opposite direction from the one observed in practice.

Proof of Proposition 2

The equilibrium contract maximizes

$$u_A(c_1^{NS}) + \alpha u_D\left(c_D^{NS}\right) + (1-\alpha)\, u_A\left(c_A^{NS}\right) \tag{20}$$

subject to

$$l\left[c_1^S + \alpha c_D^S + (1-\alpha)c_A^S\right] + (1-l)\left[c_1^{NS} + \alpha c_D^{NS} + (1-\alpha)c_A^{NS}\right] = W + I_1 + I_2 - lL \qquad \text{(Zero Profits)}$$

$$u_A(c_1^S) + \alpha u_D(c_D^S) + (1-\alpha)u_A(c_A^S) \geq u_A(c_1^{NS} - L) + \alpha u_D(c_D^{NS}) + (1-\alpha)u_A(c_A^{NS}) \qquad (IC_S)$$

$$u_A(c_1^{NS}) + \alpha u_D(c_D^{NS}) + (1-\alpha)u_A(c_A^{NS}) \geq u_A(c_1^S + L) + \alpha u_D(c_D^S) + (1-\alpha)u_A(c_A^S) \qquad (IC_{NS})$$

$$u_A(c_1^{NS}) + \alpha u_D(c_D^{NS}) + (1-\alpha)u_A(c_A^{NS}) \geq \mathscr{V}(I_1 + I_2) \qquad (RP_{NS})$$

$$u_A(c_1^S) + \alpha u_D(c_D^S) + (1-\alpha)u_A(c_A^S) \geq \mathscr{V}(I_1 + I_2 - L) \qquad (RP_S)$$

Ignore (IC_{NS}) and (RP_{NS}) for the moment (we will verify that these constraints hold later). Note that

(IC_S) does not bind if the solution satisfies

$$\mathscr{V}(I_1+I_2-L) > u_A(c_1^{NS}-L)+\alpha u_D(c_D^{NS})+(1-\alpha)u_A(c_A^{NS}),$$

and (RP_S) does not bind if the reverse inequality holds.

Case 1. (IC_S) **does not bind.**

Since (IC_S) does not bind, the equilibrium contract maximizes (20) subject to *(Zero Profits)* and (RP_S). By the same duality argument as in the proof of Lemma 3 this program has a a unique solution, and there is full insurance conditional on the income shock:

$$u_A'(c_1^S) = u_D'(c_D^S) = u_A'(c_A^S) \text{ and } u_A'(c_1^{NS}) = u_D'(c_D^{NS}) = u_A'(c_A^{NS}).$$

Substituting in (RP_S), which must bind, we find that the total expected consumption after the shock equals the expected income after the shock:

$$c_1^S+\alpha c_D^S+(1-\alpha)c_A^S = I_1+I_2-L. \tag{21}$$

This means that the insurance company makes a profit of the initial income W if the individual has an income shock. By *(Zero Profits)*, the individual's expected consumption when there is no shock equals

$$c_1^{NS}+\alpha c_D^{NS}+(1-\alpha)c_A^{NS} = \frac{W}{1-l}+I_1+I_2, \tag{22}$$

so the insurance company loses $-\frac{l}{1-l}W$ if there is no income shock.

Since there is perfect smoothing conditional on both S and NS, we can use the definition of $\mathscr{V}$ to rewrite (IC_S) as:

$$u_A(c_1^{NS})-u_A(c_1^{NS}-L) \geq \mathscr{V}\left(\frac{W}{1-l}+I_1+I_2\right)-\mathscr{V}(I_1+I_2-L).$$

We now verify that the omitted constraints (RP_{NS}) and (IC_{NS}) are satisfied. Because there is perfect smoothing conditional on NS, (RP_{NS}) becomes

$$c_1^{NS}+\alpha c_D^{NS}+(1-\alpha)c_A^{NS} \geq I_1+I_2.$$

Substituting (22), verifies that this inequality holds. To verify that (IC_{NS}) holds, note that

$$\begin{array}{rcl} u_A(c_1^{NS})+\alpha u_D(c_D^{NS})+(1-\alpha)u_A(c_A^{NS}) & = & \mathscr{V}\left(\frac{W}{1-l}+I_1+I_2\right) \\ & \geq & \mathscr{V}\left(\frac{W}{1-l}+I_1+I_2-L\right) \\ & \geq & u_A(c_1^{NS}-L)+\alpha u_D(c_D^{NS})+(1-\alpha)u_A(c_A^{NS}) \end{array},$$

where the first line uses full insurance conditional on NS and (22), the second line uses the fact that $\mathscr{V}$ is increasing, and the third line uses (22) and revealed preference.

Case 2. (RP_S) **does not bind.**

Since (RP_S) does not bind, the equilibrium contract maximizes (20) subject to *(Zero Profits)* and (IC_S). Again, we can use a duality argument to rewrite the program as the minimization of a continuous and strictly convex function subject to linear constraints, establishing existence and uniqueness of the solution. Calculating the first-order conditions, we find that there is full insurance conditional on the shock:

$$u_A'(c_1^S) = u_D'(c_D^S) = u_A'(c_A^S), \tag{23}$$

and imperfect intertemporal smoothing:

$$u_A'(c_1^{NS}) > u_A'(c_A^{NS}) = u_D'(c_D^{NS}). \tag{24}$$

We now verify that the omitted constraint (IC_{NS}) is satisfied. Use the binding (IC_S) to rewrite (IC_{NS}) as

$$u_A(c_1^{NS}) - u_A(c_1^{NS} - L) \geq u_A(c_1^S + L) - u_A(c_1^S),$$

which, by the concavity of u_A, holds if and only if $c_1^S \geq c_1^{NS} - L$. That is, (IC_{NS}) holds as long as reporting a loss would increase period-1 consumption relative to absorbing the income loss. In fact, this inequality is strict in the solution.

Suppose for the sake of contradiction that $c_1^S \leq c_1^{NS} - L$. Recall that there is full insurance against mortality conditional on both S and NS:

$$u_A'(c_A^S) = u_D'(c_D^S) \text{ and } u_A'(c_A^{NS}) = u_D'(c_D^{NS}).$$

Since (IC_S) holds with equality, it then follows that $c_A^S \geq c_A^{NS}$ and $c_D^S \geq c_D^{NS}$. Then, because u_A is concave, we have

$$u_A'(c_1^{NS}) \leq u_A'(c_1^{NS} - L) \leq u_A'(c_1^S)$$

and

$$u_A'(c_A^S) \leq u_A'(c_A^{NS}).$$

Moreover, perfect smoothing conditional on the shock (23) gives

$$u_A'(c_1^{NS}) \leq u_A'(c_1^S) = u_A'(c_A^S) \leq u_A'(c_A^{NS}),$$

which contradicts (24). Thus, $c_1^S > c_1^{NS} - L$, $c_A^S < c_A^{NS}$, $c_D^S < c_D^{NS}$, and (IC_{NS}) holds.

To conclude the proof, we need to show that $\pi^S > 0 > \pi^{NS}$ and verify that (RP_{NS}) does not bind. Let

$\bar{c}_1^{NS}$, $\bar{c}_A^{NS}$, and $\bar{c}_D^{NS}$ solve

$$\max u_A(c_1^{NS}) + \alpha u_D\left(c_D^{NS}\right) + (1-\alpha)u_A\left(c_A^{NS}\right)$$

subject to

$$c_1^{NS} + \alpha c_D^{NS} + (1-\alpha)c_A^{NS} \leq W + I_1 + I_2.$$

Note that the profile $\left(c_1^{NS}, c_D^{NS}, c_A^{NS}, c_1^S, c_D^S, c_A^S\right) = \left(\bar{c}_1^{NS}, \bar{c}_A^{NS}, \bar{c}_D^{NS}, \bar{c}_1^{NS} - L, \bar{c}_A^{NS}, \bar{c}_D^{NS}\right)$ is feasible in the maximization of (20) subject to *(Zero Profits)* and (IC_S). By revealed preference, it cannot attain a higher objective:

$$\begin{array}{rcl} u_A(c_1^{NS}) + \alpha u_D\left(c_D^{NS}\right) + (1-\alpha)u_A\left(c_A^{NS}\right) & \geq & u_A(\bar{c}_1^{NS}) + \alpha u_D\left(\bar{c}_D^{NS}\right) + (1-\alpha)u_A\left(\bar{c}_A^{NS}\right) \\ & = & \mathscr{V}\left(W + I_1 + I_2\right) \end{array},$$

showing that (RP_{NS}) holds. Moreover, since $\mathscr{V}\left(W + I_1 + I_2\right)$ is the highest utility that can be provided at cost $W + I_1 + I_2$, it follows that:

$$c_1^{NS} + \alpha c_D^{NS} + (1-\alpha)c_A^{NS} \geq W + I_1 + I_2. \tag{25}$$

In fact, because there is imperfect smoothing (24), this inequality must be strict. Hence, the insurance company makes negative profits if there is no shock, so, by zero expected profits, it must make positive profits if there is a shock:

$$\pi^S > 0 > \pi^{NS}.$$

Model with a Continuum of Losses (Theorem 1)

Formulation of the Program

To keep the notation close to the optimal control literature, we associate each possible *loss* with a "type" t. Types are distributed according to a differentiable PDF f with full support in the interval of possible losses $[0, T]$, where $0 < T < I_1 + I_2$. For each possible loss $t \in [0, T]$, let $c(t)$ denote the consumption in period 1, let $c_A(t)$ denote the consumption in period 2 if the individual is alive, and let $c_D(t)$ denote the bequests in period 2 (i.e., the resources repaid if the consumer dies). Let $V(t) \equiv \alpha u(c_D(t)) + (1-\alpha)u(c_A(t))$ denote the continuation payoff of a consumer who gets a loss of t.

Since firms do not observe the loss, contracts must be incentive compatible. The incentive-compatibility (IC) constraints are:

$$u(c(t)) + V(t) \geq u(c(\hat{t}) - t + \hat{t}) + V(\hat{t}) \;\; \forall t, \hat{t}.$$

Following standard nomenclature from mechanism design, let

$$\mathscr{U}(t) \equiv u(c(t)) + V(t)$$

denote the indirect utility of type t. The following lemma provides a standard characterization of incentive compatibility.

Lemma 4. *IC is satisfied if and only if $\dot{\mathscr{U}}(t) = -u'(c(t))$ and $c(t)+t$ is non-decreasing in t.*

Proof. Let $X \equiv c+t$. Using the taxation principle, IC can be written as

$$(X(t), V(t)) \in \arg\max_{X,V} u(X-t)+V.$$

Note that the objective function satisfies single crossing:

$$\frac{d^2}{dXdt}[u(X-t)+V] = -u''(X-t) > 0.$$

Therefore, $X(t)$ must be non-decreasing, meaning that $c(t)+t$ is non-decreasing in t. By the Envelope Theorem, $\dot{\mathscr{U}}(t) = -u'(c_1(t)) < 0$. The argument for sufficiency is standard given the validity of the single-crossing condition. □

The previous lemma shows that incentive compatibility alone has the following implications:

- The amount paid in period 1, $I_1 - c(t) - t$ is decreasing in the size of the shock: people with larger shocks pay a lower premium at period 1 if they do not lapse (that is, they borrow from their policies).
- Conversely, $\dot{V} = -u'(c(t))[1+\dot{c}(t)] \leq 0$, meaning that a lower premium at period 1 is associated with less consumption in the future. Since types with larger shocks borrow more in period 1, they have to repay what they borrowed in period 2.

We now turn to constraints imposed by the possibility of lapsing. The best contract that an individual can obtain if he lapses solves:

$$\max_{c,c_A,c_D} u(c)+\alpha u(c_D)+(1-\alpha)u(c_A)$$

subject to

$$c+\alpha c_D+(1-\alpha)c_A \leq I_1+I_2-t.$$

The solution features perfect smoothing conditional on the income shock: $c = c_A = c_D = \frac{I_1+I_2-t}{2}$. Therefore, the renegotiation proofness constraints are:

$$\mathscr{U}(t) \geq 2u\left(\frac{I_1+I_2-t}{2}\right) \ \forall t.$$

The zero profits constraint is

$$\int [c(t)+\alpha c_D(t)+(1-\alpha)c_A(t)+t]\, f(t)dt \leq W+I_1+I_2.$$

The solution of the program must entail full insurance in the second period: $c_D(t) = c_A(t) =: c_2(t)$ (otherwise, it would be possible to keep the same promised continuation utility at a lower cost). As before, it is helpful to rewrite the zero profits constraint in terms of the indirect utility $\mathscr{U}$ instead of period-2 consumption c_2:

$$\mathscr{U}(t) = u(c(t)) + u(c_2(t)) \therefore c_2(t) = u^{-1}(\mathscr{U}(t) - u(c(t))).$$

Substituting in the zero profits constraint, we obtain

$$\int \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) + t\right] f(t)dt \leq W + I_1 + I_2.$$

By the exact argument as in Lemma 2, the equilibrium must solve:

$$\max_{c,\mathscr{U}} \mathscr{U}(0)$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)), \tag{IC}$$

$$\mathscr{U}(t) \geq 2u\left(\frac{I_1 + I_2 - t}{2}\right), \tag{RP}$$

$$\int \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) + t\right] f(t)dt \leq W + I_1 + I_2,$$

and $c(t) + t$ non-decreasing. In what follows, we will follow the standard approach of ignoring the monotonicity constraint, which can be verified ex-post.

To simplify notation, let $I \equiv I_1 + I_2$. It is convenient to work with the dual program:

$$\min_{c,\mathscr{U}} \int_0^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) + t\right]$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)), \tag{IC}$$

$$\mathscr{U}(t) \geq 2u\left(\frac{I_1 + I_2 - t}{2}\right), \tag{RP}$$

$$\mathscr{U}(0) \geq \bar{u}, \tag{26}$$

where $\bar{u} > 2u\left(\frac{I}{2}\right)$ so the feasible set is non-empty.[49]

This is an optimal control problem, where $\mathscr{U}$ is a state variable and c is a control variable. Equation (26), which must hold as an equality in any solution, gives the initial condition for $\mathscr{U}$. The terminal condition $\mathscr{U}(T)$ is free. Equation (IC) is a standard constraint, which gives the law of motion for the

[49] To see that $\bar{u} > 2u\left(\frac{I}{2}\right)$ implies that the set of functions $(c, \mathscr{U})$ that satisfy the constraints is non-empty, note that $\mathscr{U}(t) = 2u\left(u^{-1}\left(\frac{\bar{u}}{2}\right) - \frac{t}{2}\right)$ and $c(t) = u^{-1}\left(\frac{\bar{u}}{2}\right) - \frac{t}{2}$ satisfies all the constraints.

state variable as a function of the control variable c.

Equation (RP) is a first-order pure state inequality constraint — see Hartl, Sethi, and Vickson (1995, Section 5) and Grass et al. (2008, Section 3.6). It is a pure state inequality constraint because it involves the state variable $\mathscr{U}$ and "time" t but not the control variable c. To see why it has order one, rewrite it in its canonical form

$$h(\mathscr{U},t) \equiv \mathscr{U} - 2u\left(\frac{I-t}{2}\right),$$

so (RP) becomes

$$h(\mathscr{U}(t),t) \geq 0.$$

Total differentiation, gives:

$$\frac{d}{dt}\left[h(\mathscr{U}(t),t)\right] = \dot{\mathscr{U}}(t) + u'\left(\frac{I-t}{2}\right) = u'\left(\frac{I-t}{2}\right) - u'(c(t)),$$

where the last equality used the law of motion (IC). Therefore, differentiating h once allows us to express h as a constraint involving the control variable.

Notation

Let $V(\bar{u}_a, a)$ denote the "continuation cost" at time a with initial state $\bar{u}_a$:

$$V(\bar{u}_a, a) := \min_{c,\mathscr{U}} \int_a^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t)))\right] f(t)dt$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)) \ \forall t, \tag{IC}$$

$$\mathscr{U}(t) \geq 2u\left(\frac{I-t}{2}\right) \ \forall t, \tag{RP}$$

$$\mathscr{U}(a) = \bar{u}_a,$$

where $\bar{u}_a \geq 2u\left(\frac{I-a}{2}\right)$ so the feasible set is non-empty. It is straightforward to verify that $V(\cdot, a)$ is a non-decreasing function of $\bar{u}_a$.

Below, we will use the fact that, by the Principle of Optimality, if $(\mathscr{U}, c)$ solves the original program then, for any $t_L, t_H \in (0,T)$ with $0 \leq t_L \leq t_H < T$, the restriction of $(\mathscr{U}, c)$ to $[t_L, t_H]$ must solve:

$$\min_{\tilde{c},\tilde{\mathscr{U}}} \int_{t_L}^{t_H} \left[\tilde{c}(t) + u^{-1}\left(\tilde{\mathscr{U}}(t) - u(\tilde{c}(t))\right)\right] f(t)dt + V(\tilde{\mathscr{U}}(t_H), t_H)$$

subject to

$$\dot{\tilde{\mathscr{U}}}(t) = -u'(\tilde{c}(t)) \ \forall t, \tag{IC}$$

$$\tilde{\mathscr{U}}(t) \geq 2u\left(\frac{I-t}{2}\right) \; \forall t, \tag{RP}$$

$$\tilde{\mathscr{U}}(t_L) = \mathscr{U}(t_L). \tag{B}$$

As usual, for a given $\mathscr{U}$, we say that RP *binds* at t if (RP) holds with equality at t. It is helpful to introduce some standard notation from optimal control:

Definition 1. The point τ is called an *entry point* if there exists $\varepsilon > 0$ such that RP binds for $t \in (\tau, \tau+\varepsilon)$ but not for $t \in (\tau-\varepsilon, \tau)$; τ is called an *exit point* if there exists $\varepsilon > 0$ such that RP binds for $t \in (\tau-\varepsilon, \tau)$ but not for $t \in (\tau, \tau+\varepsilon)$; and τ is called a *contact point* if there exists $\varepsilon > 0$ such that RP binds for $t = \tau$ but not for $t \in (\tau-\varepsilon, \tau+\varepsilon)\backslash\{\tau\}$. A point is called a *junction point* if it either an entry point, an exit point, or a contact point.

