Online Appendix of
Selling Consumer Data for Profit:
Optimal Market-Segmentation Design and its Consequences

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OA.1 Omitted Proofs

Proof of Corollary 1

Let (σ, τ) be any optimal mechanism. By Theorem 1, (σ, τ) must be a $\overline{\phi}$-quasi-perfect mechanism and induces $\overline{\phi}$-quasi-perfect price discrimination. Therefore, for any selection $\widehat{\boldsymbol{p}}$ of $\boldsymbol{P}$, for G-almost all $c \in C$ and for $\sigma(c)$-almost all $D \in \mathcal{D}$, $D(p) = 0$ for all $p > \widehat{\boldsymbol{p}}_D(c)$ and thus consumer surplus is

$$\begin{aligned}
&\int_C \left(\int_{\mathcal{D}} \left(\int_{\{v \geq \widehat{\boldsymbol{p}}_D(c)\}} (v - \widehat{\boldsymbol{p}}_D(c)) D(\mathrm{d}v) \right) \sigma^*(\mathrm{d}D|c) \right) G(\mathrm{d}c) \\
=& \int_C \left(\int_{\mathcal{D}} \left(\int_{\widehat{\boldsymbol{p}}_D(c)}^{\overline{v}} D(z)\, \mathrm{d}z \right) \sigma^*(\mathrm{d}D|c) \right) G(\mathrm{d}c) \\
=& 0,
\end{aligned}$$

as desired. ■

Proof of Lemma 4

Since $\boldsymbol{P}_{D_0}(c)$ is a singleton for (Lebesgue)-almost all $c \in C$ and since G is absolutely continuous, consumer surplus under uniform pricing does not depend which selection of $\boldsymbol{P}$ is used. Therefore, by Theorem 1, the

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difference between the data broker's optimal revenue and the consumer surplus under uniform pricing is

$$\begin{aligned}
&\int_C \left(\int_{\{v \geq \overline{\phi}(c)\}} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) \\
&\quad - \int_C \left(\int_{\{v \geq \boldsymbol{p}_0(c)\}} (v - \boldsymbol{p}_0(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) - \bar{\pi} \\
=& \int_C \left((\boldsymbol{p}_0(c) - \phi(c)) D_0(\boldsymbol{p}_0(c)) + \int_{[\overline{\phi}(c), \boldsymbol{p}_0(c))} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) - \bar{\pi} \\
=& \int_C \left(\int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z)) \, \mathrm{d}z - \frac{G(c)}{g(c)} D_0(\boldsymbol{p}_0(c)) \right) G(\mathrm{d}c) \\
&\quad + \int_C \left(\int_{[\overline{\phi}(c), \boldsymbol{p}_0(c))} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) \\
=& \int_C \left(\int_{[\overline{\phi}(c), \boldsymbol{p}_0(c))} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) \\
\geq& 0.
\end{aligned}$$

where the first equality follows from the fact that $\overline{\phi}(c) < \boldsymbol{p}_0(c)$ if and only if $\phi(c) < \boldsymbol{p}_0(c)$, and the third equality follows from changing the order of integrals. This completes the proof. ■

Proof of Lemma 5

To prove Lemma 5, first notice that by the revelation principle, it is without loss to restrict attention to the collection of incentive feasible mechanisms $(\boldsymbol{q}, t)$, where $\boldsymbol{q}(c)$ stands for the quantity purchased for each report $c \in C$ and $t(c)$ stands for the amount of payment from the exclusive retailer to the producer for each report $c \in C$. $(\boldsymbol{q}, t)$ is incentive compatible if for any $c, c' \in C$,

$$t(c) - c\boldsymbol{q}(c) \geq t(c') - c\boldsymbol{q}(c') \tag{IC**}$$

and is individually rational if for any $c \in C$,

$$t(c) - c\boldsymbol{q}(c) \geq \pi_{D_0}(c). \tag{IR**}$$

Meanwhile, notice that given any quantity $q \in [0, 1]$, it is optimal for the exclusive retailer to perfectly price discriminate the consumers with values above $D_0^{-1}(q)$.[1] Together, the exclusive retailer's problem is then to choose $(\boldsymbol{q}, t)$ to maximize

$$\int_C \left(\int_0^{\boldsymbol{q}(c)} D_0^{-1}(q) \, \mathrm{d}q - t(c) \right) G(\mathrm{d}c)$$

subject to (IC**) and (IR**).

Proof of Lemma 5. Consider the exclusive retailer's problem. First notice that by standard arguments, $(\boldsymbol{q}, t)$ is incentive compatible if and only if $\boldsymbol{q}$ is nonincreasing and there exists a constant $\bar{t}$ such that

$$t(c) = c\boldsymbol{q}(c) + \int_c^{\bar{c}} \boldsymbol{q}(z) \, \mathrm{d}z - \bar{t},$$

[1] See the formal proof of this argument in Lemma OA.2 below.

for all $c \in C$. Moreover, any incentive compatible mechanism must give the producer indirect utility

$$\bar{t} + \int_c^{\bar{c}} \boldsymbol{q}(z)\,\mathrm{d}z$$

when her cost is $c \in C$. Together, the exclusive retailer's profit maximization problem can be written as

$$\begin{aligned} &\max_{\boldsymbol{q}\in\mathcal{Q}} \int_C \left(\int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \phi(c))\,\mathrm{d}q \right) G(\mathrm{d}c) \\ &\text{s.t. } \bar{\pi} + \int_c^{\bar{c}} \boldsymbol{q}(z)\,\mathrm{d}z \geq \bar{\pi} + \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z))\,\mathrm{d}z, \end{aligned} \tag{OA.1}$$

where $\mathcal{Q}$ is the collection of nonincreasing functions that map from C to $[0,1]$. Thus, by Lemma OA.3 below, the exclusive retailer's profit maximization problem is equivalent to the price-controlling data broker's revenue maximization problem. This completes the proof. ■

Proof of Proposition 2

To solve for the price-controlling data broker's optimal mechanism, it is useful to introduce the revenue-equivalence formula for the price-controlling data broker.

Lemma OA.1. *For the price-controlling data broker, a mechanism $(\sigma, \tau, \boldsymbol{\gamma})$ is incentive compatible if and only if:*

1. *There exists some $\bar{\tau} \in \mathbb{R}$ such that for any $c \in C$,*

$$\begin{aligned} \tau(c) = &\int_{\mathcal{D}} \int_{\mathbb{R}_+} (p - c) D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, c)\sigma(\mathrm{d}D|c) \\ &- \int_c^{\bar{c}} \int_{\mathcal{D}} \int_{\mathbb{R}_+} D(p)\boldsymbol{\gamma}(\mathrm{d}p|D, z)\sigma(\mathrm{d}D|z)\,\mathrm{d}z - \bar{\tau}. \end{aligned}$$

2. *The function $c \mapsto \int_{\mathcal{D}} \int_{\mathbb{R}_+} D(p)\boldsymbol{\gamma}(\mathrm{d}p|D, c)\sigma(\mathrm{d}D|c)$ is nonincreasing.*

The proof of Lemma OA.1 follows directly from the standard envelope arguments and therefore is omitted.

In what follows, let Γ be the collection of transition kernels that map from $\mathcal{D}$ to $\Delta(\mathbb{R}_+)$. Let $s^{\mathrm{VR}} \in \mathcal{S}$ denote the value-revealing segmentation and let $\sigma^{\mathrm{VR}} : C \to \mathcal{S}$ be the segmentation scheme such that $\sigma^{\mathrm{VR}}(c) = s^{\mathrm{VR}}$ for all $c \in C$. Furthermore, for any $q \in [0,1]$, let $\rho_q := D_0^{-1}(q)$. Notice that by definition of D_0^{-1},

$$q \in [D_0(\rho_q^+), D_0(\rho_q)].$$

If $D_0(\rho_q) = D_0(\rho_q^+)$, then let $\tilde{\gamma}^q : V \to \Delta(\mathbb{R}_+)$ be defined as

$$\tilde{\gamma}^q(\cdot|v) := \delta_{\{v\}},\ \forall v \in V.$$

Meanwhile, if $D_0(\rho_q) > D_0(\rho_q^+)$, then define $\tilde{\gamma}^q : V \to \Delta(\mathbb{R}_+)$ as

$$\tilde{\gamma}^q(\cdot|v) := \begin{cases} \delta_{\{v\}}, & \text{if } v \neq \rho_q \\ \frac{q - D_0(\rho_q^+)}{D_0(\rho_q) - D_0(\rho_q^+)}\delta_{\{v\}} + \frac{D_0(\rho_q) - q}{D_0(\rho_q) - D_0(\rho_q^+)}\delta_{\{\bar{v}\}}, & \text{if } v = \rho_q \end{cases},\ \forall v \in V.$$

Finally, let $\gamma^q \in \Gamma$ be defined as

$$\gamma^q(A|D) := \int_V \tilde{\gamma}^q(A|v) D(\mathrm{d}v),$$

for any measurable $A \subseteq V$ and for any $D \in \mathcal{D}$. By construction, under the segmentation s^{VR} and the randomized price γ^q, all the consumers with values above the $D_0^{-1}(q)$ buy the product by paying exactly their values while the other consumers do not buy, so that the traded quantity is exactly q (if the consumers with value $v = D_0^{-1}(q)$ has a mass, then some of them buy and some of them do not buy, so that the total quantity sold is exactly q). That is,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \gamma^q(\mathrm{d}p|D) \right) s^{\mathrm{VR}}(\mathrm{d}D) = q. \tag{OA.2}$$

With this notation, I now introduce another auxiliary lemma as follows:

Lemma OA.2. *For any $q \in [0,1]$, let $\bar{R}(q)$ be the value of the maximization problem*

$$\begin{aligned} \sup_{s \in \mathcal{S}, \gamma \in \Gamma} & \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\gamma(\mathrm{d}p) \right) s(\mathrm{d}D) \\ \text{s.t.} & \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\gamma(\mathrm{d}p) \right) s(\mathrm{d}D) \leq q. \end{aligned} \tag{OA.3}$$

Then

$$\bar{R}(q) = \int_0^q D_0^{-1}(y)\,\mathrm{d}y.$$

Moreover, $(s^{\mathrm{VR}}, \gamma^q)$ is a solution of (OA.3).

Proof. Consider the dual problem of (OA.3). That is, for any $\nu \geq 0$, let

$$\begin{aligned} d(\nu) &:= \sup_{s \in \mathcal{S}, \gamma \in \Gamma} \left[\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\gamma(\mathrm{d}p|D) \right) s(\mathrm{d}D) + \nu \left(q - \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\gamma(\mathrm{d}p|D) \right) s(\mathrm{d}D) \right) \right] \\ &= \sup_{s \in \mathcal{S}, \gamma \in \Gamma} \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-\nu)D(p)\gamma(\mathrm{d}p|D) \right) s(\mathrm{d}D) + \nu q. \end{aligned}$$

Clearly, $d(\nu) \geq \bar{R}(q)$ for any $\nu \geq 0$. Thus, by weak duality, to solve (OA.3), it suffices to find ν^* and (s^*, γ^*) such that (s^*, γ^*) is feasible in the primal problem (OA.3), (s^*, γ^*) solves the dual problem

$$\sup_{s \in \mathcal{S}, \gamma \in \Gamma} \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-\nu^*)D(p)\gamma(\mathrm{d}p|D) \right) s(\mathrm{d}D) \tag{OA.4}$$

and that the complementary slackness condition

$$\nu^* \left[q - \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\gamma^*(\mathrm{d}p|D) \right) s^*(\mathrm{d}D) \right] = 0 \tag{OA.5}$$

holds. Since this would imply that

$$\bar{R}(q) \leq d^* = \inf_{\lambda \geq 0} d(\lambda) \leq d^*(\nu^*) = \bar{R}(q)$$

and hence (s^*, γ^*) must be a solution to (OA.3).

To this end, let

$$\nu^* := D_0^{-1}(q).$$

and consider the pair $(s^{\mathrm{VR}}, \gamma^q)$. Notice that by definition, $(s^{\mathrm{VR}}, \gamma^q)$ perfectly price-discriminates all the consumers with $v > \nu^*$ and does not sell to any consumers with $v < \nu^*$. Therefore,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p - \nu^*) D(p) \gamma(\mathrm{d}p|D) \right) s^{\mathrm{VR}}(\mathrm{d}D) = \int_V (v - \nu^*)^+ D_0(\mathrm{d}v)$$

Furthermore, notice that for any $s \in \mathcal{S}$ and any $\gamma \in \Gamma$

$$\begin{aligned}
&\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p - \nu^*) D(p) \gamma(\mathrm{d}p|D) \right) s(\mathrm{d}D) \\
\leq &\int_{\mathcal{D}} \max_{p \in \mathbb{R}_+} (p - \nu^*) D(p) s(\mathrm{d}D) \\
\leq &\int_V (v - \nu^*)^+ D_0(\mathrm{d}v).
\end{aligned}$$

Therefore, $(s^{\mathrm{VR}}, \gamma^q)$ solves the dual problem (OA.4). Meanwhile, by (OA.2),

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \gamma^q(\mathrm{d}p|D) \right) s^{\mathrm{VR}}(\mathrm{d}D) = q.$$

Thus, the complementary slackness condition (OA.5) holds and $(s^{\mathrm{VR}}, \gamma^q)$ is feasible in the primal problem (OA.3). Together, $(s^{\mathrm{VR}}, \gamma^q)$ is a solution to the primal problem (OA.3).

Finally, notice that by the definition of D_0^{-1} and $(s^{\mathrm{VR}}, \gamma^q)$,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} p D(p) \gamma^q(\mathrm{d}p|D) \right) s^{\mathrm{VR}}(\mathrm{d}D) = \int_0^q D_0^{-1}(y)\, \mathrm{d}y.$$

This completes the proof. ■

Notice that since both prices and market segmentations can be contracted by the price-controlling data broker, and since the producer's private information is one-dimensional, the price controlling data broker's problem can effectively be summarized by a one-dimensional screening problem where the data broker contracts on quantity (sold via perfect price discrimination), as stated in Lemma OA.3 below.

Lemma OA.3. *There exists an incentive feasible mechanism that maximizes the price-controlling data broker's revenue. Furthermore, the price-controlling data broker's revenue maximization problem is equivalent to the following:*

$$\begin{aligned}
&\sup_{\boldsymbol{q} \in \mathcal{Q}} \int_C \left(\int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \phi(c))\, \mathrm{d}q \right) G(\mathrm{d}c) - \bar{\pi} \\
&\text{s.t. } \bar{\pi} + \int_c^{\bar{c}} \boldsymbol{q}(z)\, \mathrm{d}z \geq \bar{\pi} + \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z))\, \mathrm{d}z,
\end{aligned} \tag{OA.6}$$

where $\mathcal{Q}$ is the collection of nonincreasing functions that map from C to $[0, 1]$.

