D. Quint and K. Hendricks, *A Theory of Indicative Bidding* Online Appendix

This appendix contains proofs of Lemmas 1 and 2, the constructive proof of Theorem 1 (equilibrium existence), a discussion of how uniqueness can be proved in special cases, and the proofs of the "large-N" bidder surplus results discussed in footnote 27. Throughout, ε will refer to the value defined in Lemma 3, i.e., a value such that $s_i \leq s_j + \varepsilon$ implies $u_2(s_i, s_j) < c$.

B.1 Proof of Lemma 1

Lemma 1 (in the text) says that any symmetric equilibrium must use a strategy which is weakly increasing and has support $\{0, 1, 2, \ldots, M\}$ for some $M < \infty$.

Weakly Increasing

Suppose τ is a symmetric equilibrium strategy, with $m \in \operatorname{supp} \tau(s)$ and $m' \in \operatorname{supp} \tau(s')$; we'll show that if $s < s'$ but $m > m'$, this leads to a contradiction.

As in the text, let $v_\tau(m; s)$ denote the expected payoff to bidder 1 if $S_1 = s$, he sends message m, and bidders 2 through N play the strategy τ. Let $\mathbf{m}$ denote a profile of messages sent by 1's opponents, and write v_τ as

$$v_\tau(m; s_1) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \Pr(adv|m, \mathbf{m}) V(s_1, \mathbf{m})$$

where $\Pr(\mathbf{m})$ is the probability (given τ) that 1's opponents send the messages $\mathbf{m}$, $\Pr(adv|m, \mathbf{m})$ is 1's probability of advancing to the auction if he sends message m and his opponents send message profile $\mathbf{m}$, and $V(s_1, \mathbf{m})$ is his expected payoff from advancing to the auction, given true type s_i, in expectation over all the different type profiles that would have generated $\mathbf{m}$. (What's significant here is that this last term $V(s_1, \mathbf{m})$ does not depend on m, the message sent by bidder 1: conditional on advancing *and on* $\mathbf{m}$, he faces the same distribution of opponent types regardless of the message he himself sends.)

Now, suppose $m > m'$. There are two cases to consider: either m and m' offer bidder 1 the same probability of advancing, or they do not. If they do not, then we can write

$$v_\tau(m; s_1) - v_\tau(m'; s_1) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m, \mathbf{m}) - \Pr(adv|m', \mathbf{m})\right] V(s_1, \mathbf{m})$$

We consider two sub-cases: either for every $\mathbf{m}$ at which $\Pr(\mathbf{m}) [\Pr(adv|m, \mathbf{m}) - \Pr(adv|m', \mathbf{m})] > 0$, there is probability 1 that at least one of the other bidders advancing has type $S_j \geq s$; or at least one of these message profiles puts positive probability on $\max\{S_j\} < s$ among the other bidders j who advance.

- In the first case, the "small rents" assumption guarantees that $V(s, \mathbf{m}) < 0$ for every $\mathbf{m}$ in the sum, meaning $v_\tau(m; s) - v_\tau(m'; s) < 0$, contradicting the assumption that m is a best-response for a bidder with type $S_1 = s$.

- In the second case, the monotonicity assumption (Assumption 1(b)) guarantees that $V(s', \mathbf{m}) \geq V(s, \mathbf{m})$ for every $\mathbf{m}$, with strict inequality holding for some of them, so $v_\tau(m; s') - v_\tau(m'; s') > v_\tau(m; s) - v_\tau(m'; s)$; this means that either the first expression is strictly positive or the second is strictly negative, contradicting either m' being a best-response when $S_1 = s'$ or m being a best-response when $S_1 = s$.