Let $S \subset [0,T]$ denote the subset of points where RP binds. A contact point is an isolated point of S. Therefore, if all junction points in $[0,T]$ are contact points, then S must be a countable set, which has Lesbegue measure zero (i.e., RP does not bind for almost all t).

In the proofs below, we will use the following result:

Lemma 5. *Let $\phi(\cdot)$ be a Lipschitz continuous function. Suppose $f'(t) < \phi(f(t))$, $g'(t) \geq \phi(g(t))$, and $f(a) = g(a) = \alpha$. Then, $f(t) < g(t)$ for all $t > a$.*

Proof. Suppose not. Then, there is $b > a$ such that $f(b) \geq g(b)$. Let

$$c \equiv \inf\{x > a : \; f(x) \geq g(x)\}.$$

If $c > a$, we must have $f'(c) \geq g'(c)$, which contradicts the fact that

$$f'(c) < \phi(f(c)) = \phi(g(c)) \leq g'(c).$$

If $c = a$, then we must have $f'(a) = g'(a)$, contradicting the fact that

$$f'(a) < \phi(f(a)) = \phi(\alpha) = \phi(g(a)) \leq g'(a).$$

□

Results

As a *benchmark*, consider a version of the original program without the IC constraint:

$$\min_{c,\mathscr{U}} \int_0^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t)))\right] f(t)dt$$

subject to

$$\mathscr{U}(t) \geq 2u\left(\frac{I-t}{2}\right) \; \forall t, \tag{RP}$$

$$\mathscr{U}(0) = \bar{u}. \tag{B}$$

The solution can be obtained by minimizing the objective pointwise, which gives:

$$1 - \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} = 0 \therefore \mathscr{U}(t) = 2u(c(t))$$

for almost all t.

We are now ready to present the proof of Theorem 1 using a series of lemmas. The first lemma shows that introducing the IC constraint can only distort the control $c(t)$ downwards relative to the benchmark:

Lemma 6. *Suppose $(\mathscr{U}, c)$ solves the program. Then, $2u(c(t)) \leq \mathscr{U}(t)$ for all t.*

The proof will be given in four separate claims.

Claim 1. Let $(\mathscr{U}, c)$ satisfy IC and RP and suppose $2u(c(t)) > \mathscr{U}(t)$ for $t \in (t_L, t_H)$. Then, RP does not bind for any $t \in (t_L, t_H)$.

Proof. Suppose, for the sake of contradiction, that RP binds for some $t \in (t_L, t_H)$. Let

$$t_- = \inf\{t \in (t_L, t_H) : \; \mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right) = 0\}.$$

There are two cases: $t_- > t_L$ and $t_- = t_L$. If $t_- > t_L$, then we must have

$$\frac{d}{dt}\left[\mathscr{U}(t_-) - 2u\left(\frac{I-t_-}{2}\right)\right] \leq 0.$$

Use IC to rewrite this condition as

$$\dot{\mathscr{U}}(t_-) + u'\left(\frac{I-t_-}{2}\right) \leq 0 \therefore \frac{I-t_-}{2} \geq c(t_-). \tag{27}$$

Therefore, we have

$$\mathscr{U}(t_-) < 2u(c(t_-)) \leq 2u\left(\frac{I-t_L}{2}\right),$$

where the first inequality follows from $t_- > t_L$ and the second condition follows from 27. But this contradicts the hypothesis that $(\mathscr{U}, c)$ satisfies RP.

If $t_- = t_L^*$, then there must exist $\varepsilon > 0$ such that, for all $t \in (t_L^*, t_L^* + \varepsilon)$,

$$\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right) = 0.$$

Differentiate this condition and use IC to write

$$\frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right] = \dot{\mathscr{U}}(t) + u'\left(\frac{I-t}{2}\right) = -u'(c(t)) + u'\left(\frac{I-t}{2}\right) = 0$$

$$\therefore c(t) = \frac{I-t}{2}$$

for $t \in (t_L^*, t_L^* + \varepsilon)$, so that $2u(c^*(t)) = \mathscr{U}^*(t)$ in this interval, a contradiction, since $2u(c^*(t)) > \mathscr{U}^*(t)$ for all $t \in (t_L^*, t_H^*)$. $\square$

Claim 1 shows that t_H cannot be an exit point. There are three possibilities: (i) RP does not bind at t_H, (ii) t_H is an entry point, or (iii) t_H is a contact point. We address each of them separately:

Claim 2. Let $(\mathscr{U}, c)$ satisfy IC and RP and suppose $2u(c(t)) > \mathscr{U}(t)$ for $t \in (t_L, t_H)$. If RP does not bind at t_H, then $(\mathscr{U}, c)$ is not optimal.

Proof. Let $(\mathscr{U}, c)$ satisfy IC and RP. Suppose $2u(c(t)) > \mathscr{U}(t)$ for $t \in (t_L, t_H)$ and $\mathscr{U}(t_H) > 2u\left(\frac{I-t_H}{2}\right)$. Then, the restriction of $(\mathscr{U}, c)$ to $[t_L, t_H]$ must solve the following continuation program:

$$\min_{c,\mathscr{U}} \int_{t_L}^{t_H} \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t)))\right] f(t)dt + V(\mathscr{U}(t_H), t_H)$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)) \ \forall t, \tag{IC}$$

$$\mathscr{U}(t_L) = \bar{u}_L,$$

where $\mathscr{U}^*(t_L) = \bar{u}_L$. The optimality conditions are:

$$\lambda(t) = -\left[1 - \frac{u'(c(t))}{u'(u^{-1}(\mathscr{U}(t) - u(c(t))))}\right] \frac{f(t)}{u''(c(t))}, \tag{28}$$

$$\dot{\lambda}(t) = \frac{f(t)}{u'(u^{-1}(\mathscr{U}(t) - u(c(t))))} > 0, \tag{29}$$

and the transversality condition

$$\lambda(t_H) = -\frac{\partial V}{\partial \mathscr{U}}(\mathscr{U}(t_H), t_H) \leq 0.$$

Transversality and (29) imply $\lambda(t) < 0$ for $t < t_H$, which, by (28), implies

$$\mathscr{U}(t) > 2u(c(t)),$$

contradicting the fact that $2u(c(t)) > \mathscr{U}(t)$ for all $t \in (t_L, t_H)$. $\square$

Claim 3. Let $(\mathscr{U}, c)$ satisfy IC and RP and suppose $2u(c(t)) > \mathscr{U}(t)$ for $t \in (t_L, t_H)$. If t_H is an entry point, then $(\mathscr{U}, c)$ is not optimal.

Proof. Let $(\mathscr{U}^*, c^*)$ be a solution. Suppose first that RP binds for all $t \in [t_H^*, L)$, that is $2u(c^*(t)) = \mathscr{U}^*(t)$

for (t_H^*, T). Construct $(\mathscr{U}, c)$ as follows. Let $\mathscr{U}(t) = \mathscr{U}^*(t)$ and $c(t) = c^*(t)$ for $t < t_L^*$. For $t \geq t_L^*$, let

$$\dot{\mathscr{U}}(t) = -u'\left(u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)\right),$$

with $\mathscr{U}(t_L^*) = \mathscr{U}^*(t_L^*)$ and let $c(t) = u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)$. Note that $c(t) = u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)$ and $c^*(t) > u^{-1}\left(\frac{\mathscr{U}^*(t)}{2}\right)$. Then, since $-u'(c)$ is a strictly increasing function of c, IC and the initial conditions give:

$$\dot{\mathscr{U}}(t) < -u'\left(u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)\right),$$

$$\dot{\mathscr{U}}^*(t) > -u'\left(u^{-1}\left(\frac{\mathscr{U}^*(t)}{2}\right)\right),$$

and

$$\mathscr{U}(t_L^*) = \mathscr{U}^*(t_L^*).$$

It then follows from Lemma 5 that $\mathscr{U}(t) < \mathscr{U}^*(t)$ for all $t \in (t_L^*, t_H^*]$. Moreover, for $t \geq t_H^*$, we have

$$\dot{\mathscr{U}}(t) = -u'\left(u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)\right) \text{ and } \dot{\mathscr{U}}^*(t) = -u'\left(u^{-1}\left(\frac{\mathscr{U}^*(t)}{2}\right)\right),$$

with

$$\mathscr{U}(t_H^*) < \mathscr{U}^*(t_H^*),$$

which implies $\mathscr{U}(t) < \mathscr{U}^*(t)$ for all $t > t_H^*$. Therefore, we have shown that $\mathscr{U}(t) < \mathscr{U}^*(t)$ for all $t > t_L^*$ and $2u(c(t)) = \mathscr{U}(t)$. But this implies that

$$c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) = u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right) < u^{-1}\left(\frac{\mathscr{U}^*(t)}{2}\right) \leq c^*(t) + u^{-1}(\mathscr{U}^*(t) - u(c^*(t)))$$

for all $t > t_L^*$, contradicting the optimality of $(\mathscr{U}^*, c^*)$.

Next, suppose that RP binds for $t \in [t_H^*, \tau)$ but RP does not bind on $\tau < L$. Construct the restriction of $(\mathscr{U}, c)$ to $[0, \tau)$ as in the previous case, and let the restriction of $(\mathscr{U}, c)$ to $[\tau, L]$ be given by the solution to the continuation program

$$\min_{c,\mathscr{U}} \int_\tau^L \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t)))\right] f(t)dt$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)) \ \forall t, \tag{IC}$$

$$\mathscr{U}(t) \geq 2u\left(\frac{I-t}{2}\right) \ \forall t, \tag{RP}$$

$$\mathscr{U}(\tau) = \mathscr{U}(\tau^-), \tag{B}$$

where $\mathscr{U}(\tau^-) \equiv \lim_{t\nearrow\tau} \mathscr{U}(t)$. By the same argument as in the previous case, we find that $(\mathscr{U}, c)$ yields the same cost for all $t < t_L^*$, has a strictly lower cost for all points in (t_L^*, τ). Moreover, $\mathscr{U}(\tau) < \mathscr{U}^*(\tau)$, which implies that the cost that solves the continuation program above is weakly lower than the cost under $(\mathscr{U}^*, c^*)$. Therefore, the total cost under $(\mathscr{U}, c)$ is strictly lower than under $(\mathscr{U}^*, c^*)$, contradicting the optimality of $(\mathscr{U}^*, c^*)$. □

Claim 4. Let $(\mathscr{U}, c)$ satisfy IC and RP and suppose $2u(c(t)) > \mathscr{U}(t)$ for $t \in (t_L, t_H)$. If t_H is a contact point, then $(\mathscr{U}, c)$ is not optimal.

Proof. Since τ is a contact point, we must have

$$\lim_{t\nearrow\tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right] \leq 0.$$

Use IC to rewrite this as

$$c(\tau^-) \leq \frac{I-\tau}{2}. \tag{30}$$

Therefore,

$$\mathscr{U}(\tau) < 2u\left(c\left(\tau^-\right)\right) \leq 2u\left(\frac{I-\tau}{2}\right) = \mathscr{U}(\tau),$$

where the first inequality follows from the fact that $\mathscr{U}(t) < 2u(c(t))$ for all $t < \tau$ and $\mathscr{U}$ is continuous, the second inequality uses (30), and the equality at the end uses the fact that τ is a contact point. But this is a contradiction. □

We follow the indirect adjoining approach and refer the reader to Hartl, Sethi, and Vickson (1995) for a description of the method. The Hamiltonian and Lagrangian functions are defined as follows:

$$H(c, \mathscr{U}, \lambda, t) = -\left[c + u^{-1}(\mathscr{U} - u(c)) + t\right] f(t) - \lambda u'(c)$$

$$L(c, \mathscr{U}, \lambda, \eta, t) = H(c, \mathscr{U}, \lambda, t) + \nu\left[u'\left(\frac{I-t}{2}\right) - u'(c)\right].$$

The necessary optimality conditions are:[50]

$$\lambda(t) + \nu(t) = -\left[1 - \frac{u'(c(t))}{u'(u^{-1}(\mathscr{U}(t) - u(c(t))))}\right] \frac{f(t)}{u''(c(t))}, \tag{31}$$

$$\dot{\lambda}(t) = \frac{f(t)}{u'(u^{-1}(\mathscr{U}(t) - u(c(t))))}, \tag{32}$$

$$\nu(t) \geq 0 \text{ with } = \text{if either } c(t) > \frac{I-t}{2} \text{ or } \mathscr{U}(t) > 2u\left(\frac{I-t}{2}\right), \tag{33}$$

$$\dot{\nu}(t) \leq 0, \tag{34}$$

[50] In terms of the conditions in Grass et al. (2008), it is easy to verify that the problem is normal (i.e., $\lambda_0 = 1$ using their notation).

and

$$\lambda(T^-) \geq 0 \text{ with } = \text{ if } \mathscr{U}(T^-) > 2u\left(\frac{I-T}{2}\right). \tag{35}$$

If τ is an entry or contact point, then

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau), \tag{36}$$

$$\begin{aligned} &\left[c(\tau^-) + u^{-1}\left(\mathscr{U}(\tau) - u\left(c(\tau^-)\right)\right)\right] f(t) + \lambda(\tau^-)u'(c(\tau^-)) = \\ &= \left[c(\tau^+) + u^{-1}\left(\mathscr{U}(\tau) - u\left(c(\tau^+)\right)\right)\right] f(t) + \lambda(\tau^+)u'(c(\tau^+)) + \eta(\tau)u'\left(\tfrac{I-t}{2}\right) \end{aligned}, \tag{37}$$

and

$$\eta(\tau) \geq 0 \text{ with } = \text{ if } \mathscr{U}(t) > 2u\left(\frac{I-t}{2}\right). \tag{38}$$

Moreover, at any entry point τ_1,

$$\eta(\tau_1) \geq \nu(\tau_1^+), \tag{39}$$

and $\lambda(\tau_2)$ is continuous at any exit point τ_2.

We first show that there is "no distortion at the top":

Lemma 7. *$\mathscr{U}(T) = 2u\left(\frac{I-T}{2}\right)$ and $\lambda(T^-) = \nu(T^-) = 0$.*

Proof. From (35), there are two possible cases. If RP does not bind in a neighborhood of T, then $\lambda(T^-) = \nu(T^-) = 0$ and the result follows from (31). If RP binds in a neighborhood of T, then

$$\mathscr{U}(T) = 2u\left(\frac{I-T}{2}\right).$$

Differentiate of RP and use IC to establish that $c(T^-) = \frac{I-T}{2}$. Substituting in (31), gives

$$\lambda(T^-) + \nu(T^-) = 0,$$

and, since $\lambda(T^-) \geq 0$ (35) and $\nu(t) \geq 0$ for all t (33), we find $\lambda(T^-) = \nu(T^-) = 0$. □

Lemma 8. *Let τ_2 be an exit point. Then $c(\cdot)$ is continuous at τ_2.*

Proof. Let τ_2 be an exit point, so that RP binds in $(\tau_2 - \varepsilon, \tau_2]$ and does not bind in $(\tau_2, \tau_2 + \varepsilon)$. Differentiate RP at $t \in (\tau_2 - \varepsilon, \tau_2]$ and use IC to find that $2u(c(t)) = \mathscr{U}(t)$ for all $t \in (\tau_2 - \varepsilon, \tau_2)$. In particular, $2u\left(c\left(\tau_2^-\right)\right) = \mathscr{U}(\tau_2)$.