Proof. By Lemma OA.1, the producer's expected profit under an incentive compatible mechanism $(\sigma, \tau, \boldsymbol{\gamma})$ of the price-controlling data broker can be written as

$$U(c) = U(\bar{c}) + \int_c^{\bar{c}} \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, z) \right) \sigma(\mathrm{d}D|z) \, \mathrm{d}z.$$

As such, an incentive compatible mechanism is individually rational if and only if

$$U(\bar{c}) + \int_c^{\bar{c}} \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, z) \right) \sigma(\mathrm{d}D|z) \, \mathrm{d}z \geq \bar{\pi} + \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z)) \, \mathrm{d}z.$$

Also, for any incentive compatible mechanism $(\sigma, \tau, \boldsymbol{\gamma})$, the price-controlling data broker's expected revenue can be written as

$$\mathbb{E}[\tau(c)] = \int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p - \phi(c)) D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, c) \right) \sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - U(\bar{c}).$$

Therefore, the price-controlling data broker's revenue maximization problem can be written as

$$\begin{aligned}
&\sup_{\sigma, \boldsymbol{\gamma}} \int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p - \phi(c)) D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, c) \right) \sigma(\mathrm{d}D|c) \right) G(\mathrm{d}c) - \bar{\pi} \\
&\text{s.t. } c \mapsto \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, c) \right) \sigma(\mathrm{d}D|c) \text{ is nonincreasing,} \\
&\qquad \int_c^{\bar{c}} \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, z) \right) \sigma(\mathrm{d}D|z) \, \mathrm{d}z \geq \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z)) \, \mathrm{d}z, \, \forall c \in C,
\end{aligned}$$

where the supremum is taken over all segmentation schemes $\sigma : C \to \mathcal{S}$ and all measurable functions $\boldsymbol{\gamma}$ that map from C to the collection of transition kernels from $\mathcal{D}$ to $\Delta(\mathbb{R}_+)$.

Now consider any incentive feasible mechanism $(\sigma, \tau, \boldsymbol{\gamma})$ for the price-controlling data broker, I will first show that there exists $\boldsymbol{q} : C \to [0, 1]$ such that the mechanism $(\sigma^{\mathrm{VR}}, \tau^{\boldsymbol{q}}, \boldsymbol{\gamma}^{\boldsymbol{q}})$ generates weakly higher revenue for the price-controlling data broker and is incentive feasible, where

$$\boldsymbol{\gamma}^{\boldsymbol{q}}(c) := \gamma^{\boldsymbol{q}(c)}, \, \forall \, c \in C$$

and $\tau^{\boldsymbol{q}}$ is the transfer determined by $(\sigma^{\mathrm{VR}}, \boldsymbol{\gamma}^{\boldsymbol{q}})$ according to Lemma OA.1, with the constant being chosen so that the producer with cost $\bar{c}$ obtains profit $\bar{\pi}$ when reporting truthfully. Next, I will show that maximizing revenue across the family of incentive feasible mechanisms $(\sigma^{\mathrm{VR}}, \tau^{\boldsymbol{q}}, \boldsymbol{\gamma}^{\boldsymbol{q}})$ is equivalent to solving (OA.6). Finally, the existence of the optimal mechanism can then be ensured by the existence of the solution of (OA.6), which will be proved at the end.

To this end, for any $c \in C$, let

$$\boldsymbol{q}(c) := \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p|D, c) \right) \sigma(\mathrm{d}D|c).$$

By Lemma OA.1, incentive compatibility of $(\sigma, \tau, \boldsymbol{\gamma})$ implies that $\boldsymbol{q} : C \to [0, 1]$ is nonincreasing and, by (OA.2), for any $c \in C$,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D, c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}D|c) = \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p) \gamma^{\boldsymbol{q}(c)}(\mathrm{d}p|D) \right) s^{\mathrm{VR}}(\mathrm{d}D) = \boldsymbol{q}(c).$$

Thus, by Lemma OA.2, $(\sigma^{\mathrm{VR}}(c), \boldsymbol{\gamma}^{\boldsymbol{q}}(c))$ solves the problem (OA.3) with the quantity constraint being $\boldsymbol{q}(c)$ and hence, since $(\sigma(c), \boldsymbol{\gamma}(c))$ is also feasible in this problem,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\boldsymbol{\gamma}(\mathrm{d}p|D,c) \right) \sigma(\mathrm{d}D|c) \leq \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}D|c) = \bar{R}(\boldsymbol{q}(c)). \tag{OA.7}$$

As a result,

$$\begin{aligned}
&\int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-\phi(c))D(p)\boldsymbol{\gamma}(\mathrm{d}p|D,c) \right) \sigma(\mathrm{d}x|c) \right) G(\mathrm{d}c) \\
=& \int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\boldsymbol{\gamma}(\mathrm{d}p|D,c) \right) \sigma(\mathrm{d}x|c) \right) G(\mathrm{d}c) - \int_C \phi(c)\boldsymbol{q}(c)G(\mathrm{d}c) \\
\leq& \int_C (\bar{R}(\boldsymbol{q}(c)) - \phi(c)\boldsymbol{q}(c))G(\mathrm{d}c) \\
=& \int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} pD(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}x|c) \right) G(\mathrm{d}c) - \int_C \phi(c)\boldsymbol{q}(c)G(\mathrm{d}c) \\
=& \int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-\phi(c))D(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}x|c) \right) G(\mathrm{d}c),
\end{aligned}$$

where the first and the third equalities follows from the definition of $\boldsymbol{q}(c)$ and from (OA.2), and the inequality and the second equality follows from (OA.7). Moreover, by (OA.2), since $\boldsymbol{q}$ is nonincreasing, the function

$$c \mapsto \int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}D|c)$$

is nonincreasing. Together with Lemma OA.1 and individual rationality of $(\sigma, \tau, \boldsymbol{\gamma})$, for any $c \in C$,

$$\begin{aligned}
\int_c^{\bar{c}} \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,z) \right) \sigma^{\mathrm{VR}}(\mathrm{d}D|z) \right) \mathrm{d}z &= \int_c^{c'} \boldsymbol{q}(z)\,\mathrm{d}z \\
&= \int_c^{\bar{c}} \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} D(p)\boldsymbol{\gamma}(\mathrm{d}p|D,z) \right) \sigma(\mathrm{d}D|z) \right) \mathrm{d}z \\
&\geq \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z))\,\mathrm{d}z,
\end{aligned}$$

these imply that $(\sigma^{\mathrm{VR}}, \tau^q, \boldsymbol{\gamma}^{\boldsymbol{q}})$ is incentive feasible.

Now notice that by (OA.2) and Lemma OA.2, for any $\boldsymbol{q} : C \to [0,1]$ and for any $c \in C$,

$$\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-\phi(c))D(p)\boldsymbol{\gamma}^{\boldsymbol{q}}(\mathrm{d}p|D,c) \right) \sigma^{\mathrm{VR}}(\mathrm{d}D|c) = \bar{R}(\boldsymbol{q}(c)) - \phi(c)\boldsymbol{q}(c) = \int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \phi(c))\,\mathrm{d}q.$$

Meanwhile, by (OA.2) and by Lemma OA.2, $(\sigma^{\mathrm{VR}}, \tau^{\boldsymbol{q}}, \boldsymbol{\gamma}^{\boldsymbol{q}})$ is incentive feasible if and only if $\boldsymbol{q}$ is nonincreasing and

$$\int_c^{\bar{c}} \boldsymbol{q}(z)\,\mathrm{d}z \geq \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z))\,\mathrm{d}z, \, \forall c \in C.$$

Therefore, maximizing revenue among all incentive feasible mechanism is equivalent to solving (OA.6).

Finally, notice that for the maximization problem (OA.6), endow the set of nonincreasing functions with the L^1 norm. Helly's selection theorem and the Lebesgue dominated convergence theorem then imply that this set is compact. Moreover, for any sequence $\{\boldsymbol{q}_n\} \subset \mathcal{Q}$ such that $\{\boldsymbol{q}_n\} \to \boldsymbol{q}$, consider any subsequence

$\{\boldsymbol{q}_{n_k}\}$ of $\{\boldsymbol{q}_n\}$, by the Riesz-Fischer theorem, there exists a further subsequence $\{\boldsymbol{q}_{n_{k,l}}\}$ such that $\{\boldsymbol{q}_{n_{k,l}}\} \to \boldsymbol{q}$ pointwise. By the dominated convergence theorem,

$$\lim_{l\to\infty} \int_C \left(\int_0^{\boldsymbol{q}_{n_{k,l}}(c)} (D_0^{-1}(q) - \phi(c))\,\mathrm{d}q \right) G(\mathrm{d}c) = \int_C \left(\int_0^{\boldsymbol{q}(c)} D_0^{-1}(q) - \phi(c)\,\mathrm{d}q \right) G(\mathrm{d}c)$$

and

$$\lim_{l\to\infty} \int_c^{\overline{c}} \boldsymbol{q}_{n_{k,l}}(z)\,\mathrm{d}z = \int_c^{\overline{c}} \boldsymbol{q}(z)\,\mathrm{d}z,\ \forall c \in C.$$

As a result, since every any subsequence of $\{\int_C \int_0^{\boldsymbol{q}_n(c)} (D_0^{-1}(q) - \phi(c))\,\mathrm{d}q\, G(\mathrm{d}c)\}$ ($\{\int_c^{\overline{c}} \boldsymbol{q}_n(z)\,\mathrm{d}z\}$, resp.) has a convergent subsequence that converges to $\int_C \int_0^{\boldsymbol{q}(c)} D_0^{-1}(q) - \phi(c)\,\mathrm{d}q\, G(\mathrm{d}c)\}$ ($\int_c^{\overline{c}} \boldsymbol{q}(z)\,\mathrm{d}z$, resp.), it the follows that

$$\lim_{n\to\infty} \int_C \left(\int_0^{\boldsymbol{q}_n(c)} D_0^{-1}(q) - \phi(c)\,\mathrm{d}q \right) G(\mathrm{d}c) = \int_C \left(\int_0^{\boldsymbol{q}(c)} D_0^{-1}(q) - \phi(c)\,\mathrm{d}q \right) G(\mathrm{d}c)$$

and

$$\lim_{n\to\infty} \int_c^{\overline{c}} \boldsymbol{q}_n(z)\,\mathrm{d}z = \int_c^{\overline{c}} \boldsymbol{q}(z)\,\mathrm{d}z,\ \forall c \in C.$$

Together, the feasible set of (OA.6) is compact and the objective is continuous (under the L^1 norm) and hence the solution must exist. This completes the proof. ∎

With Lemma OA.1 and Lemma OA.3, the price-controlling data broker's revenue maximization problem can be solved explicitly.

Proof of Proposition 2. Let R^* be the value of (OA.6) and consider the dual problem of (OA.6). By weak duality, it suffices to find a Borel measure μ^* on C and a feasible $\boldsymbol{q}^* \in \mathcal{Q}$ such that $\boldsymbol{q}^*$ is a solution of

$$\begin{aligned} \sup_{\boldsymbol{q}\in\mathcal{Q}} \Bigg[&\int_C \left(\int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \phi(c))\,\mathrm{d}q \right) G(\mathrm{d}c) - \overline{\pi} \\ &+ \int_C \left(\int_c^{\overline{c}} (\boldsymbol{q}(z) - D_0(\boldsymbol{p}_0(z)))\,\mathrm{d}z \right) \mu^*(\mathrm{d}c) \Bigg] \end{aligned} \tag{OA.8}$$

and that

$$\int_C \left(\int_c^{\overline{c}} (\boldsymbol{q}^*(z) - D_0(\boldsymbol{p}_0(z)))\,\mathrm{d}z \right) \mu^*(\mathrm{d}c) = 0. \tag{OA.9}$$

To this end, define $M^* : C \to [0,1]$ as the following:

$$M^*(c) := \lim_{z\downarrow c} g(z)(\phi(z) - \boldsymbol{p}_0(z))^+,\ \forall c \in C. \tag{OA.10}$$

By definition, M^* is right-continuous. Also, by Assumption 1, M^* is nondecreasing and hence M^* a CDF. Let μ^* be the Borel measure induced by M^*. Notice that $\mathrm{supp}(\mu^*) = [c^*, \overline{c}]$, where $c^* := \inf\{c \in C : \phi(c) > \boldsymbol{p}_0(c)\}$.

For any $\boldsymbol{q} \in \mathcal{Q}$, by interchanging the order of integrals and then rearranging, (OA.8) can be written as

$$\sup_{\boldsymbol{q}\in\mathcal{Q}} \left[\int_C \left(\int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \overline{\phi}(c))\,\mathrm{d}q \right) G(\mathrm{d}c) - \overline{\pi} - \int_C M^*(c) D_0(\boldsymbol{p}_0(c))\,\mathrm{d}c \right]. \tag{OA.11}$$

To solve (OA.11), notice that for any $\boldsymbol{q} \in \mathcal{Q}$,

$$\int_C \left(\int_0^{\boldsymbol{q}(c)} (D_0^{-1}(q) - \overline{\phi}(c))\, \mathrm{d}q \right) G(\mathrm{d}c)$$
$$\leq \int_C \left(\int_0^{D_0(\overline{\phi}(c))} (D_0^{-1}(q) - \overline{\phi}(c))\, \mathrm{d}q \right) G(\mathrm{d}c). \tag{OA.12}$$

Thus, as $\overline{\phi}$ is nondecreasing, $D_0 \circ \overline{\phi}$ is indeed a solution of (OA.11) and hence a solution of (OA.8). Moreover, since $\overline{\phi} \leq \boldsymbol{p}_0$, for all $c \in C$, $\int_c^{\overline{c}} D_0(\overline{\phi}(z))\, \mathrm{d}z \geq \int_c^{\overline{c}} D_0(\boldsymbol{p}_0(z))\, \mathrm{d}z$. Therefore, $D_0 \circ \overline{\phi} \in \mathcal{Q}$ is feasible in the primal problem (OA.6). Meanwhile, since $M^*(c) = 0$ for all $c \in [\underline{c}, c^*)$ and since $\overline{\phi}(c) = \boldsymbol{p}_0(c)$ for all $c \in (c^*, \overline{c}]$, the complementary slackness condition (OA.9) is also satisfied. Together, $D_0 \circ \overline{\phi}$ is indeed a solution of (OA.6). Finally, by definition of D_0^{-1}, it then follows that

$$R^* = \int_C \left(\int_0^{D_0(\overline{\phi}(c))} (D_0^{-1}(q) - \phi(c))\, \mathrm{d}q \right) G(\mathrm{d}c) - \bar{\pi}$$
$$= \int_C \left(\int_{\{v \geq \overline{\phi}(c)\}} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) - \bar{\pi}.$$

To see that any solution of the price-controlling data broker's problem must induce $\overline{\phi}(c)$-quasi-perfect price discrimination for G almost all $c \in C$, consider any optimal mechanism $(\sigma, \tau, \boldsymbol{\gamma})$ of the price-controlling data broker. By optimality, it must be that $\mathbb{E}[\tau(c)] = R^*$ and that the indirect utility of the producer with marginal cost $\overline{c}$ is $\bar{\pi}$. Thus, by Lemma OA.3, it must be that

$$\int_C \left(\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p - \phi(c)) D(p) \boldsymbol{\gamma}(\mathrm{d}p | D, c) \right) \sigma(\mathrm{d}D | c) \right) G(\mathrm{d}c)$$
$$= \int_C \left(\int_{\{v \geq \overline{\phi}(c)\}} (v - \phi(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c), \tag{OA.13}$$

which is equivalent to

$$\int_C \left(\int_{\mathcal{D} \times \mathbb{R}_+} (p - \overline{\phi}(c)) D(p) \boldsymbol{\gamma}(\mathrm{d}p | D, c) \sigma(\mathrm{d}D | c) \right) G(\mathrm{d}c)$$
$$+ \int_C (\overline{\phi}(c) - \phi(c)) \boldsymbol{q}_{\boldsymbol{\gamma}}^{\sigma}(c) G(\mathrm{d}c)$$
$$= \int_C \left(\int_{\{v \geq \overline{\phi}(c)\}} (v - \overline{\phi}(c)) D_0(\mathrm{d}v) \right) G(\mathrm{d}c) + \int_C (\overline{\phi}(c) - \phi(c)) D_0(\overline{\phi}(c)) G(\mathrm{d}c), \tag{OA.14}$$

where $\boldsymbol{q}_{\boldsymbol{\gamma}}^{\sigma}(c) := \int_{\mathcal{D} \times \mathbb{R}_+} D(p) \boldsymbol{\gamma}(\mathrm{d}p | D, c) \sigma(\mathrm{d}D | c)$ for all $c \in C$. Moreover, since for any $c \in C$,

$$\int_{\mathcal{D} \times \mathbb{R}_+} (p - \overline{\phi}(c)) D(p) \boldsymbol{\gamma}(\mathrm{d}p | D, c) \sigma(\mathrm{d}D | c)$$
$$\leq \int_{\mathcal{D}} \max_{p \in \mathbb{R}_+} [(p - \overline{\phi}(c)) D(p)] \sigma(\mathrm{d}D | c)$$
$$\leq \int_V (v - \overline{\phi}(c))^+ D_0(\mathrm{d}v), \tag{OA.15}$$

it must be that

$$\int_C (\overline{\phi}(c) - \phi(c))\boldsymbol{q}^{\sigma}_{\boldsymbol{\gamma}}(c)G(\mathrm{d}c) \geq \int_C (\overline{\phi}(c) - \phi(c))D_0(\overline{\phi}(c))G(\mathrm{d}c). \tag{OA.16}$$