This ensures that τ must satisfy monotonicity among messages that give different probabilities of advancing. What's left is to rule out non-monotonicities among multiple messages giving the same probability of advancing. That is, we need to rule out the possibility that m and m' are such that $\Pr(adv|m, \mathbf{m}) = \Pr(adv|m', \mathbf{m})$ for every $\mathbf{m}$ with $\Pr(\mathbf{m}) > 0$, since that would allow $m \in \operatorname{supp} \tau(s)$ and $m' \in \operatorname{supp} \tau(s')$ without contradicting monotonicity of V.

So suppose that were the case, that is, $\Pr(adv|m, \mathbf{m}) = \Pr(adv|m', \mathbf{m})$ for every $\mathbf{m}$ with $\Pr(\mathbf{m}) > 0$. If bidders with types s and s' were both playing strategies giving this probability of advancing, then all bidders with types $S_i \in (s, s')$ would also have to play messages giving this probability of advancing with probability 1, since otherwise this would violate the type of monotonicity we showed above. This means at least a measure $s' - s$ of types send messages giving this same probability of advancing. Let $\underline{m}$ denote the lowest such message, and let $\overline{m}$ denote a message giving the same probability of advancing but such that a positive measure of bidders play messages in $\{\underline{m}, \ldots, \overline{m}\}$ with positive probability. (If a positive measure of bidders only play messages giving this probability of advancing, such a message $\overline{m}$ must exist.) It's easy to show that $\overline{m}$ and $\underline{m}$ can't give the same probability of advancing, yielding a contradiction; this proves that symmetric equilibrium must be monotonic.

Finite Support

Now that we have monotonicity, define α_m as the supremum of the set of types s such that $\tau(s)$ puts positive probability on message m or lower (or as 0 if this set is empty). This means that $\alpha_{m-1} = \alpha_m$ if $m \notin \operatorname{supp} \tau$, and if $\alpha_{m-1} < \alpha_m$ then τ has types in (α_{m-1}, α_m) sending message m for sure.

Now, consider two messages m and $m' > m$, both of which are in the support of τ, and which have no message between them in the support of τ. For δ small enough, this means $\tau(\alpha_m - \delta) = m$ and $\tau(\alpha_m + \delta) = m'$, which requires $v_\tau(m; \alpha_m - \delta) \geq v_\tau(m'; \alpha_m - \delta)$ and $v_\tau(m; \alpha_m + \delta) \leq v_\tau(m'; \alpha_m + \delta)$. Since v_τ is a weighted sum of u_k terms, which are each continuous in a bidder's own type, they're continuous, and therefore $v_\tau(m; \alpha_m) = v_\tau(m'; \alpha_m)$.

As noted before,

$$v_\tau(m'; \alpha_m) - v_\tau(m; \alpha_m) \quad = \quad \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m})$$

Now, all the message profiles $\mathbf{m}$ that show up in the sum have at least n opponents sending message m or higher, since if fewer than that did, then $\Pr(adv|m', \mathbf{m}) = \Pr(adv|m, \mathbf{m}) = 1$. This means that there are two types of terms in the sum:

- Profiles $\mathbf{m}$ that involve at least one opponent sending message m' or higher, and therefore have at least one opponent with type α_m or higher advancing, which by the small rents assumption have $V(\alpha_m, \mathbf{m}) < 0$

- Profiles $\mathbf{m}$ that involve at least n opponents sending message m and none sending a higher message, which therefore have $n - 1$ opponents with types in $[\alpha_{m-1}, \alpha_m]$

Since the overall sum must be equal to 0, the latter terms must be positive (since the former are negative). This requires $u_2(\alpha_m, \alpha_{m-1}) > c$; by Lemma 3, this requires $\alpha_m - \alpha_{m-1} > \varepsilon$.

So if m and $m' > m$ are both played in equilibrium, there must be an interval of types of width at least ε who send message m. This applies to every message in the support of τ other than the highest and 0, so the maximal number of messages in the support of τ is $2 + \lceil \frac{1}{\varepsilon} \rceil$.