Recall that λ must be continuous at any exit point. Since RP does not bind at $t > \tau_2$ but binds at $t < \tau_2$, condition (33) gives:

$$\nu(\tau_2^-) \geq 0 = \nu(\tau_2^+).$$

From (31), we have

$$\lambda(\tau_2)+\nu(\tau_2^-) = -\left[1-\frac{u'\left(c\left(\tau_2^-\right)\right)}{u'\left(u^{-1}\left(\mathscr{U}\left(\tau_2\right)-u\left(c\left(\tau_2^-\right)\right)\right)\right)}\right]\frac{f(\tau_2)}{u''\left(c\left(\tau_2^-\right)\right)} = 0,$$

so $\lambda(\tau_2) \leq 0$, and

$$\lambda(\tau_2) = -\left[1-\frac{u'\left(c\left(\tau_2^+\right)\right)}{u'\left(u^{-1}\left(\mathscr{U}\left(\tau_2\right)-u\left(c\left(\tau_2^+\right)\right)\right)\right)}\right]\frac{f(\tau_2)}{u''\left(c\left(\tau_2^+\right)\right)}.$$

Combining both, we find that

$$-\left[1-\frac{u'\left(c\left(\tau_2^+\right)\right)}{u'\left(u^{-1}\left(\mathscr{U}\left(\tau_2\right)-u\left(c\left(\tau_2^+\right)\right)\right)\right)}\right]\frac{f(\tau_2)}{u''\left(c\left(\tau_2^+\right)\right)} \leq 0.$$

But this condition is equivalent to $\mathscr{U}(\tau_2) \geq 2u\left(c\left(\tau_2^+\right)\right)$. Substituting $\mathscr{U}(\tau_2) = 2u\left(\frac{I-\tau_2}{2}\right)$ and using the monotonicity of u, we obtain

$$\frac{I-\tau_2}{2} \geq c\left(\tau_2^+\right).$$

Therefore, if $c(\cdot)$ is discontinuous at the exit point τ_2, it must jump downwards. But jumping downwards is not possible at an exit point since, to make RP no longer bind at a neighborhood to the right of τ_2, we must have

$$0 \leq \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right]_{t\searrow\tau_2} = -u'(c(\tau_2^+)) + u'\left(\frac{I-\tau_2}{2}\right),$$

$$\therefore \frac{I-\tau_2}{2} \leq c(\tau_2^+).$$

So $c(\cdot)$ must be continuous at any exit point. $\square$

We now show that there are no exit points, so that if τ is an entry point, RP must bind for all $t > \tau$ (i.e., there can be at most one entry point).

Lemma 9. *There are no exit points.*

Proof. Suppose τ_2 is an exit point, so RP binds in $(\tau_2-\varepsilon, \tau_2]$ but not in $(\tau_2, \tau_2+\varepsilon)$. From condition (31), for $t \in (\tau_2, \tau_2+\varepsilon)$, we must have

$$\lambda(t) = -\left[1-\frac{u'\left(c\left(t\right)\right)}{u'\left(u^{-1}\left(\mathscr{U}\left(t\right)-u\left(c\left(t\right)\right)\right)\right)}\right]\frac{f(t)}{u''(c)}.$$

Since $c(t)$ is continuous at τ_2 (by the previous lemma) and $\mathscr{U}(\tau_2) = 2u\left(\frac{I-\tau_2}{2}\right)$ (since τ_2 is a junction point), this condition yields $\lambda\left(\tau_2^+\right) = 0$. Then, because $\dot{\lambda}(t) > 0$ for all $t \in (\tau_2, \tau_2+\varepsilon)$ by condition (32), it follows that $\lambda(t) > 0$ for $t \in (\tau_2, \tau_2 + \varepsilon)$, implying that

$$2u(c(t)) > \mathscr{U}(t)$$

in this interval, a contradiction by Lemma 6. $\square$

Lemma 10. *Let τ be an entry point. Then, $c(\cdot)$ is continuous at τ.*

Proof. From the previous lemma, there are no exit points. Therefore, if τ is an entry point, RP must bind for all $t \in [\tau, T]$. From (35), $\lambda(T^-) = 0$. Moreover, from (32), $\dot{\lambda}(t) = \frac{f(t)}{u'\left(\frac{I-t}{2}\right)}$ for all $t > \tau$. Therefore, we must have

$$\lambda(\tau^+) < 0. \tag{40}$$

As shown in Grass et al. (2008, pp. 151), the following condition must hold at any junction point, so it must also hold at the entry point τ:

$$\eta(\tau)\left[u'\left(\frac{I-\tau}{2}\right) - u'\left(c\left(\tau^-\right)\right)\right] = 0. \tag{41}$$

By (39), we must have

$$\eta(\tau) \geq \nu(\tau^+). \tag{42}$$

As in the proof of Lemma 8, differentiate RP and use IC to find that $2u(c(t)) = \mathscr{U}(t)$ for $t > \tau$. Then, by (31), we must have

$$\lambda(t) + \nu(t) = 0 \therefore \nu(t) = -\lambda(t)$$

for all $t > \tau$.

Taking the limit as $t \searrow \tau$ and using the fact that $\lambda(\tau^+) < 0$ (equation 40), we obtain:

$$\nu(\tau^+) = -\lambda(\tau^+) > 0. \tag{43}$$

Substituting in (42), we find that $\eta(\tau) \geq \nu(\tau^+) > 0$. Therefore, (41) implies that

$$u'\left(\frac{I-\tau}{2}\right) - u'\left(c\left(\tau^-\right)\right) = 0 \therefore c\left(\tau^-\right) = \frac{I-\tau}{2},$$

showing that c is continuous at τ. $\square$

Lemma 11. *Let τ be an entry point. Then, $2u(c(t)) < \mathscr{U}(t)$ for all $t < \tau$ and $2u(c(t)) = \mathscr{U}(t)$ for $t > \tau$.*

The proof will use the following claim:

Claim 5. Let $\hat{\tau}$ be a contact point. Then, $\lambda(\cdot)$ is continuous at $\hat{\tau}$ and $c(\cdot)$ jumps upwards at $\hat{\tau}$ ($c(\hat{\tau}^-) < c(\hat{\tau}^+)$).

Proof. This proof uses results Grass et al. (2008, page 151). Let $\hat{\tau}$ be a contact point, so that

$$\lim_{t \nearrow \tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right] \leq 0 \leq \lim_{t \searrow \tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right].$$

Use IC to rewrite this as

$$c(\tau^-) \leq \frac{I-\tau}{2} \leq c(\tau^+). \tag{44}$$

At any contact time, we must have

$$\lambda(\hat{\tau}^-) = \lambda(\hat{\tau}^+) + \eta(\hat{\tau}),$$

where $\eta(\hat{\tau}) \geq 0$ and

$$\eta(\hat{\tau})\left[u'\left(\frac{I-\hat{\tau}}{2}\right) - u'\left(c\left(\hat{\tau}^-\right)\right)\right] = 0.$$

Suppose, in order to obtain a contradiction, that $\eta(\hat{\tau}) > 0$, so that $c(\hat{\tau}^-) = \frac{I-\hat{\tau}}{2}$. Since the Hamiltonian evaluated at contact times must be continuous, we must have:

$$\left\{\left[c(\tau^+) + u^{-1}\left(\mathscr{U} - u\left(c(\tau^+)\right)\right) + t\right] - (I - \hat{\tau} + t)\right\} f(t) = \lambda^+\left[u'\left(\frac{I-\hat{\tau}}{2}\right) - u'\left(c(\tau^+)\right)\right] + \eta(\hat{\tau})u'\left(\frac{I-\hat{\tau}}{2}\right).$$

Note that if $c(\cdot)$ were continuous at τ, so that $c(\tau^+) = c(\tau^-) = \frac{I-\hat{\tau}}{2}$, this condition would become:

$$\eta(\hat{\tau})u'\left(\frac{I-\hat{\tau}}{2}\right) = 0,$$

contradicting our assumption that $\eta(\hat{\tau}) > 0$. Thus, $c(\cdot)$ must be discontinuous at $\hat{\tau}$, which, by (44), requires

$$c(\tau^-) = \frac{I-\tau}{2} < c(\tau^+)$$

$$\therefore \mathscr{U}(\hat{\tau}) = 2u\left(\frac{I-\hat{\tau}}{2}\right) < 2u\left(c\left(\tau^+\right)\right),$$

where the second line uses the fact that $\hat{\tau}$ is a contact point. But this contradicts Lemma 6. Thus, the solution entails $\eta(\hat{\tau}) = 0$, so $\lambda(\cdot)$ is continuous at $\hat{\tau}$. □

We are now ready to present the proof of the lemma:

Proof of Lemma 11. Let τ be an entry point. Because RP binds at $t > \tau$, we have:

$$\mathscr{U}(\tau) = 2u\left(\frac{I-\tau}{2}\right)$$

and

$$\dot{\mathscr{U}}(\tau) = -u'(c(\tau^+)) = -u'\left(\frac{I-\tau}{2}\right) \therefore c(\tau^+) = \frac{I-\tau}{2}.$$

Then, by the continuity of $c(\cdot)$ at τ (Lemma 10), we have $c(\tau^-) = \frac{I-\tau}{2}$. By Lemma 9, any $t < \tau$ is either a point in which RP does not bind or a contact point, so $\nu(t) = 0$ for $t \in (\tau - \varepsilon, \tau)$. It then follows from (31) that

$$\lambda(\tau^-) = -\left[1 - \frac{u'\left(\frac{I-\tau}{2}\right)}{u'\left(u^{-1}\left(2u\left(\frac{I-\tau}{2}\right) - u\left(\frac{I-\tau}{2}\right)\right)\right)}\right] \frac{f(\tau)}{u''\left(\frac{I-\tau}{2}\right)} = 0.$$

By our previous claim, $\lambda(\cdot)$ is continuous at $[0,\tau)$. Since $\dot{\lambda}(t) > 0$ at all points of differentiability of $\lambda(\cdot)$ and $\lambda(\tau^-) = 0$, it follows that $\lambda(t) < 0$ for all $t < \tau$. But this implies that $2u(c(t)) < \mathscr{U}(t)$ for all $t < \tau$. □

Interpretation of Results

The equilibrium must be such that insurance is lapse based. That is, if the lapsing region is non-empty ($\tau < T$), insurance companies must make a profit of W when individuals lapse and lose $\frac{W}{\int_\tau^T f(t)dt}$ when individuals do not lapse.

The fact that insurance is lapse based is not surprising since consumers put zero weight on the chance they will get any shock $t > 0$, so they do not think they will ever lapse. The distortion effect is more interesting. If there were no ICs (i.e., if shocks were observable), the most profitable way to exploit different beliefs would be to extract the entire surplus when there is a loss while always providing full insurance. With unobservable losses, the contract must ensure that consumers report losses truthfully. Then, distorting their consumption profile by shifting consumption to the future reduces their incentives to misreport losses. This is equivalent to offering policy loans at above market rates. So, the high interest rates on policy loans are a consequence of the unobservability of income shocks.

We conclude by showing that profits increase with types. Let π denote the firm's profits:

$$\pi(t) \equiv W + I_1 + I_2 - t - c(t) - u^{-1}\left(\mathscr{U}(t) - u(c(t))\right).$$

Differentiation gives, at all points of differentiability,

$$\dot{\pi}(t) = -\left[1 + \dot{c}(t)\right] - \frac{\dot{\mathscr{U}}(t) - u'(c(t))\,\dot{c}(t)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)}.$$

Using IC, we obtain

$$\begin{aligned}\dot{\pi}(t) &= -\left[1 + \dot{c}(t)\right] + \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)}\left[1 + \dot{c}(t)\right] \\ &= -\left[1 + \dot{c}(t)\right]\left[1 - \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)}\right].\end{aligned}$$

By monotonicity, $1 + \dot{c}(t) \geq 0$. Moreover,

$$1 - \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \leq 0 \iff \mathscr{U}(t) \geq 2u(c(t)).$$

Then, using Lemma 6, we find that $\dot{\pi}(t) \geq 0$. Note also that $\pi(t) = W$ for all $t > \tau$.

Log Utility and Uniform Distribution (Proposition 3)

Let $u(c) = \ln c$, and suppose losses are uniformly distributed in $[0, \bar{L}]$, so that $f(t) = \frac{1}{\bar{L}}$. Let τ denote the entry point. Consider the relaxed program in which we ignore RP for $t < \tau$, so we can ignore contact points for now. We will verify that RP does not bind in this interval later.

The optimality conditions (31) and (32) become:

$$\lambda(t) = \frac{c(t)^2 - \exp \mathscr{U}(t)}{\bar{L}} \text{ for all } t < \tau \tag{45}$$

where we used the fact that $\nu(t) = 0$ if (RP) does not bind, and

$$\dot{\lambda}(t) = \frac{1}{\bar{L}} \frac{\exp \mathscr{U}(t)}{c(t)} \text{ for all } t. \tag{46}$$

Differentiate (45) to obtain:

$$\dot{\lambda}(t) = \frac{2c(t) \cdot \dot{c}(t) - \exp \mathscr{U}(t) \cdot \dot{\mathscr{U}}(t)}{\bar{L}}.$$

Recall that, from IC, we have $\dot{\mathscr{U}}(t) = -\frac{1}{c(t)}$. Therefore, the previous condition becomes:

$$\dot{\lambda}(t) = \frac{2c(t) \cdot \dot{c}(t)}{\bar{L}} + \frac{1}{\bar{L}} \frac{\exp \mathscr{U}(t)}{c(t)}.$$

Substituting in (46), gives $\dot{c}(t) = 0$. Therefore, $c(t) = \bar{c}$ constant for $t < \tau$. Moreover, since $c(\cdot)$ is continuous at the entry point τ (Lemma 10) and $c(t) = \frac{I-t}{2}$ for all $t > \tau$, we must have $\bar{c} = \frac{I-\tau}{2}$.

We now verify that RP does not bind at $t < \tau$, so there are no contact points. Using the logarithmic utility, (RP) becomes:

$$\mathscr{U}(t) - 2\ln\left(\frac{I-t}{2}\right) \geq 0.$$

Since, by construction, this inequality binds at the entry point τ, to show that it does not bind at any $t < \tau$, it suffices to verify that

$$\frac{d}{dt}\left\{\mathscr{U}(t) - 2\left[\ln(I-t) - \ln 2\right]\right\} \leq 0.$$

Differentiate and use IC to rewrite this inequality as:

$$-\frac{1}{\bar{c}} \leq -\frac{2}{I-t} \iff \frac{I-t}{2} \geq \bar{c}.$$

Using $\bar{c} = \frac{I-\tau}{2}$, this condition becomes:

$$\frac{I-t}{2} \geq \frac{I-\tau}{2} \iff t \leq \tau.$$

Therefore, RP does not bind at any $t < \tau$.

We now obtain the expressions for the equilibrium consumption. As shown previously, first-period consumption equals

$$c_1(t) = \begin{cases} \frac{I-\tau}{2} & \text{if } t < \tau \\ \frac{I-t}{2} & \text{if } t \geq \tau \end{cases}.$$

Since $\mathscr{U}(0) = \bar{u}$ and, by IC, $\dot{\mathscr{U}}(t) = -\frac{1}{c(t)}$, we can recover the indirect utility:

$$\mathscr{U}(t) = \bar{u} - \int_0^t \frac{1}{c(s)} ds = \begin{cases} \bar{u} - \frac{2t}{I-\tau} & \text{for } t \leq \tau \\ \bar{u} - \frac{2\tau}{I-\tau} + 2\ln\left(\frac{I-t}{I-\tau}\right) & \text{for } t > \tau \end{cases}.$$

Since RP binds at all $t \geq \tau$, we must have

$$\mathscr{U}(t) = 2\ln\left(\frac{I-t}{2}\right) \text{ for all } t \geq \tau$$

Thus,

$$\bar{u} - \frac{2\tau}{I-\tau} + 2\ln\left(\frac{I-t}{I-\tau}\right) = 2\ln\left(\frac{I-t}{2}\right) \therefore \bar{u} = 2\left[\frac{\tau}{I-\tau} + \ln\left(\frac{I-\tau}{2}\right)\right].$$

Substituting back in the indirect utility, gives

$$\mathscr{U}(t) = \begin{cases} 2\left[\ln\left(\frac{I-\tau}{2}\right) + \frac{\tau-t}{I-\tau}\right] & \text{for } t \leq \tau \\ 2\ln\left(\frac{I-t}{2}\right) & \text{for } t > \tau \end{cases}.$$

Using the definition of the indirect utility, $\mathscr{U}(t) \equiv \ln(c(t)) + \ln(c_2(t))$, we can recover the second-period consumption for $t < \tau$:

$$c_2(t) = \frac{I-\tau}{2} \exp\left(2\frac{\tau-t}{I-\tau}\right).$$

For $t \geq \tau$, the individual lapses and gets perfect smoothing $c_1(t) = c_2(t) = \frac{I-t}{2}$. Finally, note that $c(t) + t$ is non-decreasing in t, so the omitted monotonicity constraint does not bind.

Rational Expectations (Subsection 4.2.5)

We now show that the equilibrium contract has the opposite pattern when consumers have rational expectations about the distribution of losses. Formally:

Proposition 5. *With rational expectations about the income shock, there exists $\tau \in [0,T]$ such that RP binds for $t \in [0,\tau)$ and does not bind for almost all $t \in (\tau,T]$. Moreover, $\mathscr{U}(t) < 2u(c(t))$ for almost all $t \in (\tau,T]$.*

The intuition for why the conclusions flip when consumers have rational expectations is as follows:

- Without IC and RP, the optimal contract when consumers have rational expectations provides full insurance (they are fully compensated for a loss without having to repay at all). This violates IC because consumers would always report a large loss, consuming the additional reimbursement. If RP fails, it fails for consumers with small losses who have to subsidize those with large losses. Thus, in equilibrium, policies offer subsidized loans (individuals have to repay part of it to maintain IC, but they are given below-market interest rates to smooth consumption), causing RP to bind at the bottom (with below-market interest rates, those with small loans subsidize those with large loans). That is, the key incentive issue with rational expectations is to provide insurance against unobservable liquidity shocks while ensuring that individuals do not inflate their losses.

- In contrast, when consumers think that they will not have any losses, the insurance company would like to charge as much as possible from those with positive losses, since consumers think that those states would not happen. This corresponds to charging an infinitely high interest rate for everyone with a loss, which gives consumers an incentive to pretend that they not have any income loss. To restore IC, the company cannot charge infinite rates, so it charges an above market but still finite interest rate. This disproportionately hurts consumers with larger losses, who now cross subsidize others by paying above-market interest rates. Thus, RP binds at for those with large enough losses.