Furthermore, since $\overline{\phi}(c) = \boldsymbol{p}_0(c) \leq \phi(c)$ for all $c \in (c^*, \overline{c}]$ and $\overline{\phi}(c) = \phi(c)$, for all $c \in [\underline{c}, c^*]$, by the definition of M^* given by (OA.10), together with integration by parts, (OA.16) is equivalent to

$$\int_C \left(\int_c^{\overline{c}} \left(\boldsymbol{q}^{\sigma}_{\boldsymbol{\gamma}}(z) - D_0(\boldsymbol{p}_0(z)\right) \mathrm{d}z \right) M^*(\mathrm{d}c) \leq 0 \tag{OA.17}$$

Lastly, since $(\sigma, \tau, \boldsymbol{\gamma})$ is individually rational, for any $c \in C$,

$$\int_c^{\overline{c}} \left(\boldsymbol{q}^{\sigma}_{\boldsymbol{\gamma}}(z) - D_0(\boldsymbol{p}_0(z))\right) \mathrm{d}z \geq 0.$$

Thus, as M^* is the CDF of a Borel measure, (OA.17) must hold with equality, which in turn implies that (OA.16) must hold with equality. Together with (OA.14), (OA.15) must hold with equality for G-almost all $c \in C$. Therefore, $(\sigma, \tau, \boldsymbol{\gamma})$ must induce $\overline{\phi}(c)$-quasi-perfect price discrimination for G-almost all $c \in C$, as desired. ∎

Proof of Theorem 3

By Lemma 5, it suffices to prove the outcome-equivalence between data brokership and price-controlling data brokership. By Proposition 2 and Theorem 1, both the data broker and the price-controlling data broker have optimal revenue R^*. Furthermore, for any optimal mechanism (σ, τ) of the data broker and any optimal mechanism $(\hat{\sigma}, \hat{\tau}, \hat{\boldsymbol{\gamma}})$ of the price-controlling data broker, both of them must induce $\overline{\phi}(c)$-quasi-perfect price discrimination for G-almost all $c \in C$. In particular, for G-almost all $c \in C$, all the consumers with $v \geq \overline{\phi}(c)$ buy the product by paying their values and all the consumers with $v < \overline{\phi}(c)$ do not buy the product. Thus, the consumer surplus and the allocation of the product induced by (σ, τ) and $(\hat{\sigma}, \hat{\tau}, \hat{\boldsymbol{\gamma}})$ are the same.

In addition, for any optimal mechanism (σ, τ) of the data broker, Theorem 1 implies that σ must be a $\overline{\phi}$-quasi-perfect scheme and hence by assertions 3 and 4 of Lemma 6, and by Lemma 1, for Lebesgue almost all $c \in C$,

$$\begin{aligned}\int_{\mathcal{D}} \pi_D(c)\sigma(\mathrm{d}D|c) - \tau(c) =& \bar{\pi} + \int_c^{\overline{c}} \left(\int_{\mathcal{D}} D(\boldsymbol{p}_D(z))\sigma(\mathrm{d}D|z) \right) \mathrm{d}z \\ =& \bar{\pi} + \int_c^{\overline{c}} D_0(\overline{\phi}(z))\, \mathrm{d}z.\end{aligned} \tag{OA.18}$$

Meanwhile, for the price-controlling data broker's optimal mechanism $(\hat{\sigma}, \hat{\tau}, \hat{\boldsymbol{\gamma}})$, since, by Proposition 2, it induces $\overline{\phi}(c)$-quasi-perfect price discrimination for almost all $c \in C$, it must be that $\boldsymbol{q}^{\hat{\sigma}}_{\hat{\boldsymbol{\gamma}}}(c) = D_0(\overline{\phi}(c))$. Together with Lemma OA.1, for any $c \in C$,

$$\begin{aligned}&\int_{\mathcal{D}} \left(\int_{\mathbb{R}_+} (p-c)D(p)\hat{\boldsymbol{\gamma}}(\mathrm{d}p|D,c) \right) \hat{\sigma}(\mathrm{d}D|c) - \hat{\tau}(c) \\ =& \bar{\pi} + \int_c^{\overline{c}} \boldsymbol{q}^{\hat{\sigma}}_{\hat{\boldsymbol{\gamma}}}(z)\, \mathrm{d}z \\ =& \bar{\pi} + \int_c^{\overline{c}} D_0(\overline{\phi}(z))\, \mathrm{d}z.\end{aligned} \tag{OA.19}$$

Thus, the producer's profit under both (σ, τ) and $(\hat{\sigma}, \hat{\tau}, \hat{\boldsymbol{\gamma}})$ are the same. This completes the proof. ∎

Proof of Lemma 6

For any nondecreasing function $\psi : C \to \mathbb{R}_+$ with $c \leq \psi(c)$ for all $c \in C$, since for any $c \in C$, $\sigma(c) \in \mathcal{S}$ is a $\psi(c)$-quasi-perfect segmentation for c, by definition,

$$\int_{\mathcal{D}} D(p)\sigma(\mathrm{d}D|c) = D_0(p), \tag{OA.20}$$

for all $p \in V$, which proves assertion 1. Furthermore, since ψ is nondecreasing and is thus continuous except at countably many points, $\sigma : C \to \Delta(\mathcal{D})$ is measurable, which establishes assertion 2. For assertion 3, notice that for any $c \in C$, since $\sigma(c) \in \mathcal{S}$ is a $\psi(c)$-quasi-perfect segmentation for c, for any $D \in \operatorname{supp}(\sigma(c))$ such that $D(\boldsymbol{p}_D(c)) > 0$,

$$D(\boldsymbol{p}_D(c)) = D(\max(\operatorname{supp}(D))) = D(\psi(c)),$$

while for any $D \in \operatorname{supp}(\sigma(c))$ such that $D(\boldsymbol{p}_D(c)) = 0$, it must be that $D(\psi(c)) = 0$ as well. Therefore,

$$\int_{\mathcal{D}} D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) = \int_{\mathcal{D}} D(\psi(c))\sigma(\mathrm{d}D|c) = D_0(\psi(c)),$$

where the last equality follows from (OA.20). This proves assertion 3. Finally, to prove assertion 4, consider any $c \in C$. First notice that if $D_0(c) = 0$, then assertion 4 clearly holds as both sides would be zero. Now suppose that $D_0(c) > 0$. The fact that $\sigma(c) \in \mathcal{S}$ is a $\psi(c)$-quasi-perfect segmentation for c ensures that $D_0(\psi(c)) > 0$. Then, for any $v \in [\psi(c), \overline{v}]$, let

$$H(v) := \sigma(\{D \in \mathcal{D} : \max(\operatorname{supp}(D)) \leq v\}|c).$$

Since $\sigma(c)$ is a probability measure, H is nondecreasing and right-continuous and hence induces a Borel measure μ_H on $[\psi(c), \overline{v}]$. Meanwhile, for any measurable sets $A, B \subseteq [\psi(c), \overline{v}]$, define

$$K(A|B) := \int_{\{D \in \mathcal{D}: \max(\operatorname{supp}(D)) \in A\}} m^D(B)\sigma(\mathrm{d}D|c).$$

Notice that for any measurable set $B \subseteq [\psi(c), \overline{v}]$, $K(\cdot|B)$ is a measure and is absolutely continuous with respect to μ_H and hence there exists a (essentially) unique Radon-Nikodym derivative $v \mapsto m^v(B)$ such that for any measurable $A \subseteq [\psi(c), \overline{v}]$,

$$K(A|B) = \int_{v \in A} m^v(B)H(\mathrm{d}v). \tag{OA.21}$$

In particular, by definition of K and by (OA.20), for any measurable set $B \subseteq [\psi(c), \overline{v}]$,

$$\int_{[\psi(c), \overline{v}]} m^v(B)H(\mathrm{d}v) = K([\psi(c), \overline{v}]|B) = \int_{\mathcal{D}} m^D(B)\sigma(\mathrm{d}D|c) = m^0(B). \tag{OA.22}$$

Moreover, since for any measurable set $A \subseteq [\psi(c), \overline{v}]$, $K(A|\cdot)$ is a measure on $[\psi(c), \overline{v}]$ and thus m^v is also a measure on $[\psi(c), \overline{v}]$ for μ_H-almost all $v \in [\psi(c), \overline{v}]$. Furthermore, since $\sigma(c) \in \mathcal{S}$ is a $\psi(c)$-quasi-perfect segmentation for c, for any measurable sets $A, B \subseteq [\psi(c), \overline{c}]$,

$$K(A|B) = m^0(A \cap B) = K(B|A)$$

and hence, for any measurable sets $A, B \subseteq [\psi(c), \overline{v}]$,

$$\int_A m^v(B)H(\mathrm{d}v) = \int_B m^v(A)H(\mathrm{d}v). \tag{OA.23}$$

As a result,

$$\begin{aligned}\int_{\mathcal{D}} \boldsymbol{p}_D(c)D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) &= \int_{\mathcal{D}} \boldsymbol{p}_D(c)D(\psi(c))\sigma(\mathrm{d}D|c)\\ &= \int_{\mathcal{D}} \max(\operatorname{supp}(D))m^D([\psi(c),\overline{v}])\sigma(\mathrm{d}D|c)\\ &= \int_{[\psi(c),\overline{v}]} vK(\mathrm{d}v|[\psi(c),\overline{v}])\\ &= \int_{[\psi(c),\overline{v}]} vm^v([\psi(c),\overline{v}])H(\mathrm{d}v)\\ &= \int_{v\in[\psi(c),\overline{v}]}\int_{v'\in[\psi(c),\overline{v}]} vm^v(\mathrm{d}v')H(\mathrm{d}v)\\ &= \int_{v\in[\psi(c),\overline{v}]} v\left(\int_{v'\in[\psi(c),\overline{v}]} m^{v'}(\mathrm{d}v)H(\mathrm{d}v')\right)\\ &= \int_{[\psi(c),\overline{v}]} vD_0(\mathrm{d}v),\end{aligned}$$

where the second equality follows from the fact that $\sigma(c)$ is a $\psi(c)$-quasi-perfect segmentation for c, the third equality follows from the definition of K, the fourth equality follows from (OA.21), the sixth equality follows from (OA.23), and the last equality follows from (OA.22). This completes the proof. ■

Proof of Lemma 7

Consider any optimal mechanism (σ,τ). As it is optimal and incentive compatible, by Lemma 1,

$$R^* = \mathbb{E}[\tau(c)] = \int_C \left(\int_{\mathcal{D}} (\widehat{\boldsymbol{p}}_D(c) - \phi(c))D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c) - \bar{\pi}. \tag{OA.24}$$

for any selection $\widehat{\boldsymbol{p}}$ of $\boldsymbol{P}$. Meanwhile, since (σ,τ) is individually rational, by Lemma 1, it must be that

$$\int_c^{\overline{c}} \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(z))\sigma(\mathrm{d}D|z)\right)\mathrm{d}z \geq \int_c^{\overline{c}} D_0(\boldsymbol{p}_0(z))\,\mathrm{d}z,\ \forall c\in C, \tag{OA.25}$$

for any selection $\widehat{\boldsymbol{p}}$ of $\boldsymbol{P}$.

Now suppose that (σ,τ) is not a $\overline{\phi}$-quasi-perfect mechanism or it does not induce $\overline{\phi}(c)$-quasi-perfect price discrimination for a positive G-measure of c, then there exists a selection $\widehat{\boldsymbol{p}}$ of $\boldsymbol{P}$, a positive G-measure of c and a positive $\sigma(c)$-measure of $D\in\mathcal{D}$ such that either $\widehat{\boldsymbol{p}}_D(c) < \boldsymbol{p}_D(c)$, or $D(c) > 0$ and either $\#\{v\in\operatorname{supp}(D) : v\geq\overline{\phi}(c)\}\neq 1$ or $\max(\operatorname{supp}(D))\notin \boldsymbol{P}_D(c)$, which imply that there is a positive G-measure of c and a positive $\sigma(c)$-measure of D such that

$$\begin{aligned}&\int_{\{v\geq\overline{\phi}(c)\}} (v-\overline{\phi}(c))D(\mathrm{d}v)\\ \geq&\int_{\{v\geq\widehat{\boldsymbol{p}}_D(c)\}} (v-\overline{\phi}(c))D(\mathrm{d}v)\\ =&(\widehat{\boldsymbol{p}}_D(c)-\overline{\phi}(c))D(\widehat{\boldsymbol{p}}_D(c)) + \int_{\{v\geq\widehat{\boldsymbol{p}}_D(c)\}} (v-\widehat{\boldsymbol{p}}_D(c))D(\mathrm{d}v)\\ \geq&(\widehat{\boldsymbol{p}}_D(c)-\overline{\phi}(c))D(\widehat{\boldsymbol{p}}_D(c)),\end{aligned}$$

with at least one inequality being strict. Therefore,

$$\begin{aligned}&\int_C \left(\int_{\mathcal{D}} (\widehat{\boldsymbol{p}}_D(c) - \overline{\phi}(c)) D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c)\\ &< \int_C \left(\int_V (v - \overline{\phi}(c))^+ D_0(\mathrm{d}v)\right) G(\mathrm{d}c). \end{aligned} \tag{OA.26}$$

Meanwhile, since by (OA.24),

$$\begin{aligned}&\int_C \left(\int_{\mathcal{D}} (\widehat{\boldsymbol{p}}_D(c) - \overline{\phi}(c)) D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c)\\ &\quad + \int_C (\overline{\phi}(c) - \phi(c)) \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c)\\ &= \int_{\mathcal{D}} \left(\int_{\mathcal{D}} (\widehat{\boldsymbol{p}}_D(c) - \phi(c)) D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c)\\ &= \int_C \left(\int_{\{v \geq \overline{\phi}(c)\}} (v - \phi(c)) D_0(\mathrm{d}v)\right) G(\mathrm{d}c)\\ &= \int_C \left(\int_V (v - \overline{\phi}(c))^+ D_0(\mathrm{d}v)\right) G(\mathrm{d}c) + \int_C (\overline{\phi}(c) - \phi(c)) D_0(\overline{\phi}(c)) G(\mathrm{d}c), \end{aligned}$$

(OA.26) implies that

$$\int_C (\overline{\phi}(c) - \phi(c)) \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c) > \int_C (\overline{\phi}(c) - \phi(c)) D_0(\overline{\phi}(c)) G(\mathrm{d}c).$$