Support $\{0, 1, 2, \ldots, M\}$

First, we'll show that if messages m and m' are consecutive messages in the support of τ, there can't be any other message in between them not in the support of τ. (For this reason, no symmetric equilibrium exists when the set of allowed messages is continuous: such an equilibrium would require both that a finite number of messages be used and that no unused messages exist between messages used in equilibrium, which can't both hold.)

In the sum $v_\tau(m'; \alpha_m) - v_\tau(m; \alpha_m)$, separate the opponent message profiles $\mathbf{m}$ into two sets: let M_1 denote all the profiles where at least n opponents sent message m' or higher, and M_2 the profiles where fewer than n sent message m' or higher. Write

$$\begin{aligned} v_\tau(m'; \alpha_m) - v_\tau(m; \alpha_m) &= \sum_{\mathbf{m} \in M_1} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \\ &\quad + \sum_{\mathbf{m} \in M_2} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \\ &= \sum_{\mathbf{m} \in M_1} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - 0 \right] V(\alpha_m, \mathbf{m}) \\ &\quad + \sum_{\mathbf{m} \in M_2} \Pr(\mathbf{m}) \left[1 - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \end{aligned}$$

Now, by small rents, $V(\alpha_m, \mathbf{m}) < 0$ for every $\mathbf{m} \in M_1$, so the first sum is negative; which means since the entire expression must be zero, the second sum must be positive.

But the second sum is exactly the benefit that a bidder with type α_m would get if he deviated from message m to a message in between m and m', since he would no longer be rationed against other bidders sending message m, but would still not be selected when at least n others sent m' or higher. So for a bidder with type close to α_m, and therefore close to indifferent between m and m', another message in between them would give a strictly higher payoff than either one; so no such message can exist.

Next, I show that the lowest opt-in message must be used. Suppose not. Consider the lowest type sending any opt-in message. Since any opponent he faces will have type higher than him, small rents implies his payoff in the auction is negative whenever anyone else opts in. So if there was an opt-in message lower than the one he's using, he would deviate to it, since he would still advance when everyone else opted out but would be less likely to be chosen when others opted in.

Finally, if message 1 is being used, and used by an interval of width at least ε, then there is a positive probability that all of bidder 1's opponents send message 1; so if bidder 1 has the lowest type among the types who opt in, he must be getting a positive payoff from the probability all his opponents opt out, so 0 must be in the support of τ as well. □

B.2 Proof of Lemma 2

Lemma 2 says that if τ satisfies the conditions of Lemma 1, then the following are necessary and sufficient for τ to be a symmetric equilibrium:

1. $v_\tau(m;\alpha_m) = v_\tau(m+1;\alpha_m)$ for $m = 0, 1, \ldots, M-1$, and
2. either $M = \overline{M}$ (the support of τ includes all available messages) or $v_\tau(M;1) \geq v_\tau(M+1;1)$, and $\tau(1)$ puts probability 1 on message M unless $v_\tau(M;1) = v_\tau(M+1;1)$.

Proof of Necessity

Necessity of these conditions is easy to show.

- We noted above that $v_\tau(m+1;s_1) - v_\tau(m;s_1)$ must equal 0 at $s_1 = \alpha_m$, since it must be weakly negative for $s_1 \in (\alpha_{m-1}, \alpha_m)$ and weakly positive for $s_1 \in (\alpha_m, \alpha_{m+1})$ and $v_\tau(m+1;\cdot)$ and $v_\tau(m;\cdot)$ are both continuous.
- If the second condition is violated, then $\overline{M} > M$ and $v_\tau(M+1;1) > v_\tau(M;1)$. By continuity, $v_\tau(M+1;s_i) > v_\tau(M;s_i)$ for s_i sufficiently close to 1, contradicting the equilibrium requirement that all bidders with types $s_i \in (\alpha_{M-1}, 1)$ send message M.
- Finally, if $\overline{M} = M$ or $v_\tau(M+1;1) < v_\tau(M;1)$, then $\operatorname{supp} \tau(1)$ must put probability 1 on message M, since in the first case no higher messages exist and in the second case they give lower payoff than M.