The equilibrium program is:

$$\max_{c,\mathscr{U}} \int \mathscr{U}(t) f(t) dt$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)) \ \forall t, \tag{IC}$$

$$\mathscr{U}(t) \geq 2u\left(\frac{I-t}{2}\right) \ \forall t, \tag{RP}$$

$$\int_0^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) - t\right] f(t) dt \leq W$$

As a benchmark, ignore RP:

$$\max_{c,\mathscr{U}} \int \mathscr{U}(t) f(t) dt$$

subject to

$$\dot{\mathscr{U}}(t) = -u'(c(t)) \ \forall t, \tag{IC}$$

$$W - \int_0^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) - t\right] f(t) dt \geq 0$$

Note that without IC, we have

$$\max_{c,\mathscr{U}} \int \mathscr{U}(t) f(t) dt$$

subject to

$$W - \int_0^T \left[c(t) + u^{-1}(\mathscr{U}(t) - u(c(t))) - t\right] f(t) dt \geq 0.$$

Pointwise maximization gives:

$$f(t) - \frac{\lambda}{u'\left(u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right)\right)} f(t) = 0 \therefore \mathscr{U}(t) - u\left(c(t)\right) = \text{constant}$$

$$1 = \frac{u'(c)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right)\right)} \therefore \mathscr{U}(t) = 2u\left(c(t)\right).$$

Thus, the solution entails $c(t) = \bar{c}$ constant and $\mathscr{U}(t) = 2u(\bar{c})$ (also constant) for all t. But this violates IC, which requires $\dot{\mathscr{U}}(t) = -u'(\bar{c}) > 0$.

We now incorporate IC. Introduce the auxiliary variable:

$$X(s) \equiv -\int_0^s \left[c(t) + u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right) - t\right] f(t) dt,$$

so zero profits can be written as:

$$\dot{X}(t) = -\left[c(t) + u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right) - t\right] f(t),$$

with $X(0) = 0,\ \ X(T) = W$.

The equilibrium program becomes:

$$\max_{c,\mathscr{U}} \int \mathscr{U}(t) f(t) dt$$

subject to

$$\dot{\mathscr{U}}(t) = -u'\left(c(t)\right)\ \forall t, \tag{IC}$$

$$\dot{X}(t) = -\left[c(t) + u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right) - t\right] f(t),$$

$X(0) = 0,\ \ X(T) = W$, $\mathscr{U}(0)$ and $\mathscr{U}(T)$ free.

Set up the Hamiltonian:

$$H(\mathscr{U}, c, \lambda, \mu, t) = \mathscr{U} f(t) - \lambda u'(c) - \mu\left[c(t) + u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right) - t\right] f(t).$$

The optimality conditions are:

$$\lambda(t) = -\mu\left[1 - \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right)\right)}\right] \frac{f(t)}{u''(c(t))}$$

$$\dot{\lambda}(t) = \left[\frac{\mu}{u'\left(u^{-1}\left(\mathscr{U}(t) - u\left(c(t)\right)\right)\right)} - 1\right] f(t)$$

$$\lambda(0) = \lambda(T) = 0,\ \ \mu(t) = \mu \text{ constant}$$

From the transversality condition, it follows that $\mathscr{U}(c(0)) = 2u(c(0))$ and $\mathscr{U}(c(T)) = 2u(c(T))$. This implies that $\mu > 0$ (otherwise, we would have $\dot{\lambda} < 0$ for all t, which would violate transversality). This is consistent with the fact that, by the envelope theorem, μ is the shadow cost of wealth W, which has to be positive.

We now verify that we cannot have $\lambda(t) = 0$ for all t. For this to be the case, we would need $\mathscr{U}(t) = 2u(c(t))$ for all t, so that

$$\mathscr{U}(t) - u(c(t)) = u(c(t)) \ \forall t$$

Moreover, since $\dot{\lambda}(t) = 0$ for all t, we would have

$$u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) = \mu \ \ \forall t \therefore \mathscr{U}(t) - u(c(t)) \text{ is constant.}$$

Therefore, this is only possible if $c(t) = \bar{c}$ constant in t, which implies $\mathscr{U}(t) = 2u(\bar{c})$ constant in t. But this would give $\dot{\mathscr{U}}(t) = 0$, which violates IC.

Thus, we must have $\lambda(t) \neq 0$ in some interval of types (the interval requirement comes from the continuity of λ), which requires that the sign of $\dot{\lambda}(t)$ cannot be constant. Let

$$\xi(t) \equiv \mu - u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right),$$

and note that

$$\dot{\lambda}(t) > 0 \iff \mu > u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \iff \xi(t) > 0.$$

Differentiation gives

$$\begin{aligned}
\dot{\xi}(t) &= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{d}{dt}\left[\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)\right] \\
&= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{\dot{\mathscr{U}}(t) - u'(c(t))\,c'(t)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \\
&= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{-u'(c(t)) - u'(c(t))\,c'(t)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \\
&= \frac{u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)}\left[1 + c'(t)\right].
\end{aligned}$$

By monotonicity, we must have $1 + c'(t) \geq 0$. Thus, we have $\dot{\xi}(t) < 0$, so that there exists $\tau \in (0, L)$ such that

$$\dot{\lambda}(t) \left\{ \begin{array}{c} > \\ < \end{array} \right\} 0 \iff t \left\{ \begin{array}{c} < \\ > \end{array} \right\} \tau.$$

It then follows that $\lambda(t) > 0$ for all $t \in (0, L)$:

$$\lambda(t) = -\mu\left[1 - \frac{u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)}\right]\frac{f(t)}{u''(c(t))} > 0 \iff u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) > u'(c(t))$$

$$\iff u^{-1}\left(\mathscr{U}(t)-u(c(t))\right)<c(t) \iff \mathscr{U}(t)<2u(c(t))$$

for all $t\in(0,L)$. That is, with rational liquidity shocks, loans are subsidized (below-market interest). This is because the IC binds in the opposite direction (absent IC, we would want to fully insure against losses, but then everyone would like to pretend to have had a higher shock).

To understand where RP may bind once we introduce it into the model, note that:

$$h(\mathscr{U}(t),t)=\mathscr{U}(t)-2u\left(\frac{I-t}{2}\right),$$

so that

$$\frac{d}{dt}\left[h(\mathscr{U}(t),t)\right]=\dot{\mathscr{U}}(t)+u'\left(\frac{I-t}{2}\right)=u'\left(\frac{I-t}{2}\right)-u'(c(t))\leq 0 \iff \frac{I-t}{2}\geq c(t).$$

We have seen that the solution of the relaxed program where we ignore RP entails $\mathscr{U}(t)<2u(c(t))$ for all $t\in(0,T)$. Thus, $u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)<c(t)$.

Suppose $\frac{d}{dt}\left[h(\mathscr{U}(t),t)\right]\leq 0$ for some t so that $\frac{I-t}{2}\geq c(t)$. Then, we must have

$$u^{-1}\left(\frac{\mathscr{U}(t)}{2}\right)<c(t)\leq\frac{I-t}{2}\therefore\mathscr{U}(t)<2u\left(\frac{I-t}{2}\right),$$

meaning that RP fails. Thus, if RP holds, we must have $\frac{d}{dt}\left[h(\mathscr{U}(t),t)\right]>0$. But this means that RP holds if and only if it holds at $t=0$. More generally, this shows that if the solution entails $\mathscr{U}(t)<2u(c(t))$, then any junction point must be an exit point. This is intuitive because the incentive issue here is to give insurance against large shocks so the type most willing to leave the mechanism is the one with the lowest shock.

We now formally introduce the RP constraint. Define the Hamiltonian and Lagrangian functions as:

$$H(\mathscr{U},c,\lambda,\mu,t)=\mathscr{U}f(t)-\lambda u'(c)-\mu\left[c(t)+u^{-1}\left(\mathscr{U}(t)-u(c(t))\right)-t\right]f(t)$$

$$L(c,\mathscr{U},\lambda,\mu,\eta,t)=H(c,\mathscr{U},\lambda,t)+\nu\left[u'\left(\frac{I-t}{2}\right)-u'(c)\right].$$

The optimality conditions are:

- $\lambda(t)+\nu(t)=-\mu\left[1-\frac{u'(c(t))}{u'\left(u^{-1}(\mathscr{U}(t)-u(c(t)))\right)}\right]\frac{f(t)}{u''(c(t))}$,
- $\dot{\lambda}(t)=\left[\frac{\mu}{u'\left(u^{-1}(\mathscr{U}(t)-u(c(t)))\right)}-1\right]f(t)$,
- $\mu(t)=\mu$ constant,
- $\nu(t)\geq 0$ with $=$ if either $c(t)>\frac{I-t}{2}$ or $\mathscr{U}(t)>2u\left(\frac{I-t}{2}\right)$,

- $\dot{\nu}(t) \leq 0$,
- $\lambda(T^-) \geq 0$, with $=$ if $\mathscr{U}(T^-) > 2u\left(\frac{I-T}{2}\right)$,
- $\lambda(0^+) \geq 0$, with $=$ if $\mathscr{U}(0^+) > 2u\left(\frac{I}{2}\right)$.

Moreover, at entry or contact points,

- $\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau)$
- $\left[c(\tau^-) + u^{-1}\left(\mathscr{U}(\tau) - u\left(c(\tau^-)\right)\right)\right] f(t) + \lambda(\tau^-)u'(c(\tau^-)) =$

$$= \left[c(\tau^+) + u^{-1}\left(\mathscr{U}(\tau) - u\left(c(\tau^+)\right)\right)\right] f(t) + \lambda(\tau^+)u'(c(\tau^+)) + \eta(\tau)u'\left(\frac{I-t}{2}\right)$$

- $\eta(\tau) \geq 0$ with $=$ if $\mathscr{U}(t) > 2u\left(\frac{I-t}{2}\right)$.

In addition, $\eta(\tau_1) \geq \nu(\tau_1^+)$ at any entry time τ_1 and $\lambda(\tau_2)$ is continuous at any exit time τ_2.

We now show that there cannot be an entry point.

Lemma 12. *There are no entry points.*

Proof. Let τ be the **first entry point**. Then, we must have:

$$\lim_{t \nearrow \tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right] = \lim_{t \nearrow \tau}\left[\dot{\mathscr{U}}(t) + u'\left(\frac{I-t}{2}\right)\right] \leq 0$$

$$\iff u'\left(\frac{I-\tau}{2}\right) \leq u'(c(\tau^-) \iff c(\tau^-) \leq \frac{I-\tau}{2}. \tag{47}$$

Moreover, since *RP* binds at $(\tau, \tau+\varepsilon)$, we have $c(\tau^+) = \frac{I-\tau}{2}$. That is, if c jumps, it must jump upwards.

By the first optimality condition for $t < \tau$ (where RP doesn't bind), we have

$$\lambda(\tau^-) = -\mu\left[1 - \frac{u'(c(\tau^-))}{u'\left(u^{-1}\left(2u\left(\frac{I-\tau}{2}\right) - u\left(c(\tau^-)\right)\right)\right)}\right]\frac{f(\tau)}{u''(c(\tau^-))},$$

and, because $c(\tau^-) \leq \frac{I-\tau}{2}$, we must have

$$\lambda(\tau^-) \leq 0. \tag{48}$$

By the first optimality condition for $t > \tau$ (where RP binds), we have $\lambda(\tau^+) + \nu(\tau^+) = 0$ for all such t. Recall that $\eta(\tau) \geq \nu(\tau^+)$ at any entry time. Thus,

$$\lambda(\tau^+) + \eta(\tau) \geq \lambda(\tau^+) + \nu(\tau^+) = 0.$$

Then, using $\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau)$, we obtain:

$$\lambda(\tau^-) \geq 0. \tag{49}$$

Combining (48) and (49), we find that

$$\lambda(\tau^-) = 0,$$

so that $c(\tau^-) = \frac{I-\tau}{2}$. Therefore, $c(\cdot)$ is continuous at τ.

There are two possible cases. Suppose first that there are no contact points between 0 and τ so that λ is continuous at $[0,\tau]$. Then, we have:

$$\lambda(0) = \lambda(\tau^-) = 0,$$

and

$$\dot{\lambda}(t) = \left[\frac{\mu}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} - 1\right] f(t).$$

Let

$$\xi(t) \equiv \mu - u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right),$$

and note that

$$\dot{\lambda}(t) > 0 \iff \mu > u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \iff \xi(t) > 0.$$

Differentiation gives

$$\begin{aligned}
\dot{\xi}(t) &= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{d}{dt}\left[\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)\right] \\
&= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{\dot{\mathscr{U}}(t) - u'(c(t))\,c'(t)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \\
&= -u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) \times \frac{-u'(c(t)) - u'(c(t))\,c'(t)}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \\
&= \frac{u''\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right) u'(c(t))}{u'\left(u^{-1}\left(\mathscr{U}(t) - u(c(t))\right)\right)} \left[1 + c'(t)\right].
\end{aligned}$$

By monotonicity, we must have $1 + \dot{c}(t) \geq 0$. Thus, we have $\dot{\xi}(t) < 0$. Therefore, there exists $\bar{t} \in (0,\tau)$ such that $\dot{\lambda}(t) > 0$ if and only if $t < \bar{t}$. Because $\lambda(0) = \lambda(\tau) = 0$, this implies that $\lambda(t) > 0$ for all $t \in (0,\tau)$, so that

$$c(\tau^-) > \frac{I-\tau}{2}$$

for all $t \in (0,\tau)$, a contradiction to (47).

Turning to the second possible case, suppose there exists a contact point $\hat{\tau}$ between 0 and τ. Since $\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau) \geq \lambda(\tau^+)$, it follows that λ can only jump downwards. In principle, we could then have an intermediate region with $\lambda(t) < 0$ (immediately after a contact point). This would not happen if either $\lambda(\tau^+) \geq 0$ or if $\lambda(\tau^-) \leq 0$ (so both sides of the contact point are either positive or negative).

Thus, the only possible case is when[51]

$$\lambda(\tau^-) \geq 0 \geq \lambda(\tau^+)$$

with at least one strict inequality, which happens if and only if

$$c(\tau^+) \leq \frac{I-\tau}{2} \leq c(\tau^-) \tag{50}$$

with at least one strict inequality.

But any contact point τ must satisfy

$$\lim_{t \nearrow \tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right] \leq 0 \leq \lim_{t \searrow \tau} \frac{d}{dt}\left[\mathscr{U}(t) - 2u\left(\frac{I-t}{2}\right)\right].$$

Use IC to rewrite this as

$$c(\tau^-) \leq \frac{I-\tau}{2} \leq c(\tau^+), \tag{51}$$

contradicting (50). □

To conclude the proof, note that $\lambda(\cdot)$ is continuous at any exit point, so that $\lambda(\tau) = 0$. Then, by the same argument as in the model without RP, it follows from $\lambda(0) = \lambda(\tau) = 0$ that $\lambda(\tau) > 0$ for all intermediate points.

Appendix B: LLI Survey Questions

This Appendix describes the LLI (2018) survey methodology, lists all questions and along with the responses to each of them. Using administrative data provided by LLI, it also compares responders ("Complete") and non-responders ("Incomplete") by observable characteristics. The input files (survey results) and processing files can be found in the online repository. Both surveys were emailed on February 7, 2018, specifically, at 11:28 AM EST for the New Buyers survey and at 10:52 AM EST for the Lapsers

[51] Recall that

$$\lambda(t) > 0 \iff \mathscr{U}(t) < 2u(c(t)).$$

Since RP binds at τ, we have $\mathscr{U}(\tau) = 2u\left(\frac{I-\tau}{2}\right)$. Thus,

$$\lambda(\tau^+) < 0 \iff \frac{I-\tau}{2} > c(\tau^+)$$

$$\lambda(\tau^-) > 0 \iff \frac{I-\tau}{2} < c(\tau^-)$$

Thus, the possible case is when:

$$\lambda(\tau^-) > 0 > \lambda(\tau^+) \iff c(\tau^+) < \frac{I-\tau}{2} < c(\tau^-).$$

Survey. A reminder email message was sent to people in both surveys who had not completed the survey (both non-responders and people who started but did not complete) on February 14th at 12:00 PM EST. A second reminder email was sent on February 21st at 12:00 PM EST. Both surveys were closed at 9AM EST on March 9, 2018. Following standard IRB protocols, the first page of each survey informed subjects of the purpose of the survey and gave them our contact information in case they had any concerns. Neither author received any emails.

New Buyers Survey

Below are the questions for the New Buyers Survey along with percent responses for each question shown in parentheses. This survey was sent to all LLI customers who purchased life insurance between October 2013 and end of November 2017. Subjects were asked either 7 or 8 questions. The response rate was 13.0%, producing 1,689 respondents.

1. Your term life insurance policy has about *N* years left on it. What is the chance that you might stop your policy (sometimes called lapsing) before then?