Furthermore, since $\overline{\phi}(c) = \phi(c)$ for all $c \in [\underline{c}, c^*]$ and $\overline{\phi}(c) = \boldsymbol{p}_0(c)$ for all $c \in (c^*, \overline{c}]$, it then follows that

$$\int_{c^*}^{\overline{c}} (\phi(c) - \boldsymbol{p}_0(c)) \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(c))\sigma(\mathrm{d}D|c)\right) G(\mathrm{d}c) < \int_{c^*}^{\overline{c}} (\phi(c) - \boldsymbol{p}_0(c)) D_0(\boldsymbol{p}_0(c)) G(\mathrm{d}c),$$

Using integration by parts, this is equivalent to

$$\int_{c^*}^{\overline{c}} \left(\int_c^{\overline{c}} \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(z))\sigma(\mathrm{d}D|z)\right) \mathrm{d}z\right) M^*(\mathrm{d}c) < \int_{c^*}^{\overline{c}} \left(\int_c^{\overline{c}} D_0(\boldsymbol{p}_0(z))\, \mathrm{d}z\right) M^*(\mathrm{d}c),$$

where M^* is defined in (OA.10). However, by (OA.25) and by the fact that M^* is a CDF of a Borel measure, which is due to Assumption 1,

$$\int_{c^*}^{\overline{c}} \left(\int_c^{\overline{c}} \left(\int_{\mathcal{D}} D(\widehat{\boldsymbol{p}}_D(z))\sigma(\mathrm{d}D|z)\right) \mathrm{d}z\right) M^*(\mathrm{d}c) \geq \int_{c^*}^{\overline{c}} \left(\int_c^{\overline{c}} D_0(\boldsymbol{p}_0(z))\, \mathrm{d}z\right) M^*(\mathrm{d}c),$$

a contradiction. Therefore, σ must be a $\overline{\phi}$-quasi-perfect scheme and must induce $\overline{\phi}(c)$-quasi-perfect price discrimination for G-almost all $c \in C$. Together with Lemma 1, and the fact that $U(\overline{c}) = \bar{\pi}$ under any optimal mechanism, (σ, τ) must be a $\overline{\phi}$-quasi-perfect mechanism. This completes the proof. ■

OA.2 Extension: General D_0

In the paper, I assume that D_0 is a continuously differentiable decreasing function on V and induces decreasing marginal revenue. It is noteworthy that none of these assumptions are necessary. It can be verified that, given Lemma 3, all other proofs remain valid even without these assumptions, as long as D_0 is a nonincreasing upper-semicontinuous function and D_0^{-1} is defined as

$$D_0^{-1}(q) := \sup\{p \in V : D_0(p) \geq q\},$$

for all $q \in [0,1]$. Therefore, the results can be immediately extended to the case where D_0 is any nonincreasing and upper-semicontinuous function (jointly satisfying Assumption 1 with G), provided that Lemma 3 can still be established. The proof is provided below.

Proof of Lemma 3 (General D_0)

The proof of Lemma 3 relies on the following technical lemma

Lemma OA.4. *Consider any function $\psi : C \to \mathbb{R}_+$ with $c \leq \psi(c)$ for all $c \in C$. Given any $\{D_n\} \subset \mathcal{D}$ and $\{\sigma_n\}$ such that $\sigma_n : C \to \mathcal{S}_{D_n}$ is measurable for all $n \in \mathbb{N}$. Suppose that $\{\sigma_n\} \to \sigma$ pointwise and $\{D_n\} \to D_0$ for some $\sigma : C \to \Delta(\mathcal{D})$ and $D_0 \in \mathcal{D}$. Then σ is measurable and $\sigma(c) \in \mathcal{S}_{D_0}$ for all $c \in C$. Moreover, suppose further that σ_n is a ψ-quasi-perfect scheme for all $n \in \mathbb{N}$. Then σ is a ψ-quasi-perfect scheme.*

Proof. First notice that since for all $n \in \mathbb{N}$ and for all $c \in C$, $\sigma_n(c) \in \mathcal{S}_{D_n}$ and since $\{\sigma_n\} \to \sigma$ pointwise, σ is measurable. Moreover, since $\{D_n\} \to D_0$ and $\{\sigma_n\} \to \sigma$, for any bounded continuous function $f : V \to \mathbb{R}$ and for any $c \in C$,

$$\begin{aligned}
\int_V f(v) \left(\int_{\mathcal{D}} D(\mathrm{d}v)\sigma(\mathrm{d}D|c) \right) &= \int_{\mathcal{D}} \left(\int_V f(v) D(\mathrm{d}v) \right) \sigma(\mathrm{d}D|c) \\
&= \lim_{n\to\infty} \int_{\mathcal{D}} \left(\int_V f(v) D(\mathrm{d}v) \right) \sigma_n(\mathrm{d}D|c) \\
&= \lim_{n\to\infty} \int_V f(v) \left(\int_{\mathcal{D}} D(\mathrm{d}v)\sigma_n(\mathrm{d}D|c) \right) \\
&= \lim_{n\to\infty} \int_V f(v) D_n(\mathrm{d}v) \\
&= \int_V f(v) D_0(\mathrm{d}v),
\end{aligned}$$

where the first and the third equality follow from interchanging the order of integrals, the second equality follows from the fact that the integrand in the parentheses is a bounded continuous function of D and from weak-*convergence of $\{\sigma_n(c)\}$, the fourth equality is due to the fact that $\sigma_n(c) \in \mathcal{S}_{D_n}$, and the last equality follows from the weak-* convergence of $\{D_n\}$. Thus, by the Riesz representation theorem,

$$\int_{\mathcal{D}} D(p)\sigma(\mathrm{d}D|c) = D_0(p),\ \forall p \in V,\ c \in C.$$

This proves that $\sigma(c) \in \mathcal{S}_{D_0}$ for all $c \in C$.

Now suppose that σ_n is a ψ-quasi-perfect scheme for all $n \in \mathbb{N}$ and suppose that, by way of contradiction, $\sigma : C \to \mathcal{S}_{D_0}$ is not a ψ-quasi-perfect scheme. Then there exists a positive G-measure of c and a positive $\sigma(c)$-measure of $D \in \mathcal{D}$ such that $D(c) > 0$ and either $\#\{v \in \mathrm{supp}(D) : v \geq \psi(c)\} \neq 1$ or $\max(\mathrm{supp}(D)) \notin \boldsymbol{P}_D(c)$

(i.e., $D(\boldsymbol{p}_D(c)) > 0$). As such, there is a positive G-measure of c and a positive $\sigma(c)$-measure of D such that

$$\begin{aligned}\int_{\{v\geq\psi(c)\}}(v-\psi(c))D(\mathrm{d}v) \geq &\int_{\{v\geq \boldsymbol{p}_D(c)\}}(v-\psi(c))D(\mathrm{d}v)\\ =&(\boldsymbol{p}_D(c)-\psi(c))D(\boldsymbol{p}_D(c))+\int_{\{v\geq \boldsymbol{p}_D(c)\}}(v-\boldsymbol{p}_D(c))D(\mathrm{d}v)\\ \geq&(\boldsymbol{p}_D(c)-\psi(c))D(\boldsymbol{p}_D(c)),\end{aligned}$$

with at least one inequality being strict. Thus, there exists a positive G-measure of $c \in C$ such that

$$\int_{\mathcal{D}}(\boldsymbol{p}_D(c)-\psi(c))D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c) < \int_V (v-\psi(c))^+ D_0(\mathrm{d}v).$$

However, by Theorem 12 of Hart and Reny (2019) and Corollary 2 of Yang (2020*a*), for Lebesgue almost all $c \in C$,

$$\begin{aligned}&\int_{\mathcal{D}}(\boldsymbol{p}_D(c)-\psi(c))D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c)\\ =&\int_{\mathcal{D}}\pi_D(c)\sigma(\mathrm{d}D|c)-(\psi(c)-c)\int_{\mathcal{D}}D(\boldsymbol{p}_D(c))\sigma(\mathrm{d}D|c)\\ \geq&\lim_{n\to\infty}\int_{\mathcal{D}}\pi_D(c)\sigma_n(\mathrm{d}D|c)-\liminf_{n\to\infty}(\psi(c)-c)\int_{\mathcal{D}}D(\boldsymbol{p}_D(c))\sigma_n(\mathrm{d}D|c)\\ =&\limsup_{n\to\infty}\left[\int_{\mathcal{D}}\pi_D(c)\sigma_n(\mathrm{d}D|c)-(\psi(c)-c)\int_{\mathcal{D}}D(\boldsymbol{p}_D(c))\sigma_n(\mathrm{d}D|c)\right]\\ =&\limsup_{n\to\infty}\int_V(v-\psi(c))^+D_n(\mathrm{d}v)\\ =&\lim_{n\to\infty}\int_V(v-\psi(c))^+D_n(\mathrm{d}v)\\ =&\int_V(v-\psi(c))^+D_0(\mathrm{d}v),\end{aligned} \tag{OA.27}$$

a contradiction. Here, the first inequality follows from the fact that $\{\sigma_n(c)\} \to \sigma(c)$, Theorem 12 of Hart and Reny (2019) and Corollary 2 of Yang (2020*a*); the second equality follows from the properties of the lim inf and lim sup operators;[2] the third equality follows from the fact that $\sigma_n(c) \in \mathcal{S}_{D_n}$ and is a $\psi(c)$-quasi-perfect segmentation for c; and the last two equalities follow from the fact that the function $(v-\psi(c))^+$ is bounded and continuous in v and that $\{D_n\} \to D_0$. Therefore, σ must be a ψ-quasi-perfect scheme. ■

Proof of Lemma 3 (general D_0). I first prove the lemma for D_0 being a step function with finitely many steps. Consider any step function $D \in \mathcal{D}$ with $|\mathrm{supp}(D)| < \infty$ and any nondecreasing function $\psi : C \to \mathbb{R}_+$ such that $c \leq \psi(c) \leq \boldsymbol{p}_D(c)$ for all $c \in C$ and fix any $c \in C$, let

$$V^+ := \{v \in \mathrm{supp}(D) : v \geq \psi(c)\}$$

[2] More precisely, this follows from the following properties: For any real sequences $\{a_n\}$, $\{b_n\}$,

$$-\liminf_{n\to\infty} b_n = \limsup_{n\to\infty}(-b_n).$$

Moreover, if $\{a_n\}$ is convergent, then

$$\limsup_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

and let

$$\hat{c} := \inf\{z \in C : \boldsymbol{p}_D(z) \geq \psi(c)\}.$$

Since $\boldsymbol{p}_D$ is nondecreasing, it then follows $\boldsymbol{p}_D(z) \geq \psi(c)$ for all $z \in [\hat{c}, \overline{c}]$ and $\boldsymbol{p}_D(z) \leq \psi(c)$ for all $z \in [\underline{c}, \hat{c})$. Moreover, since $\psi(c) \leq \boldsymbol{p}_D(c)$, $\hat{c} \leq c$. Furthermore, by definition of $\hat{c}$, it must be either $\hat{c} = \underline{c}$ or $\hat{c} > \underline{c}$ and $\underline{\boldsymbol{p}}_D(\hat{c}) < \psi(c) \leq \boldsymbol{p}_D(\hat{c})$, where $\underline{\boldsymbol{p}}_D(\hat{c}) := \min \boldsymbol{P}_D(\hat{c})$, since otherwise, if $\hat{c} > \underline{c}$ and $\underline{\boldsymbol{p}}_D(\hat{c}) \geq \psi(c)$, then for $\varepsilon > 0$ small enough, as $|\mathrm{supp}(D)| < \infty$, $\boldsymbol{p}_D(\hat{c} - \varepsilon) = \underline{\boldsymbol{p}}_D(\hat{c}) \geq \psi(c)$, contradicting the definition of $\hat{c}$. Consider first the case where $\hat{c} > \underline{c}$. In this case, for each $v \in V^+$, define $\hat{m}^v$ recursively as the following

$$\hat{m}^v(v') := \begin{cases} 0, & \text{if } v' \geq \psi(c) \text{ and } v' \neq v \\ m^D(v'), & \text{if } v' = v \\ \beta^*(v|v')m^D(v'), & \text{if } \underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c) \\ \alpha^*(v)m^D(v'), & \text{if } v' < \underline{\boldsymbol{p}}_D(\hat{c}) \end{cases}, \forall v' \in \mathrm{supp}(D), \forall v \in V^+,$$

where for all $v \in V^+$ and all $v' \in \mathrm{supp}(D)$ s.t. $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$,

$$\beta^*(v|v') := \frac{(v - \hat{c})m^D(v) - (v' - \hat{c})\sum_{\hat{v} > v'} \hat{m}^v(\hat{v})}{\sum_{v \geq \psi(c)} \left[(v - \hat{c})m^D(v) - (v' - \hat{c})\sum_{\hat{v} > v'} \hat{m}^v(\hat{v})\right]},$$

and for all $v \in V^+$,

$$\alpha^*(v) := \frac{\sum_{\hat{v} \geq \underline{\boldsymbol{p}}_D(\hat{c})} \hat{m}^v(\hat{v})}{\sum_{\hat{v} \geq \underline{\boldsymbol{p}}_D(\hat{c})} m^D(\hat{v})}.$$

By construction,

$$\sum_{v \in V^+} \alpha^*(v) = \sum_{v \in V^+} \beta^*(v|v') = 1 \tag{OA.28}$$

for all $v' \in \mathrm{supp}(D)$ with $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$. As such,

$$\sum_{v \in V^+} \hat{m}^v(v') = m^D(v'), \forall v' \in \mathrm{supp}(D). \tag{OA.29}$$

Notice that since $\hat{c} \leq \underline{\boldsymbol{p}}_D(\hat{c}) < \psi(c) \leq \boldsymbol{p}_D(\hat{c})$, it must be that

$$\sum_{v \geq \psi(c)} (v - \hat{c})m^D(v) \geq \sum_{v \geq \boldsymbol{p}_D(\hat{c})} (v - \hat{c})m^D(v) \geq (\boldsymbol{p}_D(\hat{c}) - \hat{c})D(\boldsymbol{p}_D(\hat{c})) = (\underline{\boldsymbol{p}}_D(\hat{c}) - \hat{c})D(\underline{\boldsymbol{p}}_D(\hat{c})). \tag{OA.30}$$