Thus, what is left to do is to prove sufficiency – i.e., that if τ satisfies these conditions, it constitutes a symmetric equilibrium.

Proof of Sufficiency

So now suppose τ exists satsifying these conditions; we need to show that when one's opponents play τ, playing $\tau(s_1)$ is a best-response for a bidder with type s_1.

The key thing to note is that for $m' > m$, $v_\tau(m';s_1) - v_\tau(m;s_1)$ is weakly increasing in s_1, since as noted above

$$v_\tau(m';s_1) - v_\tau(m;s_1) \quad = \quad \textstyle\sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(s_1, \mathbf{m})$$

Since $m' > m$, $\Pr(adv|m', \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \geq 0$; and since $V(s_1, \mathbf{m})$ is an expected value of $u_k(s_1, S_{-i,k})$ over the range of opponent types who would have generated message profile $\mathbf{m}$, $V(s_1, \mathbf{m})$ is weakly increasing in s_1.

This means, then, that $v_\tau(m+1;\alpha_m) = v_\tau(m;\alpha_m)$ immediately implies $v_\tau(m+1;s_1) \geq v_\tau(m;s_1)$ for every $s_1 > \alpha_m$, and likewise $v_\tau(m+1;s_1) \leq v_\tau(m;s_1)$ for every $s_1 < \alpha_m$. (In fact, both of these can be shown to hold strictly, but that's not important for this proof.)

Using this, then, we can show that for every bidder type $s_1 \in [0,1)$, $\tau(s_1)$ selects a best-response:

- Suppose that for $s_1 = \alpha_m$, some message lower than m gave a strictly higher payoff than m and $m+1$. Let m' be the highest message below m giving a strictly higher payoff than m; then $v_\tau(m';\alpha_m) > v_\tau(m'+1;\alpha_m)$. Since $v_\tau(m';\alpha_{m'}) = v_\tau(m'+1;\alpha_{m'})$ and $\alpha_m > \alpha_{m'}$, this violates the monotonicity of $v_\tau(m'+1;s_1) - v_\tau(m';s_1)$ shown above.

  A similar contradiction follows if a bidder with type α_m strictly preferred a message $m' > m+1$ ($m' \in \operatorname{supp} \tau$) to m and $m+1$.

- Suppose that for some $s_1 \in (\alpha_{m-1}, \alpha_m)$, some message $m' < m$ gave a higher payoff than m. By monotonicity of $v_\tau(m;s) - v_\tau(m';s)$, this would also have to hold for $S_1 = \alpha_{m-1}$.

If $m' = m - 1$, this means $v_\tau(m-1; \alpha_{m-1}) > v_\tau(m; \alpha_{m-1})$, contradicting the fact that this same statement has to hold with equality; if $m' < m-1$, this means a bidder with type α_{m-1} strictly prefers a message $m' < m-1$ to either $m-1$ or m, which was ruled out above.

Similarly, if a higher message $m' > m$ gave a higher payoff than m, monotonicity of $v_\tau(m'; s) - v_\tau(m; s)$ would require this to also hold for $s_1 = \alpha_m$, which similarly generates a contradiction.

- All that's left, then, is deviations to unused messages, which Lemma 1 implies must be above M. (Note that since no opponent is sending a message above M with positive probability, all messages above M give the same expected payoff.) The second condition in Lemma 2 implies that such messages are either unavailable or no better than M even when $S_1 = 1$; monotonicity of $v_\tau(M+1; s) - v_\tau(M; s)$ ensures they're no better than M for all other types. The final part of the second condition in Lemma 2 ensures that bidders with type $S_1 = 1$ are playing a best-response as well.