1.1. I have not given it much thought (12.0%, 202 responders)
1.2. I do not currently anticipate stopping my policy (80.4%. 1357)
1.3. I currently anticipate stopping my policy with a 10 percent or lower chance (1.4%, 24)
1.4. I currently anticipate stopping my policy with a chance of 10 – 25 percent (1.7%, 28)
1.5. I currently anticipate stopping my policy with a chance of 25 – 50 percent (2.1%, 35)
1.6. I anticipate stopping my policy with a chance greater than 50 percent (2.4%, 41)

2. If 1.3, 1.4, 1.5 or 1.6: In how many years do you anticipate potentially stopping your policy?

2.1. Unsure (20.3%, 26)
2.2. Likely in between 1-5 years (28.9%, 37)
2.3. Likely in between 6-10 years (27.3%, 35)
2.4. Likely in between 11-15 years (9.4%, 12)
2.5. More than 15 years (14.1%, 18)

3. What do estimate is the chance that <u>other people</u> with your type of life insurance policy might stop their policy (or lapse) before it expires?

3.1. I have not given it much thought (67.1%, 1131)
3.2. Between 0 – 5 percent (7.7%, 129)
3.3. Between 5 – 10 percent (6.6%, 112)
3.4. Between 10 – 25 percent (10.7%, 181)
3.5. Between 25 – 50 percent (5.8%, 97)
3.6. Over 50 percent (2.1%, 35)

4. What do estimate is the chance that <u>you</u> might someday stop your policy due to <u>divorce or death</u> of a spouse?

4.1. I have not given it much thought (34.7%, 584)

4.2. Between 0 – 5 percent (55.0%, 927)
4.3. Between 5 – 10 percent (4.9%, 83)
4.4. Between 10 – 25 percent (2.6%, 43)
4.5. Between 25 – 50 percent (1.0%, 17)
4.6. Over 50 percent (1.8%, 31)

5. What do estimate is the chance that you might someday stop your policy because you will need the money, maybe due to lower income or increased expenses?

5.1. I have not given it much thought (18.3%, 308)
5.2. Between 0 – 5 percent (63.6%, 1073)
5.3. Between 5 – 10 percent (9.0%, 151)
5.4. Between 10 – 25 percent (4.2%, 71)
5.5. Between 25 – 50 percent (3.0%, 51)
5.6. Over 50 percent (1.9%, 32)

6. What do estimate is the chance that you might someday stop your policy because you feel healthier than expected and would prefer to purchase a different policy?

6.1. I have not given it much thought (24.1%, 406)
6.2. Between 0 – 5 percent (70.2%, 1183)
6.3. Between 5 – 10 percent (0%, 0)
6.4. Between 10 – 25 percent (2.5%, 42)
6.5. Between 25 – 50 percent (1.4%, 23)
6.6. Over 50 percent (0.9%, 15)

7. At some point in the last 5 years, has your total household income decreased? This might be due to a salary cut, a job separation by you or your spouse, or because part of your total household income is partly tied to commissions or bonuses that tend to fluctuate.

7.1. Yes (32.5%, 545)
7.2. No (67.5%, 1132)

8. What are the chances that at some point in the next 5 years, your total household income would decrease substantially? This might due to a salary cut, a layoff of you or a spouse, retirement, or because part of your total household income is partly tied to commissions or bonuses that tend to fluctuate?

8.1. There is little chance that my income could fluctuate downward by a lot (43.6%, 730)
8.2. My income could fluctuate downward a lot with a chance less than 5% (13.7%, 229)
8.3. My income could fluctuate downward a lot with a chance between 5 – 10% (15.4%, 257)
8.4. My income could fluctuate downward a lot with a chance between 10 – 25% (13.7%, 229)
8.5. My income could fluctuate downward a lot with a chance between 25 – 50% (7.2%, 120)
8.6. My income could fluctuate downward a lot with a chance greater than 50% (6.5%, 108)

Responders who answered Question 1 indicating a 10% chance or more of lapsing (1.4, 1.5, or 1.6) completed the survey in 6.2 days on average in comparison to 6.7 days for responders who indicated a 10 percent or less chance (1.1, 1.2, or 1.3). The t-value for a test of equal means between the two independent samples is 0.78, thereby failing to reject a difference at conventional levels of significance.

Notice that subjects were asked to report both the total probability of lapsing (Question 1) and the probability of lapsing broken down by possible reasons (Questions 4, 5, and 6). In both cases, we find that they severely underestimate the chance of lapsing. Consistently with the Conjunction Fallacy (Tversky and Kahneman, 1983), we find that lapse probabilities broken down by each reason add up to more than the total probability of lapsing. For example, 6.2% of responders indicated a 10% or higher chance of lapsing in general (Question 1), whereas 9.1% of responders indicate a 10% or higher chance of lapsing because they might someday need money (Question 5). Overall, 13.0% of responders indicated a 10% or less chance of lapsing in Question 1 while simultaneously indicating a 10% or more chance of lapsing for any one or more specific reason when subsequently prompted in Questions 4, 5 and 6. The relevant probability for our purposes depends on whether, when purchasing life insurance, individuals think about the probability of lapsing as a whole, or if they think about each possible reason for lapsing in isolation. In either case, however, these numbers support the view that individuals severely underestimate the probability of lapsing.[52]

Figure 4 shows the average characteristics of responders ("Complete") and non-responders ("Incomplete"), with 95% confidence intervals shown with the black line. At the time of the survey, responders have had held their policy for about 26 days less than non-responders. The expected default rate of responders (55.8%) is slightly larger than non-responders (57.5%). Most other characteristics are very similar, including marital status, gender, length of policy, occupation, age at policy issuance and ultimate face value. A slightly higher proportion of college employees responded to our survey, perhaps because we identified ourselves in the opening page and stated that the survey would be used for academic research. This small difference of occupations might also explain why responders are slightly older and have slightly smaller policies, as many college employees get supplemental life insurance through their college as a voluntary group benefit, outside of the individual insurance market that we consider.

Lapsers Survey

A link to this survey was sent by email to the universe of 3,229 former LLI policyholders who lapsed their policies between 2012 and 2017, generating a response rate of 4.9 percent. Subjects were asked either 6 or 7 questions. Below are the questions for the Lapsers Survey along with percent responses for each question shown in parentheses. For Question 1, there are two percent responses shown: the first one corresponds to the raw responses and the second one corresponds to our subjective recoding for those who selected 1.9 in the raw data.

1. You have recently cancelled (or let "lapse") your life insurance policy. Many people cancel / lapse their policies for one or more of the reasons listed below. Which choice best reflects your

[52] Alternatively, subjects may infer from Question 1 that lapse probabilities are higher than expected, revising their answers to subsequent questions about lapsing upwards.

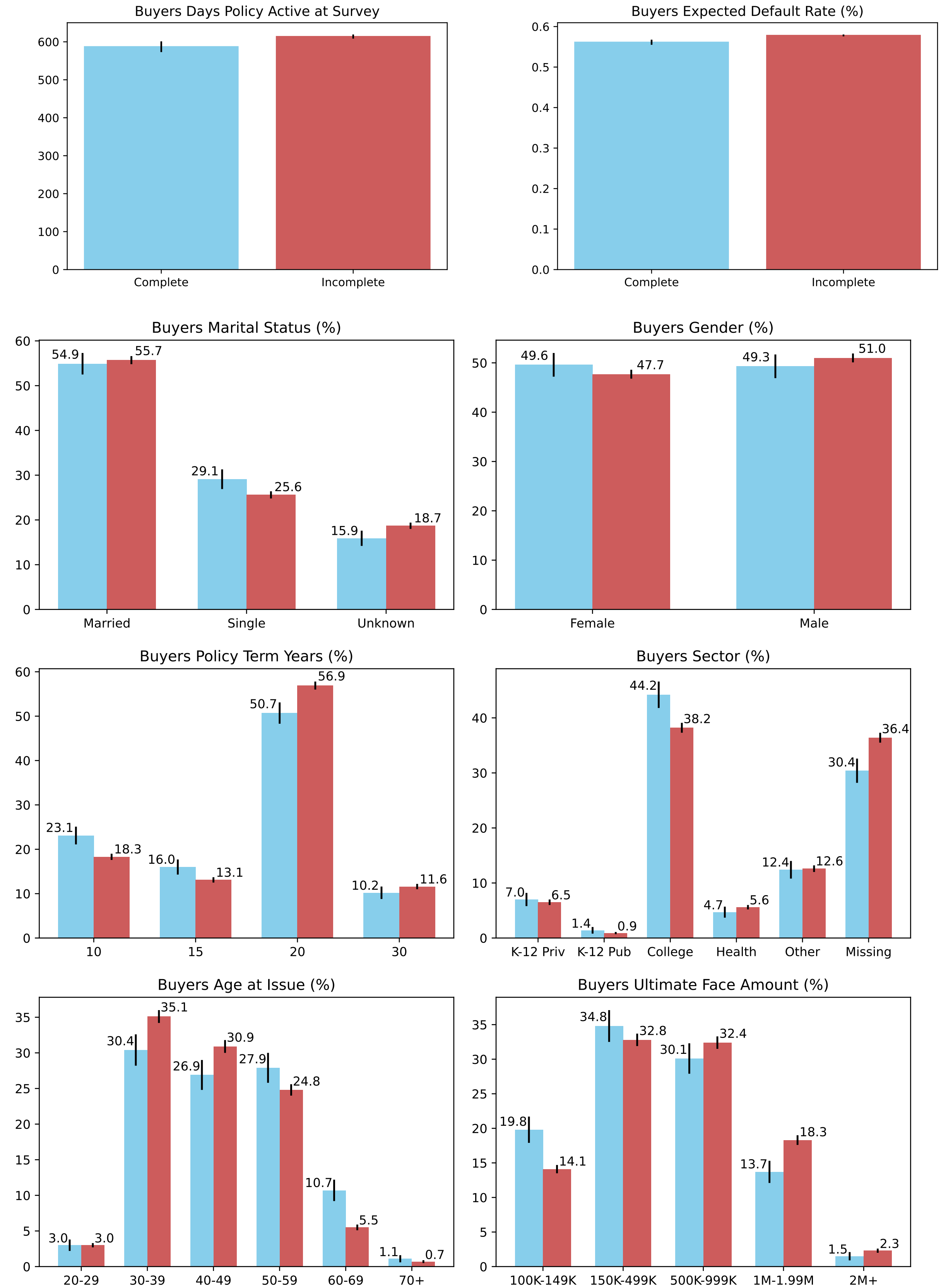

Figure 4: Descriptive statistics for respondents (blue) and non-respondents (red). Black lines represent 95% confidence intervals. Starting from the top, the figures represent days that policy has been active at point of survey; expected default rate of policies based on LLI's historical annual default rate of 5.2%; marital status; gender; term of policy; occupation; age and ultimate face amount.

reason?

1.1. My income decreased (7.6%, 12; 8.3%, 13)

1.2. I needed the money (1.3%, 2; 7.1%, 11)

1.3. My family situation changed due to divorce (4.5%, 7; 4.5%, 7)

1.4. My family situation changed due to death of spouse (0.6%, 1; 0.6%, 1)

1.5. I recently retired (12.7%, 20; 13.5%, 21)

1.6. I was healthier than expected and bought another policy (1.3%, 2; 6.4%, 10)

1.7. I forgot to make my insurance premium payments (12.1%, 19; 23.1%, 36)

1.8. I believe that I didn't cancel my policy (12.7%, 20; 14.7%, 23)

1.9. Other (please explain) (47.1%, 74; 12.8%, 20)

1.10. Family situation changed for reasons other than divorce or death of spouse (N/A%, N/A; 9.0%, 14)

2. If 1.8: Our records indicate that you cancelled your [type] insurance with a death benefit of [$$$$$$] on [DATE]. Do you recall that cancellation?

2.1. Yes (5.0%, 1)

2.2. No (95.0%, 19)

3. At some point in the last 5 years, has your total household income decreased? This might be due to a salary cut, a job separation by you or your spouse, or because part of your total household income is partly tied to commissions or bonuses that tend to fluctuate.

3.1. Yes (44.2%, 69)

3.2. No (55.8%, 87)

4. IF 3.1: By how much did your total household income decrease in the last five years?

4.1. Less than 5% (0.0%, 0)

4.2. Between 5 and 15% (23.5%, 16)

4.3. Between 15 and 25% (26.5%, 18)

4.4. Between 25 and 50% (23.5%, 16)

4.5. More than 50% (26.5%, 18)

5. Have you experienced one of the following options in the last 5 years? Check all options that apply:

5.1. A divorce (18.6%, 13)

5.2. Retirement by you or your spouse (61.4%, 43)

5.3. Hospitalization by you or your spouse (37.1%, 26)

6. Since you cancelled your policy, have you purchased a new one?

6.1. No. (82.5%, 127)

6.2. Yes. I purchased a smaller policy (9.1%, 14).

6.3. Yes. I purchased a larger policy (8.4%, 13).

7. What is your annual household income?

7.1. Less than $50,000 per year (12.2%, 19)

7.2. Between $50,000 - $125,000 per year (39.7%, 62)

7.3. Between $125,000 - $250,000 per year (25.6%, 40)
7.4. Over $250,000 per year (5.1%, 8)
7.5. I prefer to not answer (17.3%, 27)

Figure 5 shows the average characteristics of responders ("Complete") and non-responders ("Incomplete"), with 95% confidence intervals shown with the black line. Notice that responders tend to be slightly older (mean age of 49) relative to non-responders (mean age of 45), have smaller policies, and have lapsed a bit more recently. The mean number of days that policies were active before lapsing nearly are identical for responders and non-responders. Both groups are similar in terms of gender and marital status.

Appendix C: Competing Models

Model of Risk Reclassification

This section considers reclassification risk model based on Hendel and Lizzeri (2003), Daily, Hendel, and Lizzeri (2008), and Fang and Kung (2010). We demonstrate that a reasonably calibrated rational model with liquidity shocks produces back-loaded policies, the opposite loading of observable contracts.

The main distinction between the model considered here and the other ones in the literature is in the timing of shocks. Hendel and Lizzeri (2003) study a model in which consumers are subject to health shocks only. Lack of commitment on the side of the consumer motivates lapsing following positive health shocks. Preventing lapses is then welfare improving and front-loaded fees (i.e., payments before the realization of the health shock that cannot be recuperated if the consumer drops the policy) are an effective way to do so. Daily, Hendel, and Lizzeri (2008) and Fang and Kung (2010) introduce bequest shocks in this framework. In their model, there is one period in which both bequest and health shocks may happen. Lapsing is efficient if it is due to a loss of the bequest motive and is inefficient if motivated by a positive health shock. The solution then entails some amount of front loading as a way to discourage lapses.

Markov Transition Matrix (25 year old Male; 5 years)

	1	2	3	4	5	6	7	8
1	*.989*	*.001*	*.000*	*.000*	*.000*	*.000*	*.000*	*.011*
2	*.932*	*.028*	*.000*	*.000*	*.000*	*.000*	*.000*	*.039*
3	*.927*	*.030*	*.000*	*.000*	*.000*	*.000*	*.000*	*.042*
4	*.918*	*.034*	*.000*	*.000*	*.000*	*.000*	*.000*	*.046*
5	*.860*	*.056*	*.000*	*.000*	*.000*	*.000*	*.000*	*.082*
6	*.914*	*.038*	*.000*	*.000*	*.000*	*.000*	*.000*	*.048*
7	*.850*	*.060*	*.000*	*.000*	*.001*	*.000*	*.000*	*.088*

Table 1: Probability of five-year ahead changes in health states at age 25.

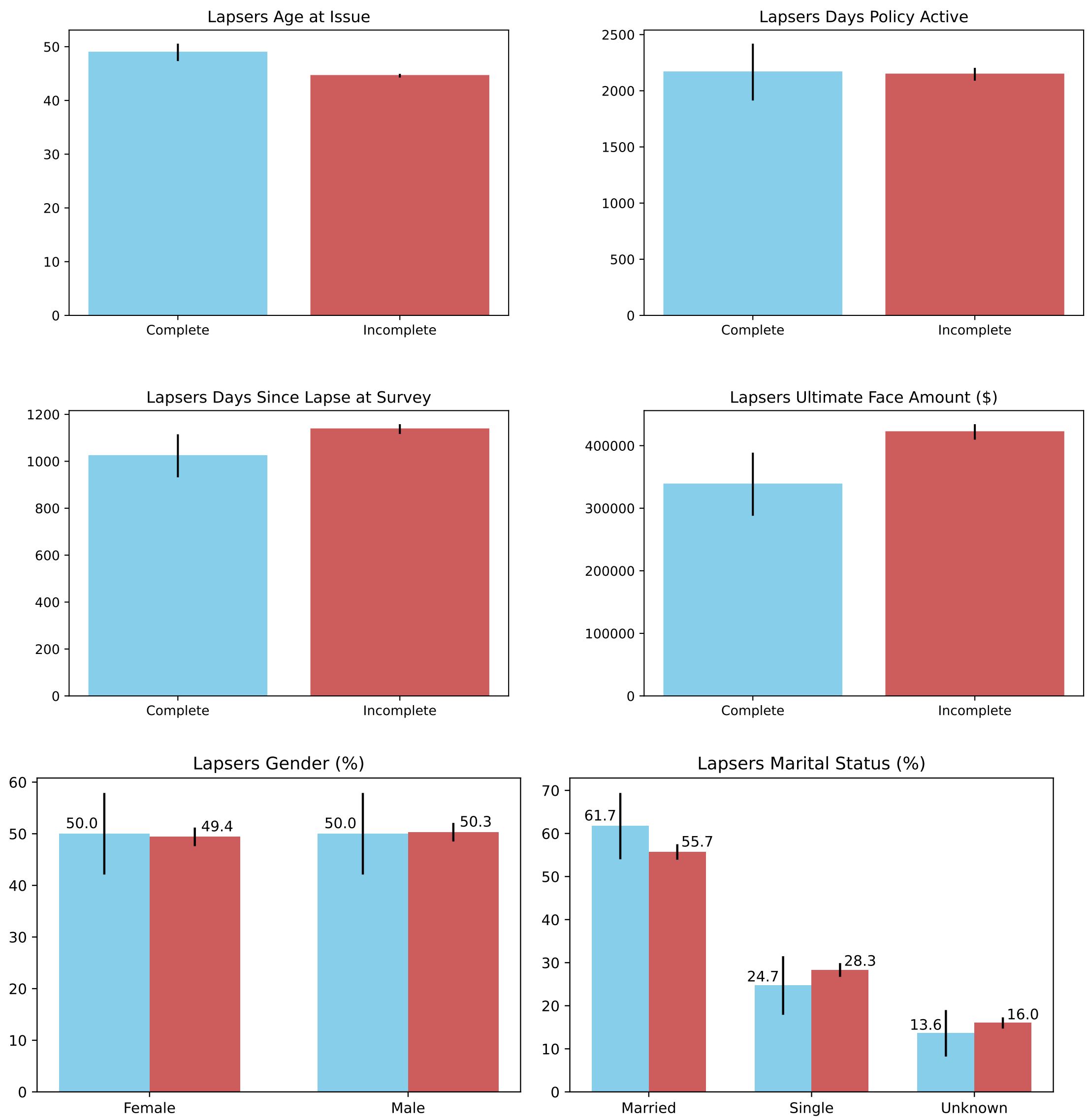

Figure 5: Descriptive statistics for respondents (blue) and non-respondents (red). Black lines represent 95% confidence intervals. Starting from the top, the figures represent age at issuance; number days before policy lapsed; number of days between lapse and survey; ultimate face amount; gender; and marital status. Differences between other variables not shown were typically not statistically significant, likely due to low response rate.