Now consider any $v' \in \mathrm{supp}(D)$ such that $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$. Notice first that

$$\begin{aligned} &\sum_{v \geq \psi(c)} \left[(v - \hat{c})m^D(v) - (v' - \hat{c})\sum_{\hat{v} > v'} \hat{m}^v(\hat{v})\right] \\ =&\sum_{v \geq \psi(c)} (v - \hat{c})m^D(v) - (v' - \hat{c})\sum_{\hat{v} > v'} m^D(\hat{v}) \\ \geq&(\underline{\boldsymbol{p}}_D(\hat{c}) - \hat{c})D(\underline{\boldsymbol{p}}_D(\hat{c})) - (v' - \hat{c})\sum_{\hat{v} > v'} m^D(\hat{v}) \\ \geq&(v' - \hat{c})\sum_{\hat{v} \geq v'} m^D(\hat{v}) - (v' - \hat{c})\sum_{\hat{v} > v'} m^D(\hat{v}) \\ =&(v' - \hat{c})m^D(v') \\ \geq&0, \end{aligned}$$

where the first equality follows from (OA.29), the first inequality follows from (OA.30), the second inequality follows from the fact that $\underline{\boldsymbol{p}}_D(\hat{c}) \in \boldsymbol{P}_D(\hat{c})$, and the last inequality follows from $\underline{\boldsymbol{p}}_D(\hat{c}) \geq \hat{c}$. As such, for any $v' \in \text{supp}(D)$ with $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$ and for any $v \in V^+$, if

$$(v - \hat{c})m^D(v) - (v' - \hat{c}) \sum_{\hat{v} > v'} \hat{m}^v(\hat{v}) \geq 0,$$

then $\beta^*(v|v') \geq 0$ and

$$\begin{aligned}
&\hat{m}^v(v') \leq \frac{(v - \hat{c})m^D(v) - (v' - \hat{c}) \sum_{\hat{v} > v'} \hat{m}^v(\hat{v})}{(v' - \hat{c})m^D(v')} m^D(v') \\
\iff& (v' - \hat{c})\hat{m}^v(v') + (v' - \hat{c}) \sum_{\hat{v} > v'} \hat{m}^v(\hat{v}) \leq (v - \hat{c})m^D(v) \\
\iff& (v' - \hat{c}) \sum_{\hat{v} \geq v'} \hat{m}^v(\hat{v}) \leq (v - \hat{c})\hat{m}^v(v),
\end{aligned}$$

which in turn implies that

$$(v - \hat{c})m^D(v) - (v'' - \hat{c}) \sum_{\hat{v} > v''} \hat{m}^v(\hat{v}) > (v - \hat{c})m^D(v) - (v' - \hat{c}) \sum_{\hat{v} \geq v'} \hat{m}^v(\hat{v}) \geq 0,$$

where $v'' \in \text{supp}(D)$ is the largest element of $\{\hat{v} \in \text{supp}(D) : \underline{\boldsymbol{p}}_D(\hat{c}) \leq \hat{v} < v'\}$. Moreover, if $v' = \max\{\hat{v} \in \text{supp}(D) : \underline{\boldsymbol{p}}_D(\hat{c}) \leq \hat{v} < \psi(c)\}$, then clearly, for all $v \in V^+$,

$$(v - \hat{c})m^D(v) - \sum_{\hat{v} > v'} \hat{m}^v(v') = (v - v')m^D(v) \geq 0.$$

Therefore, by induction, for any $v' \in \text{supp}(D)$ such that $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$, it must be that $\beta^*(v|v') \geq 0$ for all $v \in V^+$ and that

$$(v' - \hat{c}) \sum_{\hat{v} \geq v'} \hat{m}^v(\hat{v}) \leq (v - \hat{c})\hat{m}^v(v). \tag{OA.31}$$

Together with (OA.28), this also ensures that

$$\alpha^* \in \Delta(V^+) \tag{OA.32}$$

and

$$\beta^*(v') \in \Delta(V^+), \tag{OA.33}$$

for all $v' \in \text{supp}(D)$ such that $\underline{\boldsymbol{p}}_D(\hat{c}) \leq v' < \psi(c)$.

Meanwhile, for any $v' \in \text{supp}(D)$ with $v' \leq \underline{\boldsymbol{p}}_D(\hat{c})$ and any $v \in V^+$, notice that by the definition of α^*,

$$\sum_{\hat{v} \geq v'} \hat{m}^v(\hat{v}) = \alpha^*(v) \sum_{v' \leq \hat{v} < \underline{\boldsymbol{p}}_D(\hat{c})} m^D(\hat{v}) + \sum_{\hat{v} \geq \underline{\boldsymbol{p}}_D(\hat{c})} \hat{m}^v(\hat{v}) = \alpha^*(v) \sum_{\hat{v} \geq v'} m^D(\hat{v}). \tag{OA.34}$$

Thus, for any $v' \in \text{supp}(D)$ with $v' < \underline{\boldsymbol{p}}_D(\hat{c})$ and any $v \in V^+$,

$$\begin{aligned}
(v' - \hat{c}) \sum_{\hat{v} \geq v'} \hat{m}^v(\hat{v}) =& \alpha^*(v)(v' - \hat{c})D(v') \\
\leq& \alpha^*(v)(\underline{\boldsymbol{p}}_D(\hat{c}) - \hat{c})D(\underline{\boldsymbol{p}}_D(\hat{c})) \\
=& (\underline{\boldsymbol{p}}_D(\hat{c}) - \hat{c}) \sum_{\hat{v} \geq \underline{\boldsymbol{p}}_D(\hat{c})} \hat{m}^v(\hat{v}) \qquad \text{(OA.35)} \\
\leq& (v - \hat{c})\hat{m}^v(v),
\end{aligned}$$

where both equalities follow from (OA.34), the first inequality follows from the fact that $\underline{\boldsymbol{p}}_D(\hat{c}) \in \boldsymbol{P}_D(\hat{c})$, and the last inequality follows from (OA.31) by taking $v' = \underline{\boldsymbol{p}}_D(\hat{c})$.

Moreover, by (OA.34), for any $z \in [\underline{c}, \hat{c})$, and any $v \in V^+$, since $\boldsymbol{p}_D(z) \leq \underline{\boldsymbol{p}}_D(\hat{c})$, it must be that for all $v' \leq \boldsymbol{p}_D(z)$,

$$\begin{aligned}(v'-z)\sum_{\hat{v}\geq v'} \hat{m}^v(\hat{v}) =& \alpha^*(v)(v'-z)D(v') \\ &\leq \alpha^*(v)(\underline{\boldsymbol{p}}_D(z)-z)D(\underline{\boldsymbol{p}}_D(z)) \\ &= (\underline{\boldsymbol{p}}_D(z)-z)\sum_{\hat{v}\geq \underline{\boldsymbol{p}}_D(z)} \hat{m}^v(\hat{v}).\end{aligned} \tag{OA.36}$$

Finally, if $\hat{c} = \underline{c}$, then define $\{\hat{m}^v\}_{v\in V^+}$ as

$$\hat{m}^v(v') := \begin{cases} m^D(v'), & \text{if } v' = v \\ 0, & \text{if } v' \geq \psi(c) \text{ and } v' \neq v \\ \alpha^*(v)m^D(v'), & \text{if } v' < \psi(c) \end{cases}, \forall v' \in V,\, v \in V^+,\, v \geq \boldsymbol{p}_D(\underline{c})$$

and

$$\hat{m}^v(v') := \begin{cases} m^D(v'), & \text{if } v' = v \\ 0, & \text{if } v' \neq v \end{cases}, \forall v' \in V,\, v \in V^+,\, \psi(c) \leq v < \boldsymbol{p}_D(\underline{c})$$

where

$$\alpha^*(v) := \frac{m^D(v)}{\sum_{v'\geq \boldsymbol{p}_D(\underline{c})} m^D(v')}.$$

Again,

$$\sum_{v\geq \boldsymbol{p}_D(\underline{c})} \alpha^*(v) = 1 \tag{OA.37}$$

and hence

$$\sum_{v\in V^+} \hat{m}^v(v') = m^D(v'),\, \forall v' \in V. \tag{OA.38}$$

Then, for any $v \geq \boldsymbol{p}_D(\underline{c})$ and any $v' \in \text{supp}(D)$ with $v' < \psi(c)$,

$$(v'-\underline{c})\sum_{\hat{v}\geq v'} \hat{m}^v(v') = \alpha^*(v)(v'-\underline{c})D(v') \leq \alpha^*(v)(\boldsymbol{p}_D(\underline{c})-\underline{c})D(\boldsymbol{p}_D(\underline{c})) \leq (v-\underline{c})\hat{m}^v(v). \tag{OA.39}$$

Together, in both of the cases above, from the constructed $\{\hat{m}^v\}_{v\in V^+}$, for each $v \in V^+$, let

$$m^v(v') := \frac{\hat{m}^v(v')}{\sum_{\hat{v}\in V} \hat{m}^v(\hat{v})},\, \forall v' \in \text{supp}(D)$$

and let $D_v(p) := m^v([p, \overline{v}])$ for all $p \geq 0$, by (OA.32), (OA.33) and (OA.37), in each case, $D_v \in \mathcal{D}$ for all $v \in V^+$. Now define $\sigma(c) \in \Delta(\mathcal{D})$ by

$$\sigma(D_v|c) := \sum_{v'\in V} \hat{m}^v(v'),\, \forall v \in V^+.$$

By (OA.29) and (OA.38), in each case, $\sigma(c) \in \mathcal{S}_D$. Furthermore, since m^v is proportional to $\hat{m}^v$ for all $v \in V^+$, (OA.31), (OA.35) and (OA.39) ensure that in each case, $\sigma(c)$ is a $\psi(c)$-quasi-perfect segmentation for $\hat{c}$. Meanwhile, since $\hat{c} \leq c \leq \psi(c)$, $\sigma(c)$ is also a $\psi(c)$-quasi-perfect segmentation for c. Finally, since m^v is proportional to $\hat{m}^v$, (OA.36) implies that for any $z \in [\underline{c}, \hat{c})$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_D(z),\ \forall D' \in \mathrm{supp}(\sigma(c)).$$

Meanwhile, by the conclusion that $\sigma(c)$ is a $\psi(c)$-quasi-perfect segmentation for $\hat{c} \leq c$, for any $z \in [\hat{c}, c]$, since $c \leq \psi(c)$ and since $\boldsymbol{p}_D$ is nondecreasing for any $D' \in \mathcal{D}$,

$$\boldsymbol{p}_{D'}(z) \geq \boldsymbol{p}_{D'}(\hat{c}) \geq \psi(c),\ \forall D' \in \mathrm{supp}(\sigma(c)).$$

Together with the fact that ψ is nondecreasing and that $\psi \leq \boldsymbol{p}_D$, it then follows that for any $z \in [\underline{c}, c]$ and for any $D \in \mathrm{supp}(\sigma(c))$, $\psi(z) \leq \boldsymbol{p}_D(z)$. Since $c \in C$ is arbitrary, this ensures that there exists a ψ-quasi-perfect scheme $\sigma : C \to \mathcal{S}_D$ that satisfies (12).

Now consider any $D_0 \in \mathcal{D}$ and any nondecreasing $\psi : C \to \mathbb{R}_+$ with $c \leq \psi(c) \leq \boldsymbol{p}_0(c)$ for all $c \in C$. I first construct a sequence of step functions $\{D_n\} \subseteq \mathcal{D}$ such that $\{D_n\} \to D_0$ and that $c \leq \psi(c) \leq \boldsymbol{p}_{D_n}(c)$ for all $c \in C$ and for all $n \in \mathbb{N}$. To this end, for each $n \in \mathbb{N}$, first partition V by $\underline{v} = v_0 < v_1 < \ldots < v_n = \overline{v}$ and let $V_k := [v_{k-1}, v_k]$. Then define D_n by $D_n(p) := D_0(v_k)$, for all $p \in V_k$, for all $k \in \{1, \ldots, n\}$ (i.e., by moving all the masses on interval V_k to the top v_k). By construction, it must be that $\boldsymbol{p}_{D_n}(c) \geq \boldsymbol{p}_0(c)$ for all $c \in C$ and for all $n \in \mathbb{N}$ and hence $c \leq \psi(c) \leq \boldsymbol{p}_{D_n}(c)$ for all $c \in C$ and for all $n \in \mathbb{N}$. Also, by construction, $\{D_n\} \to D_0$, as desired.

As such, for each $n \in \mathbb{N}$, there exists a ψ-quasi-perfect scheme σ_n such that for all $c \in C$,

$$\psi(z) \leq \boldsymbol{p}_D(z)$$

for all $D \in \mathrm{supp}(\sigma_n(c))$ and for all $z \in [\underline{c}, c]$. Furthermore, according to Helly's selection theorem, by possibly taking a subsequence,[3] $\{\sigma_n\} \to \sigma$ for some $\sigma : C \to \Delta(\mathcal{D})$. By Lemma OA.4, $\sigma(c) \in \mathcal{S}$ for all $c \in C$ and σ is a ψ-quasi-perfect scheme.

It then remains to show that σ satisfies (12). To this end, fix any $c \in C$ and consider any $D \in \mathrm{supp}(\sigma(c))$, by definition, for any $\delta > 0$, $\sigma(\mathbb{B}_\delta(D)|c) > 0$.[4] Furthermore, since $\sigma(c)$ has at most countably many atoms, there exists a sequence $\{\delta_k\} \subset (0, 1]$ such that $\{\delta_k\} \to 0$, $\sigma(\mathbb{B}_{\delta_k}(D)|c) > 0$ and $\sigma(\partial\mathbb{B}_{\delta_k}(D)|c) = 0$ for all $k \in \mathbb{N}$. As a result, since $\{\sigma_n(c)\} \to \sigma(c)$ under the weak-* topology, $\lim_{n\to\infty} \sigma_n(\mathbb{B}_{\delta_k}(D)|c) = \sigma(\mathbb{B}_{\delta_k}(D)|c) > 0$ for all $k \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $\sigma_{n_k}(\mathbb{B}_{\delta_k}(D)|c) > 0$. Moreover, since $\sigma_n(c)$ has finite support as D_n is a step function and $\sigma_n(c) \in \mathcal{S}_{D_n}$, there must be some $D_{n_k} \in \mathbb{B}_{\delta_k}(D)$ such that $D_{n_k} \in \mathrm{supp}(\sigma_{n_k}(c))$. Notice that for the subsequence $\{n_k\}$, $\{D_{n_k}\} \to D$ and $D_{n_k} \in \mathrm{supp}(\sigma_{n_k}(c))$ for all $k \in \mathbb{N}$. As a result, since $D \mapsto \boldsymbol{p}_D(c)$ is upper-semicontinuous (see Proposition 6 of Yang (2020*a*)) and since σ_{n_k} satisfies (12) for all $k \in \mathbb{N}$, for Lebesgue almost all $z \in [\underline{c}, c]$,

$$\psi(z) \leq \limsup_{k\to\infty} \boldsymbol{p}_{D_{n_k}}(z) \leq \boldsymbol{p}_D(z).$$

Since $c \in C$ and $D \in \mathrm{supp}(\sigma(c))$ are arbitrary, this completes the proof. ■

[3] See, for instance, Porter (2005) for a generalized version of Helly's selection theorem. To apply this theorem, notice that the family of functions $\{\sigma_n\}$ is of bounded p-variation due to the quasi-perfect structure. Furthermore, for any $c \in C$, the set $\mathrm{cl}(\{\sigma_n(c)\})$ is closed in a compact metric space $\Delta(\mathcal{D})$ and hence is itself compact. As such, there exists a pointwise convergent subsequence of $\{\sigma_n\}$.

[4] $\mathbb{B}_\delta(D)$ is the δ-ball around D under the Lévy-Prokhorov metric on $\mathcal{D}$.

OA.3 Extension: Restricted Market Segmentations

Thus far, it has been assumed that the data broker is able to create *any* market segmentation, including the value-revealing segmentation that perfectly discloses consumers' values. Although it is not implausible—given the advancement of information technology—that a data broker is (or at least will soon be) able to almost perfectly predict consumers' values, it is still crucial to explore the economic implications when the data broker does not have perfect information about consumers' values. This section extends the baseline model in the paper and restricts the data broker's ability in creating market segmentations.

To model this restriction, let Θ be a finite set of consumer characteristics that can be disclosed by the data broker. Suppose that among the consumers, their characteristics $\theta \in \Theta$ are distributed according to $\beta_0 \in \Delta(\Theta)$. These characteristics are informative of the consumers' values but there may still be variations in values among the consumers who share the same characteristics. Specifically, given any $\theta \in \Theta$, suppose that among the consumers who share characteristic θ, their values are distributed according a demand $D_\theta \in \mathcal{D}$ (i.e., $D_\theta(p)$ denotes the share of consumers with values above p among those with characteristic θ). Moreover, suppose that $\{\text{supp}(D_\theta)\}_{\theta\in\Theta}$ forms a partition of V and that $\text{supp}(D_\theta)$ is an interval for all $\theta \in \Theta$. In other words, the available consumer characteristics is only partially informative of the consumers' values in a way that any particular characteristic can only identify which interval a particular consumer's value belongs to. As a result, even when θ is perfectly revealed, the producer would still be unable to perfectly identify each consumer's value. For any $p \in V$, let

$$D_0(p) := \sum_{\theta\in\Theta} D_\theta(p)\beta_0(\theta).$$

$D_0 \in \mathcal{D}$ then describes the market demand in this environment.