Thus, the conditions of Lemma 2 ensure that τ is a best-response to one's opponents playing τ, and therefore that "everyone plays τ" is an equilibrium. □

B.3 Other Preliminaries for Existence Proof

Above, we somewhat informally defined the difference $v_\tau(m+1; s_1) - v_\tau(m; s_1)$, and noted that in equilibrium, at $s_1 = \alpha_m$, it must be equal to zero. Here, we will calculate $v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m)$ more explicitly as a function of τ, and show that it has two important features. First, given the environment, it is a function only of $\{\alpha_{m-1}, \alpha_m, \alpha_{m+1}\}$; and second, it is strictly single-crossing in α_{m-1}. This means that for given values of α_m and α_{m+1}, there is a unique value of α_{m-1} that is consistent with equilibrium, which will allow us to construct the equilibrium "downwards" from the top of the type space.

Fixing an environment and a number of messages M, let $\mathcal{A}$ be the set of possible thresholds $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{M-1})$ with $0 \leq \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_{M-1} \leq 1$. Define a function Δ from $\mathcal{A} \times \{1, 2, \ldots, M-1\}$ to the reals as

$$\begin{aligned} \Delta(\alpha, m) &\equiv v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m) \\ &= \sum_{\mathbf{m}} \Pr(\mathbf{m}) \left[\Pr(adv | m+1, \mathbf{m}) - \Pr(adv | m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \end{aligned}$$

(Note that the analysis below will not hold for $m = 0$, as $v_\tau(\alpha_0, 1) - v_\tau(\alpha_0, 0)$ takes a different form; hence, $\Delta(\alpha, m)$ is defined only for $m \geq 1$.)

In order for $\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \neq 0$, it must be that $\Pr(adv|m, \mathbf{m}) < 1$, which means that in every opponent message profile $\mathbf{m}$ which shows up in the sum, there will be at least n opponents sending message m or higher. Similarly, $\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \neq 0$ requires $\Pr(adv|m+1, \mathbf{m}) > 0$, which means that at $\mathbf{m}$, fewer than n opponents are sending messages above $m+1$. Thus, if we define $\mathcal{M}_{q,r}$ as the set of message profiles at which q opponents send message $m+1$ and r send messages higher than that, we can write Δ as

$$\Delta(\alpha, m) = \sum_{r=0}^{n-1} \sum_{q=0}^{N-1-r} \left(\sum_{\mathbf{m} \in \mathcal{M}_{q,r}} \Pr(\mathbf{m}) \left[\Pr(adv|m+1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \right)$$

and define $\Delta^{q,r}(\alpha, m)$ as the inner sum, i.e., the term in large parentheses.

We will calculate $\Delta^{q,r}$ separately for the two cases $q + r < n$ and $q + r \geq n$.

First case: $q + r < n$**.**

If $q + r < n$, then fewer than n opponents are sending message $m + 1$ or higher. This means that $\Pr(adv|m + 1, \mathbf{m}) = 1$; for a message profile $\mathbf{m}$ to show up in the sum, then, it must be that $p \geq n - q - r$ opponents send message m, since otherwise $\Pr(adv|m, \mathbf{m}) = 1$ as well. If $p \geq n - q - r$ opponents send message m, then $\Pr(adv|m, \mathbf{m}) = \frac{n-q-r}{p+1}$, since counting bidder 1, there are $p + 1$ bidders sending message m competing for the $n - q - r$ spots in the auction not taken by bidders sending even higher messages. Regardless of p, if bidder 1 advances, he will face the $q + r$ opponents who sent message $m + 1$ or higher, plus $n - 1 - (q + r)$ of the opponents who sent message m.