	1	2	3	4	5	6	7	8
1	*.942*	*.014*	*.001*	*.000*	*.001*	*.001*	*.000*	*.041*
2	*.544*	*.252*	*.009*	*.004*	*.011*	*.006*	*.002*	*.172*
3	*.515*	*.259*	*.010*	*.005*	*.012*	*.006*	*.002*	*.190*
4	*.446*	*.285*	*.012*	*.007*	*.020*	*.007*	*.003*	*.219*
5	*.257*	*.273*	*.020*	*.020*	*.065*	*.007*	*.005*	*.353*
6	*.430*	*.296*	*.014*	*.009*	*.027*	*.008*	*.004*	*.212*
7	*.229*	*.267*	*.021*	*.022*	*.074*	*.007*	*.006*	*.374*

Markov Transition Matrix (50 year old Male; 5 years)

Table 2: Probability of five-year ahead changes in health states at age 50.

Markov Transition Matrix (75 year old Male; 5 years)

	1	2	3	4	5	6	7	8
1	*.645*	*.103*	*.014*	*.005*	*.014*	*.016*	*.008*	*.195*
2	*.129*	*.235*	*.038*	*.016*	*.040*	*.036*	*.024*	*.482*
3	*.094*	*.198*	*.035*	*.017*	*.048*	*.032*	*.025*	*.551*
4	*.046*	.136	*.031*	*.023*	*.078*	*.025*	*.033*	*.629*
5	*.011*	*.046*	*.016*	*.019*	*.095*	*.011*	*.032*	*.771*
6	*.052*	*.150*	*.035*	*.021*	*.079*	*.052*	*.048*	*.562*
7	*.009*	*.036*	*.013*	*.015*	*.087*	*.013*	*.039*	*.787*

Table 3: Probability of five-year ahead changes in health states at age 75.

The composition of shocks changes significantly along the life cycle. The tables above show "snap shots" across different ages of five-year ahead Markov health transition matrices based on hazard rates provided by Robinson (1996). State 1 represents the healthiest state while State 8 represents the worst (death). As the matrices show, younger individuals are unlikely to suffer negative health shocks and the ones who do experience such shocks typically recover within the next 5 years (with the obvious exception of death, which is). Older individuals are more likely to suffer negative heath shocks, and those shocks are substantially more persistent. Consistently, policyholders younger than about 65 rarely surrender due to health shocks whereas health shocks are considerably more important for older policyholders (c.f., Fang and Kung 2012).

Consistently with these observations, we consider a stylized model in which the period of shocks is broken down in two periods. In the first period, consumers are subject to non-health shocks only. In the second period, they are only subject to health shocks. As a result, optimal contracts are *back loaded*: they do not discourage lapses in the first period but discourage lapses in the second period. Because only health-related lapsing is inefficient, lapse fees should be high only in periods in which health shocks are relatively prevalent. Empirically, these periods occur much later in life.

Formally, there are 4 periods: $t = 0,1,2,3$. Period 0 is the contracting stage. Consumers are subject to a liquidity shock $L > 0$ (with probability $l > 0$) in period 1. They are subject to a health shock in period 2. The health shock is modeled as follows. With probability $\pi > 0$, the consumer finds out that he has a high risk of dying (*type H*). With complementary probability, he finds out that he has a low risk of death (*type L*). Then, in period 3, a high-risk consumer dies with probability α_H and a low-risk consumer dies with probability α_L, where $0 < \alpha_L < \alpha_H < 1$. We model lapses as motivated by liquidity/income shocks rather than bequest shocks because, as shown by First, Fang and Kung (2012), bequest shocks are responsible for a rather small proportion of lapses, whereas other (i.e. non-health and non-bequest shocks) are responsible for most of it, especially for individuals below a certain age. The assumption that mortality shocks only happen in the last period is for simplicity only. Our result remains if we assume that there is a positive probability of death in each period.

The timing of the model is as follows:

- Period 0: The consumer makes a take-it-or-leave-it offer of a contract to a non-empty set of firms. A contract is a vector of state-contingent payments to the firm

$$\left\{t_0, t_1^s, t_2^{s,h}, t_3^{d,s,h}\right\}_{s=S,NS\; h=H,L\; d=D,A},$$

  where: t_0 is paid in period 0 before any information is learned; t_1^s is paid conditional on the liquidity shocks in period 1, $s = S, NS$; $t_2^{s,h}$ is paid conditional on the health shock $h \in \{H, L\}$ in period 2 and liquidity shock s in period 1; $t_3^{d,s,h}$ is paid conditional on being either dead $d = D$ or alive $d = A$ in period 3 conditional on previous shocks s and h.

- Period 1: The consumer observes the realization of the liquidity shock s. He then decides whether to keep the original contract, thereby paying t_1^s, or obtaining a new contract in a competitive secondary market. The competitive secondary market is again modeled by having the consumer make a take-it-or-leave-it offer a (non-empty) set of firms.

- Period 2: The realization of the health shock is publicly observed. The consumer decides to keep the contract, thereby paying $t_2^{s,h}$, or substitute by a new one, obtained again in a competitive environment (in which the consumer makes a take-it-or-leave it offer to firms).

- Period 3: The mortality shock is realized. The consumer receives a payment of $-t_3^{d,s,h}$.

As before, we assume that consumers and firms discount the future at the same rate and normalize the discount rate to zero. Consumers get utility $u_A(c)$ of consuming c units (while alive). Consumers get utility $u_D(c)$ from bequeathing c units. The functions u_A and u_D satisfy the Inada condition: $\lim_{c \searrow 0} u_d(c) = -\infty$, $d = A, D$.

With no loss of generality, we can focus on period-0 contracts that the consumer never finds it optimal to drop. That is, we may focus on contracts that satisfy "non-reneging constraints." Of course, this is not to say that the equilibrium contracts will never be dropped in the same way that the revelation principle does not say that in the real world people should be "announcing their types." To wit, any allocation implemented by a non-reneging contract can also be implemented by a mechanism in which the consumer is given resources equal to the expected amount of future consumption and gets a new contract (from possibly a different firm) in each period. In particular, the model cannot distinguish between lapsing an old contract and substituting it by a new (state-contingent) contract and having an initial contract that is never lapsed and features state-dependent payments that satisfy the non-reneging constraint. However, the model determines payments in each state.

Consistently with actual (whole) life insurance policies, one can interpret the change of terms following a liquidity shock in period 1 as the lapsing of a policy at some predetermined cash value and the purchase of a new policy, presumably with a smaller coverage. We ask the following question: Is it possible for a firm to profit from lapses motivated by a liquidity shock? In other words, it is possible for the firm to get higher expected profits conditional on the consumer experiencing a liquidity shock in period 1 than conditional on the consumer not experiencing a liquidity shock? As we have seen in the evidence described in Section 2, firms do profit from such lapses, which are the most common source of lapses for policyholders below a certain age. However, as we show below, this is incompatible with the reclassification risk model described here.

The intuition for the result is straightforward. The reason why individuals prefer to purchase insurance at 0 rather than 1 is the risk of needing liquidity and therefore facing a lower wealth. If the insurance company were to profit from the consumers who suffer the liquidity shock, it would need to charge a higher premium if the consumer suffers the shock. However, this would exacerbate the liquidity shock. In that case, the consumer would be better off by waiting to buy insurance after the realization of the shock.

As in the text, there is no loss of generality in working with the space of state-contingent consumption rather than transfers. The consumer's expected utility is

$$u_A(c_0) + l \left\{ \begin{array}{l} u_A\left(c_1^S\right) + \pi \left[u_A\left(c_2^{S,H}\right) + (1-\alpha_H)\, u_A\left(c_3^{S,H,A}\right) + \alpha_H u_D\left(c_3^{S,H,D}\right) \right] \\ \quad + (1-\pi) \left[u_A\left(c_2^{S,L}\right) + (1-\alpha_L)\, u_A\left(c_3^{S,L,A}\right) + \alpha_L u_D\left(c_3^{S,L,D}\right) \right] \end{array} \right\}$$

$$+ (1-l) \left\{ \begin{array}{l} u_A\left(c_1^{NS}\right) + \pi \left[u_A\left(c_2^{NS,H}\right) + (1-\alpha_H)\, u_A\left(c_3^{NS,H,A}\right) + \alpha_H u_D\left(c_3^{NS,H,D}\right) \right] \\ \quad + (1-\pi) \left[u_A\left(c_2^{NS,L}\right) + (1-\alpha_L)\, u_A\left(c_3^{NS,L,A}\right) + \alpha_L u_D\left(c_3^{NS,L,D}\right) \right] \end{array} \right\}.$$

The equilibrium contract maximizes this expression subject to the following constraints. First, the firm cannot be left with negative profits:

$$c_0+l\left\{\begin{array}{l} c_1^S+\pi\left[c_2^{S,H}+(1-\alpha_H)\,c_3^{S,H,A}+\alpha_H c_3^{S,H,D}\right] \\ +(1-\pi)\left[c_2^{S,L}+(1-\alpha_L)\,c_3^{S,L,A}+\alpha_L c_3^{S,L,D}\right] \end{array}\right\}$$

$$+(1-l)\left\{\begin{array}{l} c_1^{NS}+\pi\left[c_2^{NS,H}+(1-\alpha_H)\,c_3^{NS,H,A}+\alpha_H c_3^{NS,H,D}\right] \\ +(1-\pi)\left[c_2^{NS,L}+(1-\alpha_L)\,c_3^{NS,L,A}+\alpha_L c_3^{NS,L,D}\right] \end{array}\right\}$$

$$\leq W+I\left[2-\pi\alpha_H-(1-\pi)\,\alpha_L\right]-lL$$

Second, allocation has to satisfy the incentive-compatibility constraints (which state that the consumer prefers the report of the liquidity shock honestly):

$$u_A\left(c_1^S\right)+\pi\left[u_A\left(c_2^{S,H}\right)+(1-\alpha_H)\,u_A\left(c_3^{S,H,A}\right)+\alpha_H u_D\left(c_3^{S,H,D}\right)\right]$$

$$+(1-\pi)\left[u_A\left(c_2^{S,L}\right)+(1-\alpha_L)\,u_A\left(c_3^{S,L,A}\right)+\alpha_L u_D\left(c_3^{S,L,D}\right)\right]\geq$$

$$u_A\left(c_1^{NS}-L\right)+\pi\left[u_A\left(c_2^{NS,H}\right)+(1-\alpha_H)\,u_A\left(c_3^{NS,H,A}\right)+\alpha_H u_D\left(c_3^{NS,H,D}\right)\right]$$

$$+(1-\pi)\left[u_A\left(c_2^{NS,L}\right)+(1-\alpha_L)\,u_A\left(c_3^{NS,L,A}\right)+\alpha_L u_D\left(c_3^{NS,L,D}\right)\right],$$

and

$$u_A\left(c_1^{NS}\right)+\pi\left[u_A\left(c_2^{NS,H}\right)+(1-\alpha_H)\,u_A\left(c_3^{NS,H,A}\right)+\alpha_H u_D\left(c_3^{NS,H,D}\right)\right]$$

$$+(1-\pi)\left[u_A\left(c_2^{NS,L}\right)+(1-\alpha_L)\,u_A\left(c_3^{NS,L,A}\right)+\alpha_L u_D\left(c_3^{NS,L,D}\right)\right]\geq$$

$$u_A\left(c_1^S+L\right)+\pi\left[u_A\left(c_2^{S,H}\right)+(1-\alpha_H)\,u_A\left(c_3^{S,H,A}\right)+\alpha_H u_D\left(c_3^{S,H,D}\right)\right]$$

$$+(1-\pi)\left[u_A\left(c_2^{S,L}\right)+(1-\alpha_L)\,u_A\left(c_3^{S,L,A}\right)+\alpha_L u_D\left(c_3^{S,L,D}\right)\right].$$

The third set of constraints requires contracts to be non-reneging after it has been agreed upon (that is, in periods 1 and 2). The period-2 non-reneging constraints are

$$u_A\left(c_2^{s,h}\right)+(1-\alpha_h)\,u_A\left(c_3^{A,NS,h}\right)+\alpha_h u_D\left(c_3^{D,s,h}\right)\geq\max_{\{\hat{c}\}}\left\{\begin{array}{c} u_A\left(\hat{c}_2\right)+(1-\alpha_h)\,u_A\left(\hat{c}_3\right)+\alpha_h u_D\left(\hat{c}_3\right) \\ \text{s.t. } \hat{c}_2+(1-\alpha_h)\,\hat{c}_3+\alpha_h\hat{c}_3=(2-\alpha_h)\,I \end{array}\right\}, \tag{52}$$

for $h=H,L$ and $s=S,NS$. The period-1 non-reneging constraints are

$$u_A\left(c_1^s\right)+\pi\left[u_A\left(c_2^{s,H}\right)+(1-\alpha_H)\,u_A\left(c_3^{s,H,A}\right)+\alpha_H u_D\left(c_3^{s,H,D}\right)\right]$$

$$+(1-\pi)\left[u_A\left(c_2^{s,L}\right)+(1-\alpha_L)\,u_A\left(c_3^{s,L,A}\right)+\alpha_L u_D\left(c_3^{s,L,D}\right)\right]\geq$$

$$\max_{\{\hat{c}\}} \begin{array}{l} u_A\left(\hat{c}_1^s\right) + \pi\left[u_A\left(\hat{c}_2^{s,H}\right) + (1-\alpha_H)\, u_A\left(\hat{c}_3^{s,H,A}\right) + \alpha_H u_D\left(\hat{c}_3^{s,H,D}\right)\right] \\ \quad + (1-\pi)\left[u_A\left(\hat{c}_2^{s,L}\right) + (1-\alpha_L)\, u_A\left(\hat{c}_3^{s,L,A}\right) + \alpha_L u_D\left(\hat{c}_3^{s,L,D}\right)\right] \end{array}$$

subject to

$$\hat{c}_1^s + \pi\left[\hat{c}_2^{s,H} + (1-\alpha_H)\,\hat{c}_3^{s,H,A} + \alpha_H \hat{c}_3^{s,H,D}\right] + (1-\pi)\left[\hat{c}_2^{s,L} + (1-\alpha_L)\,\hat{c}_3^{s,L,A} + \alpha_L \hat{c}_3^{s,L,D}\right]$$

$$\leq I\left[2 - \pi\alpha_H - (1-\pi)\,\alpha_L\right] - \chi_{s=S} L,$$

and

$$u_A\left(\hat{c}_2^{s,h}\right) + (1-\alpha_h)\, u_A\left(\hat{c}_3^{A,NS,h}\right) + \alpha_h u_D\left(\hat{c}_3^{D,s,h}\right) \geq \max_{c_2, c_3^A, c_3^D} \left\{ \begin{array}{c} u_A(c_2) + (1-\alpha_h)\, u_A\left(c_3^A\right) + \alpha_h u_D\left(c_3^D\right) \\ \text{s.t. } c_2 + (1-\alpha_h)\, c_3^A + \alpha_h c_3^D = (2-\alpha_h)\, I \end{array} \right\},$$

for $s = S, NS$, where χ_x denotes the indicator function.