In this environment, a market segmentation is defined by $s \in \Delta(\Delta(\Theta))$ such that

$$\int_{\Delta(\Theta)} \beta(\theta)s(\mathrm{d}\beta) = \beta_0(\theta),$$

for all $\theta \in \Theta$. A market segmentation s induces market segments $\{D_\beta\}_{\beta\in\text{supp}(s)}$ and

$$\int_{\Delta(\Theta)} D_\beta(p)s(\mathrm{d}\beta) = D_0(p),$$

for all $p \in V$, where $D_\beta(p) := \sum_{\theta\in\Theta} D_\theta(p)\beta(\theta)$ for any $\beta \in \Delta(\Theta)$ and any $p \in V$.

When the consumers' values can never be fully disclosed, it is clear that their surplus will increase. After all, it is no longer possible for the producer to charge the consumers their values as the additional variation in values given by D_θ always allows some consumers to buy the product at a price below their values. Nevertheless, as shown in Theorem OA.1, under any optimal mechanism, consumer surplus must be lower than the case when all the information about θ is revealed to the producer. That is, the main implication of Corollary 1— for the consumers, the presence of a data broker is no better than a scenario where their data is fully revealed to the producer—is still valid even when the consumers retain some private information.

Theorem OA.1. *For any $(\{D_\theta\}_{\theta\in\Theta}, \beta_0)$ and for any cost distribution G, an optimal mechanism always exists. Furthermore, the consumer surplus under any optimal mechanism of the data broker is lower than the case when θ is fully disclosed.*

Proof. For each $\theta \in \Theta$, write $\text{supp}(D_\theta)$ as $[l(\theta), u(\theta)]$. Also, for any $p \in V$, let θ_p be the unique θ such that $p \in (l(\theta), u(\theta)]$. Notice that since $\{(l(\theta), u(\theta))\}_{\theta\in\Theta}$ is disjoint, for any $\beta \in \Delta(\Theta)$, any $\theta \in \Theta$, and any $p \in \text{supp}(D_\beta)$,

$$D_\beta(p) = \sum_{\{\theta' : u(\theta') \geq u(\theta_p)\}} D_{\theta'}(p)\beta(\theta') = D_\theta(p)\beta(\theta) + \sum_{\{\theta' : u(\theta') > u(\theta_p)\}} \beta(\theta').$$

In particular, different prices set in $\text{supp}(D_\theta)$ do not affect the probability of trade through $\theta' \in \Theta$ such that $u(\theta') > u(\theta)$.

As a result, the construction in the proof of Lemma 3 given above is still valid, with the demands being replaced by D_β. Specifically, for any $\beta \in \Delta(\Theta)$ and any $c \in C$, there exists $\{\beta_i\}_{i=1}^n \subseteq \Delta(\Theta)$ such that:

1. $\beta \in \text{co}(\{\beta_i\}_{i=1}^n)$.
2. For each $i \in \{1, \dots, n\}$, the set
$$\{\theta \in \text{supp}(\beta_i) | u(\theta) \geq \boldsymbol{p}_{D_{\beta_i}}(c)\}$$
is nonempty and is a singleton.
3. For each $i \in \{1, \dots, n\}$,
$$\boldsymbol{P}_{D_{\beta_i}}(c) \bigcap \text{supp}(D_{\bar{\theta}_{\beta_i}}) \neq \emptyset,$$
where $\bar{\theta}_{\beta_i} := \max\{u(\theta) : \theta \in \text{supp}(\beta_i)\}$.
4. For each $i \in \{1, \dots, n\}$ and any $z \in [\underline{c}, c]$,
$$\boldsymbol{p}_{D_{\beta_i}}(z) \geq \boldsymbol{p}_{D_\beta}(z).$$

This further implies that, by Lemma 6, and by the same argument as in the proof of Lemma 3, for any $\beta \in \Delta(\Theta)$, there exists $\sigma^\beta : C \to \Delta(\Delta(\Theta))$ such that

5. For any $c \in C$,
$$\begin{aligned}&\sum_{\beta' \in \text{supp}(\sigma^\beta(c))} (\boldsymbol{p}_{D_{\beta'}}(c) - \boldsymbol{p}_{D_\beta}(c)) D_{\beta'}(\boldsymbol{p}_{D'_\beta}(c))\sigma^\beta(\beta'|c) \\ =& \sum_{\{\theta : u(\theta) \geq \theta(\boldsymbol{p}_{D_\beta}(c))\}} (\boldsymbol{p}_{D_\theta}(c) - \boldsymbol{p}_{D_\beta}(c)) D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta(\theta).\end{aligned}$$
6. For any $c \in C$, $\sum_{\beta' \in \text{supp}(\sigma^\beta(c))} D_{\beta'}(\boldsymbol{p}_{D_{\beta'}}(c))\sigma^\beta(\beta'|c) = D_\beta(\boldsymbol{p}_{D_\beta}(c))$.
7. For any $c \in C$, $\sum_{\beta' \in \text{supp}(\sigma^\beta(c))} \beta'\sigma^\beta(\beta'|c) = \beta$.
8. For any $\beta' \in \text{supp}(\sigma^\beta(c'))$,
$$\boldsymbol{p}_{D_{\beta'}}(c) \geq \boldsymbol{p}_{D_\beta}(c),$$
for any $c, c' \in C$ such that $c < c'$ and
$$\sum_{\beta' \in \text{supp}(\sigma^\beta(c))} D_{\beta'}(\boldsymbol{p}_{D_{\beta'}}(c))\sigma^\beta(\beta'|c) \geq D(\boldsymbol{p}_{D_\beta}(c)),$$
for any $c, c' \in C$ such that $c > c'$.

Now consider any mechanism (σ, τ). Suppose that there is a selection $\tilde{\boldsymbol{p}}$ of $\boldsymbol{P}$ and a positive G-measure of c such that there exists some $\beta \in \text{supp}(\sigma(c))$ and with

$$\{\theta \in \text{supp}(\beta) : u(\theta) > u(\theta_{\tilde{\boldsymbol{p}}_{D_\beta}(c)})\} \neq \emptyset. \tag{OA.40}$$

Then, for such $\tilde{\boldsymbol{p}}$, $c \in C$ and $\beta \in \text{supp}(\sigma(c))$, there exists $\sigma^\beta(c) \in \Delta(\Delta(\Theta))$ such that assertions 5 through 8 above hold. In particular, assertions 5 and 6 imply that

$$\begin{aligned}
&\sum_{\beta' \in \text{supp}(\sigma^\beta(c))} (\boldsymbol{p}_{D_{\beta'}}(c) - \phi(c)) D_{\beta'}(\boldsymbol{p}_{D_{\beta'}}(c)) \sigma^\beta(\beta'|c) \\
=& \sum_{\beta' \in \text{supp}(\sigma^\beta(c))} (\boldsymbol{p}_{D_{\beta'}}(c) - \boldsymbol{p}_{D_\beta}(c)) D_{\beta'}(\boldsymbol{p}_{D_{\beta'}}(c)) \sigma^\beta(\beta'|c) + (\boldsymbol{p}_{D_\beta}(c) - \phi(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \\
\geq& \sum_{\{\theta : u(\theta) \geq u(\theta_{\boldsymbol{p}_{D_\beta}(c)})\}} (\boldsymbol{p}_{D_\theta}(c) - \boldsymbol{p}_{D_\beta}(c)) D_\theta(\boldsymbol{p}_{D_\theta}(c)) \beta(\theta) + (\tilde{\boldsymbol{p}}_{D_\beta}(c) - \phi(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \\
>& (\tilde{\boldsymbol{p}}_{D_\beta}(c) - \phi(c)) D_\beta(\tilde{\boldsymbol{p}}_{D_\beta}(c)),
\end{aligned}$$

where the second equality follows from 5 and 6 and the inequality is strict due to (OA.40).

As such, together with assertion 7, $\sigma^\beta(c)$ induces another segmentation $\hat{\sigma}(c)$ through

$$\hat{\sigma}(\hat{\beta}|c) := \sum_{\beta \in \text{supp}(\sigma(c))} \sigma^\beta(\hat{\beta}|c) \sigma(\beta|c),\ \forall \hat{\beta} \in \bigcup_{\beta \in \text{supp}(\sigma(c))} \text{supp}(\sigma^\beta(c))$$

As (OA.40) holds with positive G-measure of $c \in C$, the induced segmentation scheme $\hat{\sigma} : C \to \Delta(\Delta(\Theta))$ strictly improves the data broker's revenue. Finally, by the revenue equivalence formula,[5] assertions 6 and 8 above and Lemma 1 ensure that there exists a transfer $\hat{\tau}$ such that $(\hat{\sigma}, \hat{\tau})$ is incentive compatible and individually rational and strictly improves the data broker's revenue.

Together, any optimal mechanism (σ, τ) must be such that for G-almost all $c \in C$ and for all $\beta \in \text{supp}(\sigma(c))$,

$$\{\theta \in \text{supp}(\beta) : u(\theta) > u(\theta_{\tilde{\boldsymbol{p}}_{D_\beta}(c)})\} = \emptyset.$$

which, together with the fact that $\sum_{\beta \in \text{supp}(\sigma(c))} \sigma(\beta|c) = \beta_0$ for all $c \in C$, implies that for G-almost all $c \in C$, the consumer surplus must be lower than the case when all the information about θ is revealed. ■

In addition to the surplus extraction result, the characterization of the optimal mechanisms can be generalized as well. With proper regularity conditions, there is an optimal mechanism analogous to the canonical $\overline{\phi}$-quasi-perfect mechanism introduced in the paper. To state this result, given any $(\{D_\theta\}_{\theta \in \Theta}, \beta_0)$, for each $\theta \in \Theta$, write $\text{supp}(D_\theta)$ as $[l(\theta), u(\theta)]$. For any $p \in V$, let $\theta_p \in \Theta$ be the unique θ such that $p \in (l(\theta), u(\theta)]$. For any $c \in C$, let $\widehat{\boldsymbol{p}}_0(c)$ be the largest optimal price for the producer with marginal cost $c \in C$ under the demand whose support contains $\boldsymbol{p}_0(c)$.[6] Also, let $\widehat{\phi}(c) := \min\{\phi(c), \widehat{\boldsymbol{p}}_0(c)\}$ for all $c \in C$. Furthermore, given any function $\psi : C \to \mathbb{R}_+$, say that a mechanism (σ, τ) is a canonical ψ-quasi-perfect

[5] See more detailed arguments in the proof of Theorem S1 of an earlier version of this paper (Yang, 2020*b*)

[6] That is, $\widehat{\boldsymbol{p}}_0(c) := \boldsymbol{p}_{D_{\theta_{\boldsymbol{p}_0(c)}}}(c)$. Notice that $\widehat{\boldsymbol{p}}_0(c) \leq \boldsymbol{p}_0(c)$ for all $c \in C$. Moreover, in the case where the data broker can disclose all the information about the value v, $\widehat{\boldsymbol{p}}_0(c) = \boldsymbol{p}_0(c)$ for all $c \in C$.

segmentation if the producer with marginal cost $\bar{c}$, when reporting truthfully, receives $\bar{\pi}$, and if for any $c \in C$, and for any $\beta \in \text{supp}(\sigma(c))$, either

$$\beta(\theta') = \beta^{\theta}_{\psi(c)}(\theta') := \begin{cases} \beta_0(\theta'), & \text{if } u(\theta') < \psi(c) \text{ and } u(\theta) \geq \psi(c) \\ \sum_{\{\hat{\theta}: u(\hat{\theta}) \geq \psi(c)\}} \beta_0(\hat{\theta}), & \text{if } u(\theta') \geq \psi(c) \text{ and } \theta' = \theta \\ 0, & \text{otherwise} \end{cases}, \tag{OA.41}$$

for any $\theta, \theta' \in \Theta$; or

$$\text{supp}(\beta) = \{\theta' : l(\theta') \leq \psi(c)\} \cup \{\theta\} \tag{OA.42}$$

for some $\theta \in \Theta$ with $l(\theta) \geq \psi(c)$ and

$$\beta(\theta') = \beta_0(\theta'). \tag{OA.43}$$

for all $\theta' \in \Theta$ such that $u(\theta') < \psi(c)$.

With these definitions, Theorem OA.2 below prescribes an optimal mechanism for the data broker.

Theorem OA.2. *For any $(\{D_\theta\}_{\theta \in \Theta}, \beta_0)$ and any distribution of marginal cost G such that the function $c \mapsto \max\{g(c)(\phi(c) - \widehat{\boldsymbol{p}}_0(c)), 0\}$ is nondecreasing and that D_0 is regular, there is a canonical $\widehat{\phi}$-quasi-perfect mechanism that is optimal.*

To prove Theorem OA.2, I first introduce two useful lemmas.

Lemma OA.5. *For any $c \in C$, any $\nu \geq c$ and any segmentation $s \in \Delta(\Delta(\Theta))$,*

$$\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \nu) D_\beta(\boldsymbol{p}_{D_\beta}(c)) s(\mathrm{d}\beta) \leq \int_{\{\theta: \boldsymbol{p}_{D_\theta}(c) \geq \nu\}} (\boldsymbol{p}_{D_\theta}(c) - \nu) D_\theta(\boldsymbol{p}_{D_\theta}) \beta_0(\mathrm{d}\theta),$$

Proof. I first show that for any segmentation $s \in \Delta(\Delta(\Theta))$, there must exist another segmentation $\hat{s}$ such that for any $\beta \in \text{supp}(\hat{s})$, either $\beta(\{\theta : u(\theta) < c\}) = 1$ or $\boldsymbol{p}_{D_\beta}(c) = \boldsymbol{p}_{D_{\bar{\theta}_\beta}}(c)$ and

$$\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta} - \nu) D(\boldsymbol{p}_{D_\beta}(c)) s(\mathrm{d}\beta) \leq \int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta} - \nu) D(\boldsymbol{p}_{D_\beta}(c)) \hat{s}(\mathrm{d}\beta),$$

where $\bar{\theta}_\beta := \max(\text{supp}(\beta))$. Indeed, consider any segmentation $s \in \Delta(\Delta(\Theta))$. For any $\beta \in \text{supp}(s)$, by definition, it must be that $\text{supp}(\beta) \cap \{\theta \in \Theta : u(\theta) \geq \boldsymbol{p}_{D_\beta}(c)\} \neq \emptyset$. Now define $\hat{\beta}^\theta$ as

$$\hat{\beta}^\theta(\theta') := \begin{cases} \beta(\theta), & \text{if } \theta' \leq \boldsymbol{p}_{D_\beta}(c) \\ \sum_{\{\hat{\theta}: u(\hat{\theta}) \geq \boldsymbol{p}_{D_\beta}(c)\}} \beta(\hat{\theta}), & \text{if } \theta' = \theta \\ 0, & \text{otherwise} \end{cases},$$

for any $\theta' \in \text{supp}(\beta)$ and for any $\theta \in \text{supp}(\beta)$ with $u(\theta) \geq \boldsymbol{p}_{D_\beta}(c)$. Notice that by construction, $\beta \in \text{co}(\{\hat{\beta}^\theta\}_{\theta \geq \boldsymbol{p}_{D_\beta}(c)})$ and hence there exists $K^\beta \in \Delta(\Delta(\Theta))$ such that $\beta = \sum_{\hat{\beta}} K^\beta(\hat{\beta})$. Therefore, by splitting every β according to K^β, and by the same arguments as in the proof of Lemma 3, the resulting segmentation $\hat{s} \in \Delta(\Theta)$ must be such that for any $\hat{\beta} \in \text{supp}(\hat{s})$, $\boldsymbol{p}_{D_{\hat{\beta}}}(c)$ is in the interval described by $\max(\text{supp}(\hat{\beta}))$. Moreover, since $\{(l(\theta), u(\theta))\}_{\theta \in \Theta}$ is disjoint, it follows that $\boldsymbol{p}_{D_{\bar{\theta}_{\hat{\beta}}}}(c) = \boldsymbol{p}_{D_{\hat{\beta}}}(c)$. Furthermore, since for any