Putting it all together, then, for $q + r < n$,

$$
\begin{aligned}
\Delta^{q,r}(\alpha, m) &= \sum_{\mathbf{m} \in \mathcal{M}_{q,r}} \Pr(\mathbf{m}) \left[\Pr(adv|m + 1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \\
&= \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r} \binom{N-1-r}{q} \binom{N-1-r-q}{p} w^{N-1-p-q-r} x^p y^q z^r \left[1 - \frac{n-q-r}{p+1} \right] V^{q,r}
\end{aligned}
$$

where $z = 1 - \alpha_{m+1}$ is the probability (under τ) that a bidder sends a message higher than $m + 1$; $y = \alpha_{m+1} - \alpha_m$ the probability a bidder sends message $m + 1$; $x = \alpha_m - \alpha_{m-1}$ the probability a bidder sends message m; $w = \alpha_{m-1}$ the probability he sends a message lower than m; and $V^{q,r}$ the expected payoff to a bidder with type $S_1 = \alpha_m$ from advancing to an auction against r opponents with types above α_{m+1}, q opponents with types in $[\alpha_m, \alpha_{m+1}]$, and $n - 1 - q - r$ opponents with types in $[\alpha_{m-1}, \alpha_m]$.

Second case: $q + r \geq n$**.**

If $q + r \geq n$, at least n opponents are sending messages $m + 1$ or higher, so $\Pr(adv|m, \mathbf{m}) = 0$, and $\Pr(adv|m + 1, \mathbf{m}) = \frac{n-r}{q+1}$, since r spots in the auction are taken by opponents sending messages above $m + 1$, leaving $n - r$ for the $q + 1$ bidders sending $m + 1$. If bidder 1 advances, he faces the r bidders who sent messages above $k + 1$, plus $n - 1 - r$ opponents who sent message $m + 1$. How many of his opponents sent message m, versus lower messages, is irrelevant, since he never advances when he sends m. So

$$
\begin{aligned}
\Delta^{q,r}(\alpha, m) &= \sum_{\mathbf{m} \in \mathcal{M}_{q,r}} \Pr(\mathbf{m}) \left[\Pr(adv|m + 1, \mathbf{m}) - \Pr(adv|m, \mathbf{m}) \right] V(\alpha_m, \mathbf{m}) \\
&= \binom{N-1}{r} \binom{N-1-r}{q} (w + x)^{N-1-q-r} y^q z^r \frac{n-r}{q+1} V^{n-1-r,r}
\end{aligned}
$$

Next, we establish several key properties of our function Δ.

Lemma 5. *Fixing an environment,*

1. $\Delta(\alpha, m)$ *is a function only of* α_{m-1}, α_m, *and* α_{m+1}, *so we can write it as* $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$
2. $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ *is continuous in each of its arguments*
3. *There is* $\varepsilon > 0$ *such that when* $\alpha_m - \alpha_{m-1} \in (0, \varepsilon)$, $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) < 0$
4. *On* $[0, \alpha_m)$, $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ *is strictly single-crossing from above in* α_{m-1}

Together, these tell us that for given values of α_m and α_{m+1}, either $\Delta(0, \alpha_m, \alpha_{m+1}) \leq 0$ – in which case there is no value of $\alpha_{m-1} > 0$ that would satisfy $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = 0$ – or else $\Delta(0, \alpha_m, \alpha_{m+1}) > 0$, in which case there is a unique value of α_{m-1} satisfying $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = 0$.

Proof of Lemma 5. Parts 1 and 2 of Lemma 5 follow directly from the expressions for $\Delta^{q,r}(\alpha, m)$: for each q and r, $\Delta^{q,r}(\alpha, m)$ is a function only of α_{m-1}, α_m, and α_{m+1}, and is continuous in each of them, so Δ inherits these properties. For part 3, note from Lemma 5 that $u_2(s_i, s_j) > c$ requires $s_i > s_j + \varepsilon$. Then $V^{q,r} < 0$ for $q + r > 0$, and $V^{0,0} < 0$ whenever $\alpha_m - \alpha_{m-1} < \varepsilon$, and so $\Delta = \sum_{q,r} \Delta^{q,r} < 0$ if $\alpha_m - \alpha_{m-1} < \varepsilon$. ($\alpha_m - \alpha_{m-1} > 0$ is needed here because when $\alpha_{m-1} = \alpha_m = \alpha_{m+1}$, $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = 0$.) Part 4 (single-crossing) requires a lot of additional algebra, and is done separately at the end. □