We will define a couple of "indirect utility" functions that will be useful in the proof by simplifying the non-reneging constraints. First, for $h = H, L$ we introduce the function $U_h : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$U_h(W) \equiv \max_{c^A, c^D} \left\{ \begin{array}{c} (2-\alpha_h)\, u_A\left(c^A\right) + \alpha_h u_D\left(c^D\right) \\ \text{s.t. } (2-\alpha_h)\, c^A + \alpha_h c^D \leq W \end{array} \right\}.$$

It is straightforward to show that U_h is strictly increasing and strictly concave. Next, we introduce the function $\mathscr{U} : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$\mathscr{U}(W) \equiv \max_{c, C^L, C^H} \left\{ \begin{array}{c} u_A(c) + \pi U(C^H) + (1-\pi)\, U(C^L) \\ s.t.\ \ c + \pi C^H + (1-\pi)\, C^L \leq W \\ (2-\alpha_H)\, I \leq C^H \\ (2-\alpha_L)\, I \leq C^L \end{array} \right\}. \tag{53}$$

It is again immediate to see that $\mathscr{U}$ is strictly increasing. The following lemma establishes that it is also strictly concave:

Lemma 13. *$\mathscr{U}$ is a strictly concave function.*

Proof. Let

$$\mathscr{U}_0(W) \equiv \max_{C^L, C^H} \left\{ u_A\left(W - \pi C^H - (1-\pi)\, C^L\right) + \pi U(C^{s,H}) + (1-\pi)\, U(C^{s,L}) \right\},$$

$$\mathscr{U}_1(W) \equiv \max_{C^L, C^H} \left\{ \begin{array}{c} u_A\left(W - \pi C^H - (1-\pi)\, C^L\right) + \pi U(C^{s,H}) + (1-\pi)\, U(C^{s,L}) \\ \text{subject to } \ (2-\alpha_L)\, I = C^H \end{array} \right\}, \text{ and}$$

$$\mathscr{U}_2(W) \equiv \max_{C^L, C^H} \left\{ \begin{array}{c} u_A\left(W - \pi C^H - (1-\pi) C^L\right) + \pi U(C^{s,H}) + (1-\pi) U(C^{s,L}) \\ \text{subject to } \ (2-\alpha_H) I = C^H \\ (2-\alpha_L) I = C^L \end{array} \right\}.$$

Notice that $\mathscr{U}_0(W) \geq \mathscr{U}_1(W) \geq \mathscr{U}_2(W)$, and $\mathscr{U}_0$, $\mathscr{U}_1$, and $\mathscr{U}_2$ are strictly concave. It is straightforward to show that there exist W_L and $W_H > W_L$ such that:

- $\mathscr{U}(W) = \mathscr{U}_0(W)$ for $W \geq W_H$,
- $\mathscr{U}(W) = \mathscr{U}_1(W)$ for $W \in [W_L, W_H]$, and
- $\mathscr{U}(W) = \mathscr{U}_2(W)$ for $W \leq W_L$.

Moreover, by the envelope theorem, $\mathscr{U}_0'(W_H) = \mathscr{U}_1'(W_H)$ and $\mathscr{U}_1'(W_L) = \mathscr{U}_2'(W_L)$. Therefore,

$$\mathscr{U}'(W) = \left\{ \begin{array}{c} \mathscr{U}_0'(W) \text{ for } W \geq W_H \\ \mathscr{U}_1'(W) \text{ for } W_L < W \leq W_H \\ \mathscr{U}_2'(W) \text{ for } W < W_L \end{array} \right. .$$

Because $\mathscr{U}'$ is strictly decreasing in each of these regions and is continuous, it then follows that $\mathscr{U}$ is strictly concave. $\square$

Let X^s be the sum of the insurance company's expected expenditure at time t=1 conditional on s in the original contract:

$$X^s \equiv c_1^s + \pi\left[c_2^{s,H} + (1-\alpha_H) c_3^{s,H,A} + \alpha_H c_3^{s,H,D}\right] + (1-\pi)\left[c_2^{s,L} + (1-\alpha_L) c_3^{s,L,A} + \alpha_L c_3^{s,L,D}\right] + \chi_{s=S} L.$$

Our main result establishes that in any optimal mechanism the insurance company gets negative profits from consumers who suffer a liquidity shock and positive profits from those who do not suffer a liquidity shock. Expected profits conditional on the liquidity shock $s = S, NS$ equal

$$\Pi^s \equiv W + I\left[2 - \pi\alpha_H - (1-\pi)\alpha_L\right] - (c_0 + X^s).$$

By zero profits, we must have $l\Pi^S + (1-l)\Pi^{NS} = 0$. We can now prove our main result:

Proposition 6. *In any equilibrium contract, the insurance company gets negative profits from consumers who suffer a liquidity shock and positive profits from those who do not suffer a liquidity shock:*

$$\Pi^S \leq 0 \leq \Pi^{NS}. \tag{54}$$

Proof. Suppose we have an initial contract in which the firm profits from the liquidity shock in period 1 (that is, inequality 54 does not hold). Then, by the definition of Π^s, we must have that the total expenditure conditional on $s = NS$ exceeds the one conditional on $s = S$: $X^{NS} > X^S$.

Consider the alternative contract that allocates the same consumption at $t = 0$ as the original one but implements the best possible renegotiated contract at $t = 1$ conditional on the liquidity shock. More precisely, consumption in subsequent periods is defined by the solution to

$$\max_{\left(c_1^s,c_2^{s,h},c_3^{s,h,d}\right)_{h=H,L,\,d=A,D}} u_A\left(c_1^s\right)+\pi\left[u_A\left(c_2^{s,H}\right)+\left(1-\alpha_H\right)u_A\left(c_3^{s,H,A}\right)+\alpha_H u_D\left(c_3^{s,H,D}\right)\right] \tag{55}$$

$$+\left(1-\pi\right)\left[u_A\left(c_2^{s,L}\right)+\left(1-\alpha_L\right)u_A\left(c_3^{s,L,A}\right)+\alpha_L u_D\left(c_3^{s,L,D}\right)\right]$$

subject to

$$\left\{\begin{array}{l} c_1^s+\pi\left[c_2^{s,H}+\left(1-\alpha_H\right)c_3^{s,H,A}+\alpha_H c_3^{s,H,D}\right] \\ +\left(1-\pi\right)\left[c_2^{s,L}+\left(1-\alpha_L\right)c_3^{s,L,A}+\alpha_L c_3^{s,L,D}\right] \end{array}\right\}\leq I\left[2-\pi\alpha_H-\left(1-\pi\right)\alpha_L\right]-\chi_{s=S}L,$$

$$u_A\left(c_2^{s,h}\right)+\left(1-\alpha_h\right)u_A\left(c_3^{A,s,h}\right)+\alpha_h u_D\left(c_3^{D,s,h}\right)\geq \tag{56}$$

$$\max_{\{\hat{c}\}}\left\{\begin{array}{c} u_A\left(\hat{c}_2\right)+\left(1-\alpha_h\right)u_A\left(\hat{c}_3\right)+\alpha_h u_D\left(\hat{c}_3\right) \\ \text{s.t. } \hat{c}_2+\left(1-\alpha_h\right)\hat{c}_3+\alpha_h\hat{c}_3=\left(2-\alpha_h\right)I \end{array}\right\},\ h=L,H.$$

By construction, this new contract satisfies the non-reneging and incentive-compatibility constraints. We claim that the solution entails full insurance conditional on the shock: $u_A'\left(c_2^{s,h}\right)=u_A'\left(c_3^{A,NS,h}\right)=u_D'\left(c_3^{D,s,h}\right)$ for all s,h (starting from any point in which this is not satisfied, we can always increase the objective function while still satisfying both the zero-profit condition and the non-reneging constraints by moving towards full insurance). Let $C^{s,h}\equiv c_2^{s,h}+\left(1-\alpha_h\right)c_3^{A,s,h}+\alpha_h c_3^{D,s,h}$ denote the total expected consumption at periods 2 and 3. Then, $c_2^{s,h}$ and $c_3^{d,s,h}$ maximize expected utility in period 2 conditional on the shocks s,h given the total expected resources:

$$u_A\left(c_2^{s,h}\right)+\left(1-\alpha_h\right)u_A\left(c_3^{A,s,h}\right)+\alpha_h u_D\left(c_3^{D,s,h}\right)=\max_{c,c^A,c^D}\left\{\begin{array}{c} u(c)+\left(1-\alpha_h\right)u_A\left(c^A\right)+\alpha_h u_D\left(c^D\right) \\ \text{s.t. } c+\left(1-\alpha_h\right)c^A+\alpha_h c^D\leq C^{s,h} \end{array}\right\}$$

$$=\max_{c^A,c^D}\left\{\begin{array}{c} \left(2-\alpha_h\right)u_A\left(c^A\right)+\alpha_h u_D\left(c^D\right) \\ \text{s.t. } \left(2-\alpha_h\right)c^A+\alpha_h c^D\leq C^{s,h} \end{array}\right\}=U_h\left(C^{s,h}\right).$$

The non-reneging constraints (56) can be written as

$$U_h\left(C^{s,h}\right)\geq U_h\left(\left(2-\alpha_h\right)I\right),\ \ h=L,H.$$

Using the fact that U_h is strictly increasing, they can be further simplified to

$$\left(2-\alpha_h\right)c_3^{A,s,h}+\alpha_h c_3^{D,s,h}\geq\left(2-\alpha_h\right)I,\ \ h=L,H.$$

With these simplifications, we can rewrite Program (55) as

$$\max_{c_1^s, C^{s,H}, C^{s,L}} u_A\left(c_1^s\right) + \pi U(C^{s,H}) + (1-\pi) U(C^{s,L})$$

subject to

$$c_1^s + \pi C^{s,H} + (1-\pi) C^{s,L} \leq I\left[2 - \pi\alpha_H - (1-\pi)\alpha_L\right] - \chi_{s=S} L,$$

$$(2-\alpha_H) I \leq C^{s,H},$$

$$(2-\alpha_L) I \leq C^{s,L}.$$

By equation (53), this expression corresponds to $\mathscr{U}\left(I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]-\chi_{s=S}L\right)$.

The consumer's expected utility from this new contract (at time 0) equals

$$u(c_0) + l\mathscr{U}\left(I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]-L\right) + (1-l)\mathscr{U}\left(I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]\right). \tag{57}$$

The utility that the consumer attains with the original contract is bounded above by the contract that provides full insurance conditional on the amount of resources that the firm gets at each state in period 1: X^S and X^{NS} (note that this is an upper bound since we do not check for incentive-compatibility or non-reneging constraints). That is, the utility under the original contract is bounded above by

$$u(c_0) + l\mathscr{U}\left(X^S - L\right) + (1-l)\mathscr{U}\left(X^{NS}\right). \tag{58}$$

By zero profits, the expected expenditure in the original and the new contracts are the same. Moreover, because $X^S < I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]$, it follows that the lottery $\{X^S - L, l; X^{NS}, 1-l\}$ is a mean-preserving spread of the lottery

$$\{I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]-L, l; I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right], 1-l\}.$$

Thus, strict concavity of $\mathscr{U}$ yields:

$$l\mathscr{U}\left(X^S - L\right) + (1-l)\mathscr{U}\left(X^{NS}\right) <$$

$$l\mathscr{U}\left(I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]-L\right) + (1-l)\mathscr{U}\left(I\left[2-\pi\alpha_H-(1-\pi)\alpha_L\right]\right).$$

Adding $u(c_0)$ to both sides and comparing with expressions (57) and (58), it follows that the consumer's expected utility under the new contract exceed his expected utility under the original contract, thereby contradicting the optimality of the original contract. $\square$

Therefore, in any equilibrium, firms cannot profit from consumers who suffer a liquidity shock and cannot lose money from those that do not.

Appendix D: Additional Empirical Evidence

HRS Data

This appendix reports the relationship between lapsing and health shocks using the data from the Health and Retirement Study (HRS) (various years). The construction of Table 4 follows the steps outlined in Fang and Kung (2012, Table 6) for the 1994 and 1996 longitudinal waves. We also included respondents in the more recent 2012 wave. As in Fang and Kung, Table 4 shows that there is a positive and statistically insignificant relationship between lapses and either the number of health conditions (Conditions) and changes in health conditions (ΔConditions). Therefore, individuals who lapse are not healthier than those who maintain coverage and they do not appear to lapse after positive health shocks, which is not in line with a reclassification risk channel.

Evidence of Lapsing in Health and Retirement Survey

	Logit Regression		Logit Marginal Effect	
Variable	*Coefficient*	*SE*	*Coefficient*	*SE*
Constant	1.73	0.987	na	na
Age	0.012	0.007	0.001	0.000
Logincome	-0.406***	0.046	-0.032***	0.003
Number of health conditions	0.008	0.026	0.007	0.002
Married	0.162	0.250	0.012	0.019
Has children	-0.775	0.657	-0.061	0.052
Age of youngest child	-0.004	0.005	0.000	0.000
ΔAge	0.202	0.327	0.016	0.026
(ΔAge)^2	-0.011	0.078	-0.001	0.006
ΔLogincome	0.076	0.051	0.006	0.004
(ΔLogincome)^2	-0.003	0.014	0.000	0.001
ΔConditions	0.106	0.139	0.008	0.011
(ΔConditions)^2	0.013	0.075	0.001	0.006
ΔMarried	-0.299	0.335	-0.023	0.026

Table 4: Logistic Regression using 1992 - 2012 RAND HRS Longitudinal (13,137 Observations) with dependent variable equal to maintaining (0) / lapsing (1) coverage. ***, **, and * represent significance at 0.001, 0.01, and 0.05, respectively.

MetLife and SBLI Data

As we examine in Subsection 5.2, in the presence of fixed costs that are proportional to coverage, such as sales commissions or illiquidity premiums, even consumers with rational expectations may demand policies with surrender fees. With commissions, one should not observe different surrender fees after all commissions have been paid, which typically happens in the first two or three years of the policy.

As described in Subsection 5.2, with an explanation based on the cost of liquidating investments, the optimal insurance policy sets a surrender fee that balances the higher return that can be obtained by facing a more predictable pattern of lapses against the cost that policyholders face when they are unable to smooth consumption after a liquidity shock. Let the liquidity premium be the proportion of the value of an investment that has to be given up in case of early liquidation. Since the (ex-ante) expected cost to policyholders of being unable to smooth consumption is increasing in the probability of a liquidity shock, this explanation predicts that, holding the liquidity premium constant, surrender fees (as a proportion of the amount invested) should *decrease* in the probability of facing a liquidity shock.

In order to evaluate the relationship between surrender fees and the probability of liquidity shocks, we hand-collected detailed whole life insurance policy offers from two national insurance companies, MetLife and SBLI, at the Lifequotes (2013) website.[53] MetLife and SBLI are two national life insurers with operations in most of the 50 states. MetLife is the largest U.S. life insurer with over $2 trillion in total life insurance coverage in force while SBLI is middle sized with $125 billion of coverage in force, thereby allowing us to ensure that premium data was not driven by idiosyncratic features associated with firm size. We collected policy information across both genders across with the following coverage amounts: $100,000; $250,000; $500,000; $750,000 and $1,000,000. We chose ages between 20 and 70 in 5-year increments and both genders. We focused on traditional whole life policies since future cash surrender values do not depend on the return of the insurer's portfolio.[54] MetLife policies mature at age 120, whereas SBLI policies mature at age 121. All policies assume no tobacco or nicotine use and excellent health ("preferred plus"). Premiums are annual, which is the most common frequency. An automation tool was used to effectively eliminate human coding error. For each policy, we obtained the cash surrender values for each of the 25 years after purchase.

Our data set covers all American States except for New York[55], where the companies did not offer these policies, for a total of 10,738 policies. MetLife offered policies for all coverage amounts noted above, for a total of 5,390 policies (2,695 per gender). SBLI did not offer policies with $100,000 coverage for individuals aged 60 and older in the states of Alabama, Alaska, Idaho, Minnesota, Montana, Nebraska, North Dakota, and Washington. In total, these missing data add up to 42 policies (21 per gender). Our results remain if we exclude these states from the sample. In sum, our data set consists of 5,348 policies (2,674 per gender) SBLI policies.

The surrender fee for each policy corresponds to the proportion of the discounted sum of insurance loads (i.e., present value of premiums paid in excess of the actuarially-fair price) that cannot be recovered as cash surrender value. Thus, the surrender fee is the fraction of pre-paid premiums that cannot be recovered if the policy is surrendered. To ensure the comparability of the policies, we kept the terms of each policy constant except for our controls (coverage, ages, and genders). We, therefore, focused on

[53]The choice of these two firms was dictated by data availability. Whole-life policies are typically used differently than Universal Life (UL) policies, as UL policies are often used a tax-preferred investment vehicle in addition to insurance.

[54]Unlike traditional whole policies, most universal life insurance policies only provide an estimate of future cash surrender values.

[55]Unfortunately, these two companies did not sell this type of policies in the state New York. SBLI does not operate in New York. MetLife whole policies in New York are issued separately from the ones in other states. In order to verify the robustness of our findings to other companies, we also collected data from other insurance firms for the state of California and obtained the same results.