$\beta \in \text{supp}(s)$,

$$\begin{aligned}(\boldsymbol{p}_{D_\beta}(c)-\nu)D_\beta(\boldsymbol{p}_\beta(c)) =&(\boldsymbol{p}_{D_\beta}(c)-\nu)\sum_{\{\theta:u(\theta)\geq \boldsymbol{p}_{D_\beta}(c)\}} D_\theta(\boldsymbol{p}_{D_\beta}(c))\beta(\theta)\\ \leq& \sum_{\{\theta:u(\theta)\geq \boldsymbol{p}_{D_\beta}(c)\}} (\boldsymbol{p}_{D_\theta}(c)-\nu)D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta(\theta)\\ =& \sum_{\hat\beta\in\text{supp}(K^\beta)} (\boldsymbol{p}_{D_{\hat\beta}}(c)-\nu)D_{\hat\beta}(\boldsymbol{p}_{D_{\hat\beta}}(c))K^\beta(\hat\beta).\end{aligned}$$

As a result, since $\hat{s}(\hat\beta)=\sum_\beta K^\beta(\hat\beta)s(\beta)$, it then follows that

$$\int_{\Delta(\Theta)}(\boldsymbol{p}_{D_\beta}(c)-\nu)D_\beta(\boldsymbol{p}_{D_\beta}(c))s(\mathrm{d}\beta)\leq\int_{\Delta(\Theta)}(\boldsymbol{p}_{D_\beta}(c)-\nu)D_\beta(\boldsymbol{p}_{D_\beta}(c))\hat{s}(\mathrm{d}\beta).$$

Finally, since for any $\beta\in\text{supp}(\hat{s})$, either $\beta(\{\theta:u(\theta)<c\})=1$ or $\boldsymbol{p}_{D_\beta}(c)=\boldsymbol{p}_{D_{\bar\theta_\beta}}(c)$, it must be that

$$\int_{\Delta(\Theta)}(\boldsymbol{p}_{D_\beta}(c)-\nu)D_\beta(\boldsymbol{p}_{D_\beta}(c))\hat{s}(\mathrm{d}\beta)\leq\int_{\{\theta:\boldsymbol{p}_{D_\theta}(c)\geq\nu\}}(\boldsymbol{p}_{D_\theta}(c)-\nu)D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\mathrm{d}\theta),$$

as desired. ■

Lemma OA.6. *Suppose that D_0 is regular. For any $c\in C$ and for any $\nu\in[c,\boldsymbol{p}_0(c)]$,*

$$D_0(\boldsymbol{p}_0(c))\leq\sum_{\{\theta:u(\theta)\geq\nu\}}D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta) \tag{OA.44}$$

and

$$D_0(\boldsymbol{p}_0(c))\geq\sum_{\{\theta:l(\theta)\geq\boldsymbol{p}_0(c)\}}D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta). \tag{OA.45}$$

Proof. Consider any $c\in C$. I first show that for any $\theta\in\Theta$ such that $l(\theta)\geq\boldsymbol{p}_0(c)$, $\boldsymbol{p}_{D_\theta}(c)=l(\theta)$. Indeed, since D_0 is regular, for any $\theta\in\Theta$ such that $l(\theta)\geq\boldsymbol{p}_0(c)$ and for any $p\in(l(\theta),u(\theta)]$,

$$\begin{aligned}&(p-c)\left[D_{\theta_p}(p)\beta_0(\theta_p)+\sum_{\{\theta':l(\theta')\geq p\}}\beta_0(\theta')\right]\\ =&(p-c)\sum_{\{\theta':u(\theta')\geq p\}}D_{\theta'}(p)\beta_0(\theta')\\ =&(p-c)D_0(p)\\ \leq&(l(\theta)-c)D_0(l(\theta))\\ =&(l(\theta)-c)\left[\sum_{\{\theta':u(\theta')\geq l(\theta)\}}D_{\theta'}(l(\theta))\beta_0(\theta')\right]\\ =&(l(\theta)-c)\left[D_\theta(l(\theta))\beta_0(\theta)+\sum_{\{\theta':l(\theta')\geq l(\theta)\}}\beta_0(\theta')\right].\end{aligned}$$

As such, since $p\in(l(\theta),u(\theta)]$ and $u(\theta_p)=u(\theta)$, it must be that

$$(p-c)D_\theta(p)<(l(\theta)-c)D_\theta(l(\theta)),$$

which then implies that $\boldsymbol{p}_{D_\theta}(c) = l(\theta)$.

Now, I show that $\boldsymbol{p}_0(c) \geq \widehat{\boldsymbol{p}}_0(c) := \boldsymbol{p}_{D_{\theta_{\boldsymbol{p}_0(c)}}}(c)$. Indeed, by definition,

$$
\begin{aligned}
&=(\widehat{\boldsymbol{p}}_0(c) - c)\left[D_{\theta_{\boldsymbol{p}_0(c)}}(\widehat{\boldsymbol{p}}_0(c))\beta_0(\theta_{\boldsymbol{p}_0(c)}) + \sum_{\{\theta' : l(\theta') \geq \widehat{\boldsymbol{p}}_0(c)\}} \beta_0(\theta')\right] \\
&=(\widehat{\boldsymbol{p}}_0(c) - c)D_0(\widehat{\boldsymbol{p}}_0(c)) \\
&\leq(\boldsymbol{p}_0(c) - c)D_0(\boldsymbol{p}_0(c)) \\
&=(\boldsymbol{p}_0(c) - c)\left[D_{\theta_{\boldsymbol{p}_0(c)}}(\boldsymbol{p}_0(c)) + \sum_{\{\theta' : l(\theta') \geq \boldsymbol{p}_0(c)\}} \beta_0(\theta')\right],
\end{aligned}
$$

and

$$
(\boldsymbol{p}_0(c) - c)D_{\theta_{\boldsymbol{p}_0(c)}}(\boldsymbol{p}_0(c)) \leq (\widehat{\boldsymbol{p}}_0(c) - c)D_{\theta_{\boldsymbol{p}(c)}}(\widehat{\boldsymbol{p}}_0(c)).
$$

As a result, it must be that $\widehat{\boldsymbol{p}}_0(c) \leq \boldsymbol{p}_0(c)$.

Consequently,

$$
\begin{aligned}
\sum_{\{\theta : l(\theta) \geq \boldsymbol{p}_0(c)\}} D_\theta(\boldsymbol{p}_\theta(c))\beta_0(\theta) = &\sum_{\{\theta : l(\theta) \geq \boldsymbol{p}_0(c)\}} \beta_0(\theta) \\
\leq &\sum_{\{\theta : l(\theta) \geq \boldsymbol{p}_0(c)\}} \beta_0(\theta) + D_{\theta_{\boldsymbol{p}_0(c)}}(\boldsymbol{p}_0(c))\beta_0(\theta_{\boldsymbol{p}_0(c)}) \\
\leq &D_0(\boldsymbol{p}_0(c)),
\end{aligned}
$$

which proves (OA.45). On the other hand, for any $\nu \in [c, \boldsymbol{p}_0(c)]$

$$
\begin{aligned}
&\sum_{\{\theta : u(\theta) \geq \nu\}} D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta) \\
= &\sum_{\{\theta : \nu \leq u(\theta) < \boldsymbol{p}_0(c)\}} D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta) + \sum_{\{\theta : u(\theta) \geq \boldsymbol{p}_0(c)\}} D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta) \\
\geq &\sum_{\{\theta : u(\theta) \geq \boldsymbol{p}_0(c)\}} D_\theta(\boldsymbol{p}_{D_\theta}(c))\beta_0(\theta) \\
= &D_{\theta_{\boldsymbol{p}_0(c)}}(\widehat{\boldsymbol{p}}_0(c)) + \sum_{\{\theta' : l(\theta') \geq \boldsymbol{p}_0(c)\}} D_{\theta'}(l(\theta'))\beta_0(\theta') \\
\geq &D_{\theta_{\boldsymbol{p}_0(c)}}(\boldsymbol{p}_0(c)) + \sum_{\{\theta' : l(\theta') \geq \boldsymbol{p}_0(c)\}} \beta_0(\theta') \\
= &D_0(\boldsymbol{p}_0(c)),
\end{aligned}
$$

which proves (OA.44) ■

With Lemma OA.5 and Lemma OA.6, the proof of Theorem OA.2 is as below.

Proof of Theorem OA.2. To prove Theorem OA.2, first notice that Lemma 1 still applies and hence the data broker's maximization problem can be written as

$$\begin{aligned}
&\max_{\sigma} \int_C \left(\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \phi(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma(\mathrm{d}\beta|c) \right) G(\mathrm{d}c) \\
&\text{s.t. } \int_c^{c'} \left(D_\beta(\boldsymbol{p}_{D_\beta}(z))(\sigma(\mathrm{d}\beta|z) - \sigma(\mathrm{d}\beta|c')) \right) \mathrm{d}z \geq 0, \, \forall c, c' \in C \qquad \text{(OA.46)} \\
&\qquad \bar{\pi} + \int_c^{\bar{c}} \left(\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(z)) \sigma(\mathrm{d}\beta|z) \right) \mathrm{d}z \geq \bar{\pi} + \int_c^{\bar{c}} D_0(\boldsymbol{p}_0(z)) \, \mathrm{d}z, \, \forall c \in C,
\end{aligned}$$

where the maximum is taken over all $\sigma : C \to \Delta(\Delta(\Theta))$ such that $\sigma(c)$ is a segmentation for all $c \in C$.

Consider first a relaxed problem of (OA.46) where the first constraint is relaxed to $\boldsymbol{D}_\sigma : C \to [0,1]$ being nonincreasing, where

$$\boldsymbol{D}_\sigma(c) := \int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma(\mathrm{d}\beta|c),$$

for all $c \in C$. By the same duality argument as in the proof of Lemma OA.3, it suffices to find a feasible σ^* and a Borel measure μ^* on C such that

$$\begin{aligned}
\sigma^* \in \operatorname*{argmax}_{\sigma \in \Sigma} \Bigg[&\int_C \left(\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \phi(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma(\mathrm{d}\beta|c) \right) G(\mathrm{d}c) \\
&+ \int_C \left(\int_c^{\bar{c}} \left(\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(z)) \sigma(\mathrm{d}\beta|z) - D_0(\boldsymbol{p}_0(z)) \right) \mathrm{d}z \right) \mu^*(\mathrm{d}c) \Bigg],
\end{aligned}$$

where Σ is the collection of segmentation schemes such that $\boldsymbol{D}_\sigma$ is nonincreasing, and that

$$\int_C \left(\int_c^{\bar{c}} \left(\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(z)) \sigma^*(\mathrm{d}\beta|z) - D_0(\boldsymbol{p}_0(z)) \right) \mathrm{d}z \right) \mu^*(\mathrm{d}c) = 0.$$

To this end, let M^* be defined as

$$M^*(c) := \lim_{c' \downarrow c} g(c)(\phi(c) - \widehat{\boldsymbol{p}}_0(c))^+.$$

Since $c \mapsto g(c)(\phi(c) - \widehat{\boldsymbol{p}}_0(c))^+$ in nondecreasing, M^* is nondecreasing and right-continuous and hence induced a Borel measure μ^* with $\mathrm{supp}(\mu^*) = [c^*, \bar{c}]$ for some $c^* \leq \bar{c}$. Then, by the same arguments as in the proof of Proposition 2,

$$\begin{aligned}
\max_{\sigma \in \Sigma} \Bigg[&\int_C \left(\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \phi(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma(\mathrm{d}\beta|c) \right) G(\mathrm{d}c) \\
&+ \int_C \left(\int_c^{\bar{c}} \left(\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(z)) \sigma(\mathrm{d}\beta|z) - D_0(\boldsymbol{p}_0(z)) \right) \mathrm{d}z \right) \mu^*(\mathrm{d}c) \Bigg]
\end{aligned}$$

is equivalent to

$$\max_{\sigma \in \Sigma} \int_C \left(\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \widehat{\phi}(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma(\mathrm{d}\beta|c) \right) G(\mathrm{d}c). \qquad \text{(OA.47)}$$

To solve (OA.47), notice that for any $c \in [\underline{c}, c^*)$,

$$\sum_{\{\theta : u(\theta) \geq \widehat{\phi}(c)\}} D_\theta(\boldsymbol{p}_{D_\theta}(c)) > D_0(\boldsymbol{p}_0(c)),$$

which is due to $\widehat{\phi}(c) = \phi(c) \leq \widehat{\boldsymbol{p}}_0(c) \leq \boldsymbol{p}_0(c)$ and (OA.45). Meanwhile, for any $c \in (c^*, \overline{c}]$, there exists a unique $\lambda(c)$ such that

$$\lambda(c) D_{\theta_{\widehat{\phi}(c)}}(\widehat{\boldsymbol{p}}_0(c)) + \sum_{\{\theta : l(\theta) \geq \widehat{\phi}(c)\}} D_\theta(\boldsymbol{p}_{D_\theta}(c)) = D_0(\boldsymbol{p}_0(c)),$$

which is due to the fact that $\widehat{\phi}(c) = \widehat{\boldsymbol{p}}_0(c)$ for all $c \in (c^*, \overline{c}]$ and (OA.44). Furthermore, Since D_0 is regular, for any $\theta \in \Theta$ such that $u(\theta) \geq \widehat{\phi}(c)$ and for any $p \leq l(\theta_{\widehat{\phi}(c)})$,

$$\begin{aligned}
(p-c) D_{\beta^\theta_{\widehat{\phi}(c)}}(p) = & \sum_{\{\theta' : u(\theta') \geq u(\theta_p)\}} (p-c) D_{\theta'}(p) \beta^\theta_{\widehat{\phi}(c)}(\theta') \\
= & (p-c) D_0(p) \\
\leq & (l(\theta_{\widehat{\phi}(c)}) - c) D_0(l(\theta_{\widehat{\phi}(c)})) \\
\leq & (l(\theta) - c) D_0(l(\theta_{\widehat{\phi}(c)})) \\
= & (l(\theta) - c) \sum_{\{\theta' : u(\theta') \geq \widehat{\phi}(c)\}} \beta^\theta_{\widehat{\phi}(c)}(\theta') \\
= & (l(\theta) - c) D_{\beta^\theta_{\widehat{\phi}(c)}}(l(\theta)) \\
= & (\boldsymbol{p}_{D_\theta}(c) - c) D_{\beta^\theta_{\widehat{\phi}(c)}}(\boldsymbol{p}_{D_\theta}(c)),
\end{aligned} \tag{OA.48}$$

where $\beta^\theta_{\widehat{\phi}(c)}$ is defined in (OA.41). In addition, by the same construction as in the proof of Lemma 3, for any $c \in (c^*, \overline{c}]$, there exists a segmentation $\tilde{\sigma}(c) \in \Delta(\Delta(\Theta))$ such that $\text{supp}(\tilde{\sigma}(c)) = \{\tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)} : l(\theta) \geq \widehat{\boldsymbol{p}}_0(c)\}$, with $\tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)}$ satisfying (OA.42) and (OA.43) and that

$$(p-c) D_{\tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)}}(p) \leq (l(\theta) - c) D_\theta(l(\theta)) = (\boldsymbol{p}_{D_\theta}(c) - c) D_\theta(\boldsymbol{p}_{D_\theta}(c)) \tag{OA.49}$$

for all $\theta \in \Theta$ such that $l(\theta) \geq \boldsymbol{p}_0(c)$, as well as

$$\boldsymbol{p}_{D_{\tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)}}}(z) \geq \boldsymbol{p}_{D_0}(z) \geq \widehat{\boldsymbol{p}}_0(z) \tag{OA.50}$$

for all $z \in [\underline{c}, c]$ and for all $\theta \in \Theta$ such that $l(\theta) \geq \boldsymbol{p}_0(c)$.