With Δ defined, and given Lemma 5, define $\alpha^*(\alpha_m, \alpha_{m+1})$ as the solution to $\Delta(x, \alpha_m, \alpha_{m+1}) = 0$ on $[0, \alpha_{m-1})$, or as 0 when no such solution exists, and note that since Δ is continuous, α^* is continuous in both its arguments. There are two other useful facts we'll need:

Lemma 6. *For any environment and α,*

1. *If $\alpha_1 - \alpha_0 \geq \varepsilon$ and $\alpha_0 \approx 0$, $v_\tau(1; \alpha_0) < 0$*
2. *If $\Delta(0, \alpha_0, \alpha_1) > 0$, then $v_\tau(1; \alpha_0) > 0$*

Proof of Lemma 6. To show the first part, note that if $\alpha_0 \approx 0$, a bidder opting in has virtually no chance of entering the auction alone, and if $\alpha_1 - \alpha_0 \geq \varepsilon$, he has at least probability $\varepsilon^{N-1} \frac{n}{N}$ of being selected; a bidder with type α_0 is assured that any opponents have types at least as high as him, ensuring (via the small rents assumption) a strictly negative payoff from entering against any competition, making $v_\tau(1; \alpha_0)$ negative.

For the second part, note that

$$
\begin{aligned}
\Delta(0, \alpha_0, \alpha_1) \quad &= \quad \sum_{q+r<n} \sum \Delta^{q,r}(0, \alpha_0, \alpha_1) + \sum_{r<n,\, q+r \geq n} \sum \Delta^{q,r}(0, \alpha_0, \alpha_1) \\
&= \quad \sum_{q+r<n} \sum \left(\sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r} \binom{N-1-r}{q} \binom{N-1-r-q}{p} w^{N-1-p-q-r} x^p y^q z^r \left[1 - \frac{n-q-r}{p+1} \right] V^{q,r} \right) \\
&\quad + \sum_{r<n,\, q+r \geq n} \sum \left(\binom{N-1}{r} \binom{N-1-r}{q} (w+x)^{N-1-q-r} y^q z^r \frac{n-r}{q+1} V^{n-1-r,r} \right)
\end{aligned}
$$

where now $w = 0$, $x = \alpha_0$, $y = \alpha_1 - \alpha_0$, and $z = 1 - \alpha_1$. Since $w = 0$, all but the $p = N - 1 - q - r$ term of the innermost sum on the first line vanish (and $w + x$ becomes x), and so

$$
\begin{aligned}
\Delta(0, \alpha_0, \alpha_1) \quad &= \quad \sum_{q+r<n} \sum \left(P(q,r) \left[1 - \frac{n-q-r}{N-q-r} \right] V^{n-1-q-r,q,r} \right) \\
&\quad + \sum_{r<n,\, q+r \geq n} \sum \left(P(q,r) \frac{n-r}{q+1} V^{0,n-1-r,r} \right)
\end{aligned}
$$

where $P(q,r)$ is shorthand for $\binom{N-1}{r}\binom{N-1-r}{q} x^{N-1-q-r} y^q z^r$ and where we now write $V^{p,q,r}$ as the expected payoff in the auction given type α_0 when facing p opponents with types in $[0, \alpha_0]$, q in

$[\alpha_0, \alpha_1]$, and r in $[\alpha_1, 1]$. On the other hand, we can calculate $v_\tau(\alpha_0, 1)$ as

$$v_\tau(1; \alpha_0) \;\; = \;\; \sum\sum_{q+r<n} P(q,r) V^{0,q,r} + \sum\sum_{r<n,\, q+r\geq n} \left(P(q,r) \frac{n-r}{q+1} V^{0,n-1-r,r} \right)$$