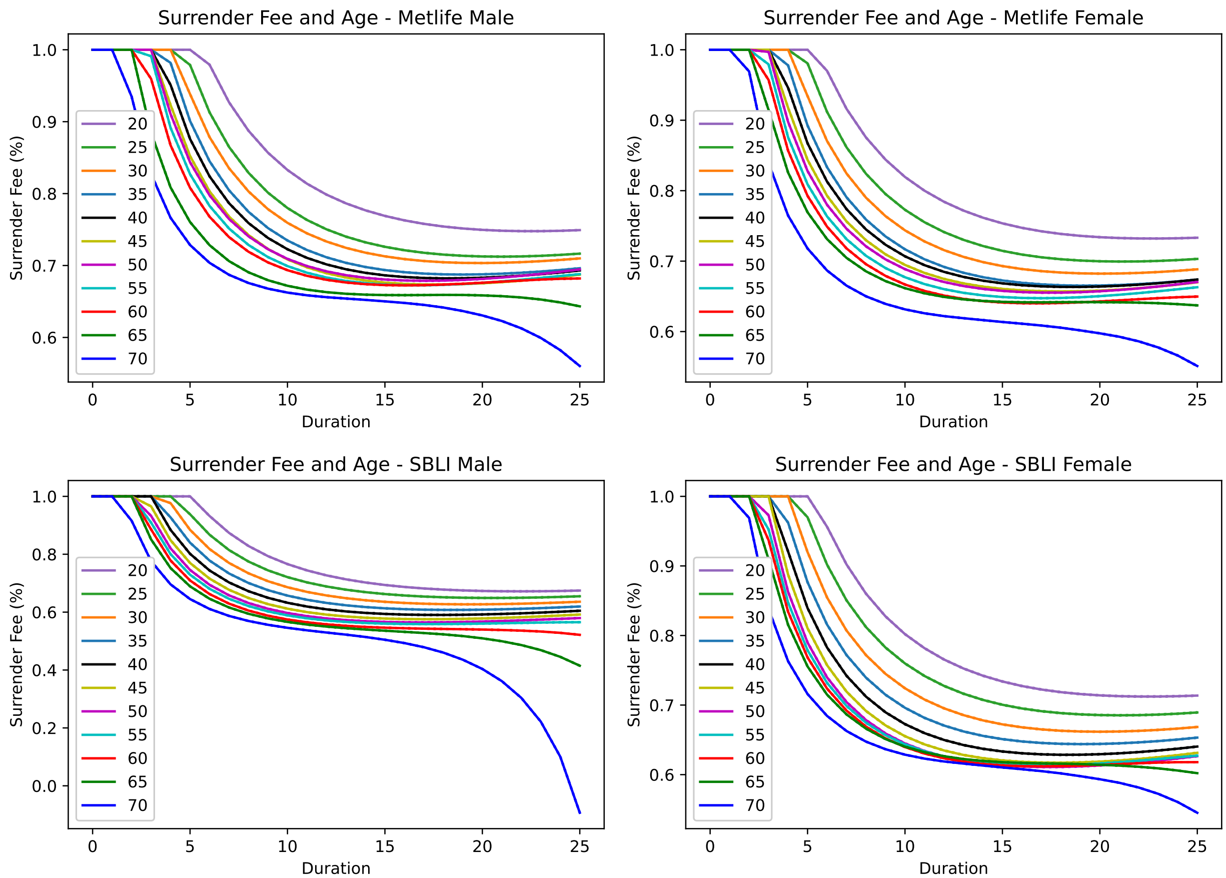

Figure 6: Mean surrender fees by policy duration for each age (different color line) and their 95% confidence intervals.

policies for the "preferred plus" health category that require a health exam.

To test the prediction that surrender fees should increase in the probability of liquidity shocks, we need observable measures of the probability of liquidity shocks. Since we have detailed policy data but no administrative information about the individuals who buy each policy, we need to proxy for the probability of liquidity shocks using the terms of the policies. We use two different proxies: age and coverage. It is widely documented that younger individuals are more likely to be liquidity constrained, and age is a frequently-used proxy for the presence of liquidity constraints.[56] In fact, consistently with these proxies, lapse rates are decreasing in both age and coverage (Section 2).

Figure 6 shows the mean surrender fees as a function of policy duration at each age along with their associated 95% confidence intervals.Because whole life policies do not have a cash surrender value during the first few years after purchasing, surrender fees start at 100% for each age. As policies mature, they accumulate cash value, reducing the surrender fee. Our main interest, however, is in the difference in surrender fees for policies sold to individuals of different ages. For both MetLife and SBLI policies, notice that the surrender fees decrease in age, at each duration. Thus, consistently with the differential

[56]See, for example, Jappelli (1990), Jappelli, Pischke, and Souleles (1998), Besley, Meads, and Surico (2010), and Zhang (2014).

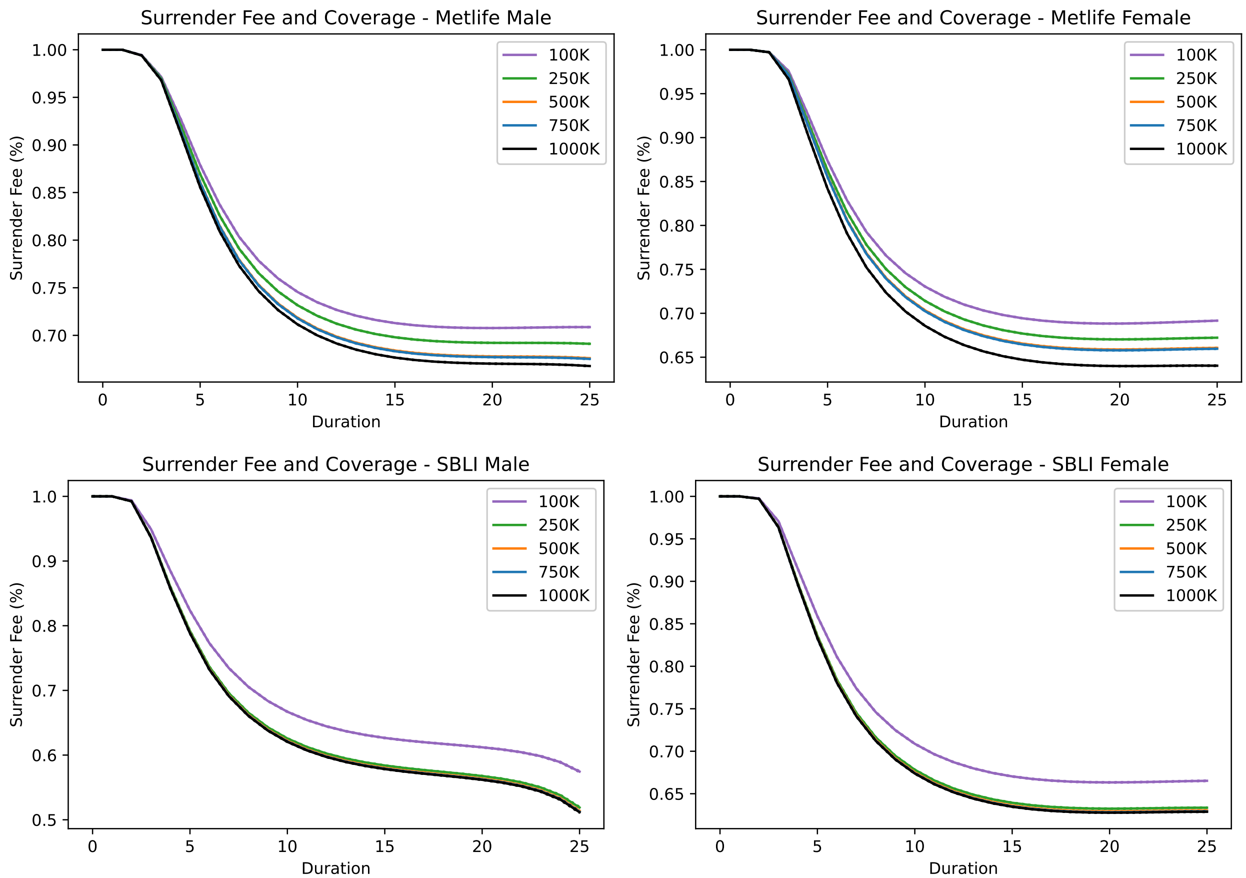

Figure 7: Mean surrender fees by policy duration for each face amount and their 95% confidence intervals.

attention and forgetfulness models, younger individuals face higher surrender fees. The differences by age are not only statistically significant; they are also economically large. For example, while a 20-year-old policyholder who surrenders after 5 years would not collect any cash value, a 70 year old collects about 30% of the amount paid in excess of the actuarially-fair prices. Figure 7 shows the mean surrender fees for different *coverage* amounts and their associated 95% confidence intervals. In contrast to the prediction of the rational model, surrender fees for both MetLife and SBLI policies are decreasing in coverage. However, while for MetLife the difference is always statistically significant, for SBLI policies with coverage above $100,000 are not statistically significant at 5% level.[57] Relative to age considered above, differences by coverage levels are slightly smaller. Nevertheless, the surrender fee on a $100,000 policy is, on average, between 5 and 10 percentage points larger than the surrender fee on a $1,000,000 policy.

[57]The lack of statistical significance for SBLI policies with more than $100,000 coverage could be due to the fact that, while lapse probabilities are much higher for smaller policies, the difference is not very large for policies with coverage above $200,000 (see Figure 2).

Compulife

Data on insurance policy quotes, current as of February 2013, were obtained from Compulife (2013). We gathered quotes for a $500,000 policy with a 20 year term for a male age 35, non-smoker, and a preferred-plus rating class. For the mortality table, we use the 2008 Valuation Basic Table (VBT) computed by the Society of Actuaries that captures the "insured lives mortality" based on the insured population. For Figure 3 in the main text, we assume a nominal interest rate of 6.5%. However, the results are very robust to the interest rate. The figures below repeat the exercise under extreme assumptions about the nominal interest rate and the inflation rate. Note that the only cost in actuarial profits is the death benefit. To obtain economic profits, one should subtract all other costs.

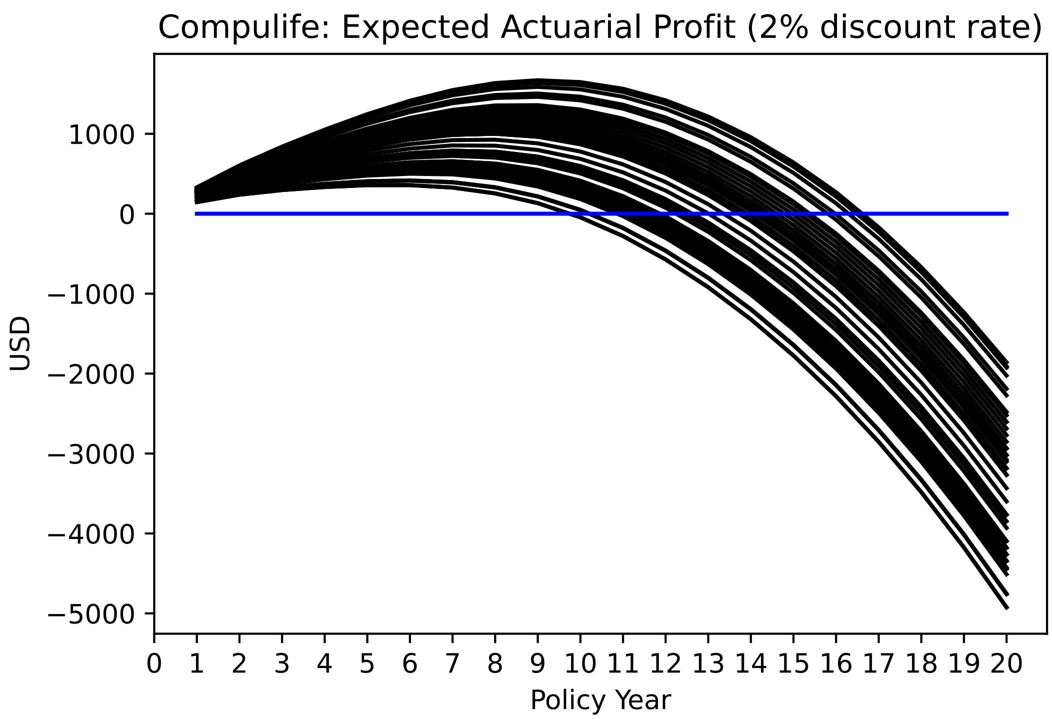

Figure 8: Insurer's profits if the consumer plans to hold policy for after N years under 2% nominal interest rate.

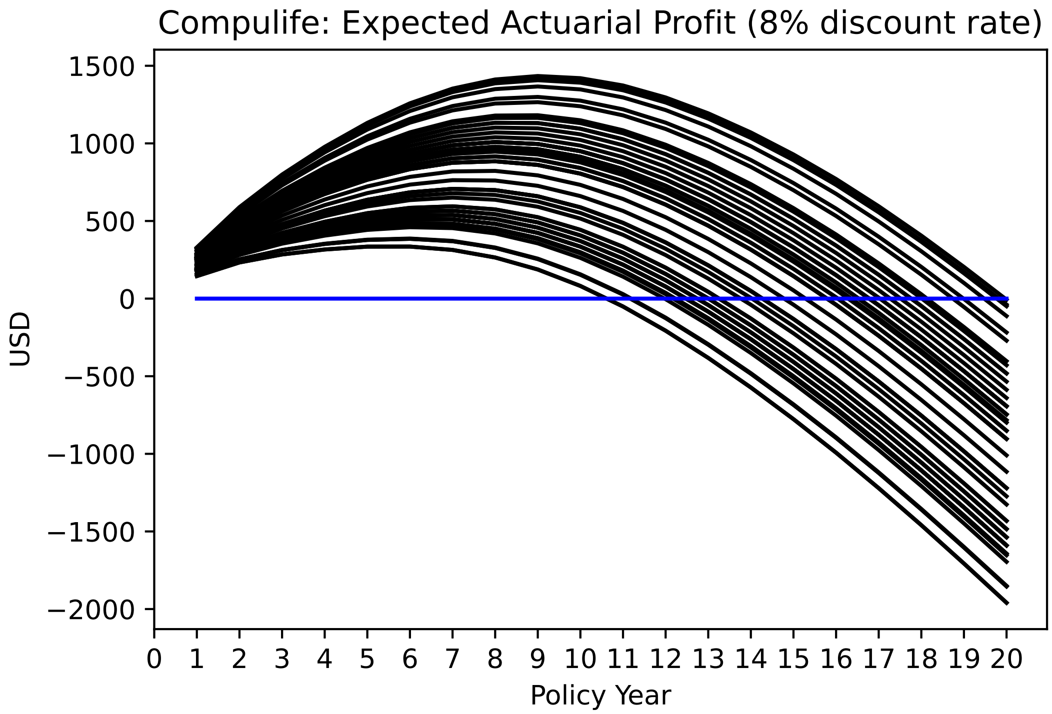

Figure 9: Insurer's profits if the consumer plans to hold policy for after N years under 8% nominal interest rate.

Large Life Insurer (LLI) Color-Coded Data

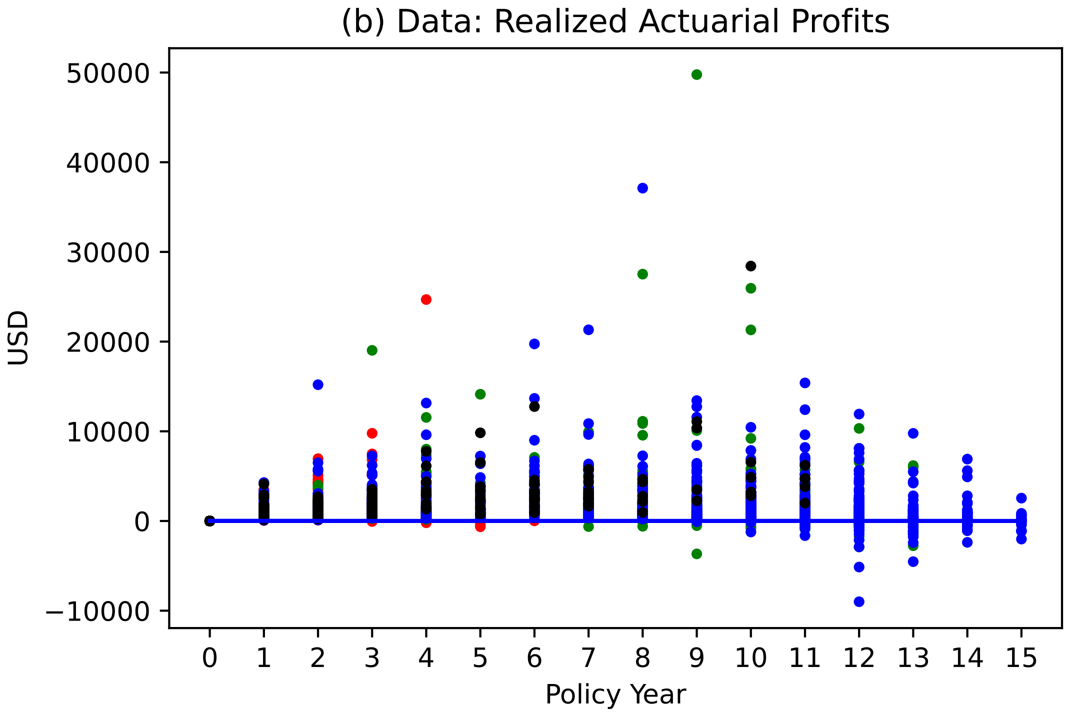

Figure 10: Figure 3(b) with color coding of policies: 10-year (red), 15-year (green), 20-year (blue) and 30-year (black).