Now define σ^* as follows.

$$\sigma^*(c) := \begin{cases} \sigma_1(c), & \text{if } c \in [\underline{c}, c^*] \\ \sigma_2(c), & \text{if } c \in (c^*, \overline{c}] \end{cases},$$

where

$$\sigma_1(\beta^\theta_{\phi(c)} | c) := \frac{\beta_0(\theta)}{\sum_{\{\theta' : u(\theta') \geq \phi(c)\}} \beta_0(\theta')}$$

for all $c \in [\underline{c}, c^*]$ and for all $\theta \in \Theta$ such that $u(\theta) \geq \phi(c)$, whereas

$$\sigma_2(\beta | c) := \begin{cases} \lambda(c) \frac{\beta_0(\theta)}{\sum_{\{\theta' : u(\theta') \geq \widehat{\boldsymbol{p}}_0(c)\}} \beta_0(\theta')}, & \text{if } \beta = \beta^\theta_{\widehat{\boldsymbol{p}}_0(c)},\ u(\theta) \geq \widehat{\boldsymbol{p}}_0(c) \\ (1 - \lambda(c)) \tilde{\sigma}(\tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)} | c), & \text{if } \beta = \tilde{\beta}^\theta_{\widehat{\boldsymbol{p}}_0(c)},\ l(\theta) \geq \widehat{\boldsymbol{p}}_0(c) \\ 0, & \text{otherwise} \end{cases},$$

for all $c \in (c^*, \bar{c}]$. It then follows that, by (OA.48) and (OA.49),

$$\begin{aligned}&\int_C \left(\int_{\Delta(\Theta)} (\boldsymbol{p}_{D_\beta}(c) - \widehat{\phi}(c)) D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma^*(\mathrm{d}\beta|c) \right) G(\mathrm{d}c) \\ =&\int_C \left(\sum_{\{\theta: \boldsymbol{p}_{D_\theta}(c) \geq \hat{\phi}_G(c)\}} (\boldsymbol{p}_{D_\theta}(c) - \widehat{\phi}(c)) D_\theta(\boldsymbol{p}_{D_\theta}(c)) \beta_0(\theta) \right) G(\mathrm{d}c),\end{aligned}$$

which, together with Lemma OA.5, implies that σ^* is a solution of (OA.47).

Furthermore, for any $c > c^*$, by the definition of $\sigma_2(c)$ and $\lambda(c)$, by (OA.48) and (OA.49), and by the fact that $\widehat{\phi}(c) = \widehat{\boldsymbol{p}}_0(c)$,

$$\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma^*(\mathrm{d}\beta|c) = D_0(\boldsymbol{p}_0(c)).$$

Therefore,

$$\int_C \left(\int_c^{\bar{c}} \left(\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(z)) \sigma^*(\mathrm{d}\beta|z) - D_0(\boldsymbol{p}_0(z)) \right) \mathrm{d}z \right) \mu^*(\mathrm{d}c) = 0.$$

Finally, by definition of $\widehat{\phi}$ and by Lemma OA.6,

$$\int_{\Delta(\Theta)} D_\beta(\boldsymbol{p}_{D_\beta}(c)) \sigma^*(\mathrm{d}\beta|c) \geq D_0(\boldsymbol{p}_0(c))$$

for all $c \in [\underline{c}, c^*)$. Together with monotonicity of $\widehat{\phi}$, $\sigma^* \in \Sigma$ and σ^* is a solution of the relaxed problem of (OA.46).

It then suffices to show that σ^* is implementable. Notice that for any $c \in C$ and for any $z \in [\underline{c}, c]$ and for any $\beta^\theta_{\widehat{\phi}(c)} \in \operatorname{supp}(\sigma^*(c))$, if

$$\boldsymbol{P}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z) \cap \operatorname{supp}(D_\theta) = \emptyset,$$

then it must be that

$$\begin{aligned}(p - z) D_0(p) =&(p - z) D_{\beta^\theta_{\widehat{\phi}(c)}}(p) \\ \leq&(\boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z) - z) D_{\beta^\theta_{\widehat{\phi}(c)}}(\boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z)) \\ =&(\boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z) - z) D_0(\boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z)),\end{aligned}$$

for all $p \leq \boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z)$. Therefore,

$$\boldsymbol{p}_{D_{\beta^\theta_{\widehat{\phi}(c)}}}(z) \geq \boldsymbol{p}_0(z) \geq \widehat{\boldsymbol{p}}_0(z) \geq \widehat{\phi}(z),$$

for all $z \in [\underline{c}, c]$. Together with (OA.50), by the same argument as the proof of Lemma 3, σ^* is indeed implementable. This completes the proof. ■

OA.4 Counterexample: Producer's Profit Is Not Single-Crossing

This example demonstrates the fact that the producer's profit, as a function of market segmentation and marginal cost, does not exhibit the single-crossing property—even when restricting the domain to the set of quasi-perfect segmentations and ordering them by the cutoff κ. Formally, let $\geq_{\mathrm{B}}$ denote the Blackwell order on $\mathcal{S}$.[7] Meanwhile, define the following two orders over the family of quasi-perfect segmentations. Let s be a κ-quasi-perfect segmentation for $c \geq 0$, and let s' be a κ'-quasi-perfect segmentation for $c' \geq 0$. Say that $s \geq_{\mathrm{QP}} s'$ if $\kappa \leq \kappa'$, and that $s \geq^*_{\mathrm{QP}} s'$ if $\kappa \leq \kappa'$ and $c \leq c'$. That is, $\geq_{\mathrm{QP}}$ is a (total) order on the family of quasi-perfect segmentations (regardless of cost, and hence regardless of pricing incentives) implied by their cutoffs κ; whereas $\geq^*_{\mathrm{QP}}$ is a (partial) order on the same family when costs (and hence pricing incentives) are further taken into account. Note that for any nondecreasing function $\psi : C \to \mathbb{R}_+$ with $\psi(c) \geq c$ for all c, a ψ-quasi-perfect scheme σ is monotone in both $\geq_{\mathrm{QP}}$ and $\geq^*_{\mathrm{QP}}$.

Below, I show that there exists a market demand D_0, two costs $c_L < c_H$, and two market segmentations s_L and s_H such that $s_L \geq_{\mathrm{B}} s_H$, $s_L \geq_{\mathrm{QP}} s_H$, $s_L \geq^*_{\mathrm{QP}} s_H$,

$$\int_{\mathcal{D}} \pi_D(c_H) s_L(\mathrm{d}D) > \int_{\mathcal{D}} \pi_D(c_H) s_H(\mathrm{d}D)$$

and yet

$$\int_{\mathcal{D}} \pi_D(c_L) s_L(\mathrm{d}D) = \int_{\mathcal{D}} \pi_D(c_L) s_H(\mathrm{d}D).$$

This means that the producer's profit is not single-crossing in general, neither under the the Balckwell order, nor when restricting attention to quasi-perfect segmentations (even when the pricing incentives are correct so that the producer induces quasi-perfect price discrimination on path).

Let the market demand D_0 be defined as

$$D_0(p) := \begin{cases} 1, & \text{if } p \in [0,1] \\ \frac{1}{4}, & \text{if } p \in (1,2] \\ \frac{1}{8}, & \text{if } p \in (2,3] \\ 0, & \text{if } p > 3 \end{cases}.$$

Now consider two costs, $c_L = 1/2$ and $c_H = 3/2$, and consider two market segmentations s_L and s_H, where $s_H = \delta_{\{D_0\}}$ is the degenerate segmentation that does not segment D_0, while s_L induces two segments, D_L^2 and D_L^3, where

$$D_L^2(p) := \begin{cases} 1, & \text{if } p \in [0,1] \\ \frac{1}{3}, & \text{if } p \in (1,2] \\ 0, & \text{if } p > 2 \end{cases}; \quad D_L^3(p) := \begin{cases} 1, & \text{if } p \in [0,1] \\ \frac{1}{5}, & \text{if } p \in (1,3] \\ 0, & \text{if } p > 3 \end{cases},$$

and $s_L(\{D_L^2\}) = 3/8$, $s_L(D_L^3) = 5/8$. Clearly $s_L \geq_{\mathrm{B}} s_H$.

Direct calculation shows $\boldsymbol{P}_0(c_H) = \{3\}$, $\boldsymbol{P}_0(c_L) = \{1\}$, $\boldsymbol{P}_{D_L^2}(c_L) = \{1,2\}$, and $\boldsymbol{P}_{D_L^3}(c_L) = \{1,3\}$, which in turn implies $\boldsymbol{P}_{D_L^2}(c_H) = \{2\}$ and $\boldsymbol{P}_{D_L^3}(c_H) = \{3\}$. Together, it follows that for any κ_L and κ_H such that $1 \leq \kappa_L \leq 2 < \kappa_H \leq 3$, s_L is a κ_L-quasi-perfect segmentation for c_L, and s_H is a κ_H-quasi-perfect segmentation for c_H. Therefore, $s_L \geq_{\mathrm{QP}} s_H$ and $s_L \geq^*_{\mathrm{QP}} s_H$.

[7] That is, $s \geq_{\mathrm{B}} s'$ if and only if s is a mean preserving spread of s'.

However,

$$\int_{\mathcal{D}} \pi_D(c_L) s_L(\mathrm{d}D) = \frac{3}{8}\cdot\left(1-\frac{1}{2}\right)\cdot D_L^2(1) + \frac{5}{8}\cdot\left(1-\frac{1}{2}\right)\cdot D_L^3(1) = \frac{1}{2} = \left(1-\frac{1}{2}\right)\cdot D_0(1) = \int_{\mathcal{D}} \pi_D(c_L) s_H(\mathrm{d}D),$$

where the first equality follows from $1 \in \boldsymbol{P}_{D_L^2}(c_L) \cap \boldsymbol{P}_{D_L^3}(c_L)$, and the third equality follows from $\boldsymbol{P}_0(c_L) = \{1\}$. Meanwhile,

$$\int_{\mathcal{D}} \pi_D(c_H) s_L(\mathrm{d}D) = \frac{3}{8}\cdot\left(2-\frac{3}{2}\right)\cdot D_L^2(2) + \frac{5}{8}\cdot\left(3-\frac{3}{2}\right) D_L^3(3) = \frac{1}{4} > \frac{3}{16} = \left(3-\frac{3}{2}\right) D_0(3) = \int_{\mathcal{D}} \pi_D(c_H) s_H(\mathrm{d}D),$$

where the first equality follows from $\boldsymbol{P}_{D_L^2}(c_H) = \{2\}$ and $\boldsymbol{P}_{D_L^3}(c_H) = \{3\}$, and the third equality follows from $\boldsymbol{P}_0(c_H) = \{3\}$. Thus, the producer's profit, as a function of market segmentation and cost, is not single-crossing in general.

In fact, this example implies that the producer's profit function does not satisfy monotone difference in general. To see this, let $c_M := 3/4$. Then $\boldsymbol{P}_0(c_M) = \{2\}$, $\boldsymbol{P}_{D_L^2}(c_M) = \{2\}$, and $\boldsymbol{P}_{D_L^3}(c_M) = \{3\}$ and thus

$$\int_{\mathcal{D}} \pi_D(c_M) s_L(\mathrm{d}D) = \frac{3}{8}\cdot\left(2-\frac{3}{4}\right) D_L^2(2) + \frac{5}{8}\cdot\left(3-\frac{3}{4}\right) D_L^3(3) = \frac{7}{16},$$

and

$$\int_{\mathcal{D}} \pi_D(c_M) s_H(\mathrm{d}D) = \left(2-\frac{3}{4}\right) D_0(2) = \frac{5}{16}.$$

Together, it follows that $c_L < c_M < c_H$, and yet

$$\int_{\mathcal{D}} \pi_D(c_L) s_L(\mathrm{d}D) - \int_{\mathcal{D}} \pi_D(c_L) s_H(\mathrm{d}D) = 0 < \frac{1}{8} = \int_{\mathcal{D}} \pi_D(c_M) s_L(\mathrm{d}D) - \int_{\mathcal{D}} \pi_D(c_M) s_H(\mathrm{d}D)$$

while

$$\int_{\mathcal{D}} \pi_D(c_M) s_L(\mathrm{d}D) - \int_{\mathcal{D}} \pi_D(c_M) s_H(\mathrm{d}D) = \frac{1}{8} > \frac{1}{16} = \int_{\mathcal{D}} \pi_D(c_H) s_L(\mathrm{d}D) - \int_{\mathcal{D}} \pi_D(c_H) s_H(\mathrm{d}D).$$

Furthermore, this example also implies that any segmentation scheme $\sigma : C \to \mathcal{S}$ with $\sigma(c_L) = s_L$ and $\sigma(c_H) = s_H$ is not implementable, even if it is monotone under $\geq_{\mathrm{B}}$, $\geq_{\mathrm{QP}}$, and $\geq_{\mathrm{QP}}^*$. Indeed, if σ can be implemented by τ, then the incentive constraint for c_L,

$$\int_{\mathcal{D}} \pi_D(c_L) \sigma(\mathrm{d}D|c_L) - \tau(c_L) \geq \int_{\mathcal{D}} \pi_D(c_L) \sigma(\mathrm{d}D|c_H) - \tau(c_H),$$

implies $\tau(c_L) \leq \tau(c_H)$. However, from the incentive constraint for c_H,

$$\int_{\mathcal{D}} \pi_D(c_H) \sigma(\mathrm{d}D|c_H) - \tau(c_H) \geq \int_{\mathcal{D}} \pi_D(c_H) \sigma(\mathrm{d}D|c_L) - \tau(c_L),$$

it follows that

$$0 < \int_{\mathcal{D}} \pi_D(c_H) \sigma(\mathrm{d}D|c_L) - \int_{\mathcal{D}} \pi_D(c_H) \sigma(\mathrm{d}D|c_H) \leq \tau(c_L) - \tau(c_H),$$

a contradiction. In particular, for any nondecreasing function ψ on $C = [0, c_H]$ such that $\psi(c) \geq c$ for all c, and that $1 < \psi(c_L) \leq 2 < \psi(c_H) \leq 3$, any ψ-quasi-perfect scheme σ with $\sigma(c_L) = s_L$ and $\sigma(c_H) = s_H$ is not implementable. This demonstrates that monotonicity of the cutoff function ψ is not sufficient for implementability of a ψ-quasi-perfect scheme.

Finally, it is noteworthy that the exact values of D_0, c_L, c_H, s_L and s_H are not essential for this counterexample. The crucial part is the fact that c_L has multiple optimal prices under both segments D_L^2 and D_L^3. This suggests the example here is generic.

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