Note that the second double-sum in $\Delta(0, \alpha_0, \alpha_1)$ and $v_\tau(\alpha_0, 1)$ are exactly the same, and the two expressions therefore differ in just two ways: the latter replaces $1 - \frac{n-q-r}{N-q-r}$ with 1, and $V^{n-1-q-r,q,r}$ with $V^{0,q,r}$, in each summand in the first double-sum. Now, fewer opponents certainly leads to a higher expected auction payoff, so this second change is an increase in value; we'll show that when $\Delta(0, \alpha_0, \alpha_1) \geq 0$, the first change must be an overall increase as well.

To show this, note first that out of all the $V^{n-1-q-r,q,r}$ and $V^{0,n-1-r,r}$ terms, the only one that isn't strictly negative (by the small-rents assumption) is $V^{n-1,0,0}$. This means the second double-sum is strictly negative, which means that if $\Delta(0, \alpha_0, \alpha_1) \geq 0$, the first sum must be strictly positive. Rewrite the first sum as

$$\sum\sum_{q+r<n} \left(P(q,r) \left[\frac{N-n}{N-q-r} \right] V^{n-1-q-r,q,r} \right)$$

and note that if this is strictly positive, then $\sum\sum P(q,r) V^{n-1-q-r,q,r}$ is strictly larger: it consists of multiplying each term by $\frac{N-q-r}{N-n} > 1$, with the term corresponding to $q = r = 0$ (which is the only positive term) therefore being multiplied by a strictly larger factor than all the other terms.

So going from $\Delta(0, \alpha_0, \alpha_1)$ to $v_\tau(1; \alpha_0)$ involves two changes: the first multiplies a bunch of terms (which collectively are positive) by factors greater than 1, with the only positive term getting multiplied by a larger factor than the others; the second replaces a bunch of terms $V^{n-1-q-r,q,r}$ with larger (less negative) $V^{0,q,r}$ terms. Thus, if $\Delta(0, \alpha_0, \alpha_1) > 0$, then $v_\tau(1; \alpha_0) > \Delta(0, \alpha_0, \alpha_1)$, giving the result. □

B.4 Proof of Theorem 1 (equilibrium construction)

Finally, we can launch into the actual construction. First, we define M^*.

1. For a given $M > 0$, define $\alpha^{(M)} = (\alpha_0^{(M)}, \alpha_1^{(M)}, \ldots, \alpha_M^{(M)})$ as follows:

   - Let $\alpha_{M-1}^{(M)} = \alpha_M^{(M)} = 1$
   - For $m = M-2, M-3, \ldots, 0$, define α_m recursively as $\alpha_m^{(M)} = \alpha^*(\alpha_{m+1}^{(M)}, \alpha_{m+2}^{(M)})$. (If at any point $\alpha_m = 0$, then stop and say the construction failed.)

2. Then define the pure strategy τ_M by $\tau_M(s_1) = 0$ for $s_1 \leq \alpha_0$ and, for $m = 1, 2, \ldots, M$, $\tau_M(s_1) = m$ for $s_1 \in (\alpha_{m-1}^{(M)}, \alpha_m^{(M)}]$.

   If $\alpha^{(M)}$ are all well-defined and greater than 0 and $v_{\tau_M}(1; \alpha_0^{(M)}) > 0$, we will say the construction succeeded; if some $\alpha_m^{(M)} = 0$ (meaning $\Delta(x, \alpha_{m+1}^{(M)}, \alpha_{m+2}^{(M)}) = 0$ had no solution) or $v_{\tau_M}(1; \alpha_0^{(M)}) \leq 0$, we will say the construction failed.

3. Define M^* as the largest value of M at which the construction succeeded.

   - Note that the construction must succeed at $M = 1$ (since at $\alpha_0 = \alpha_1 = 1$, $v_\tau(1; \alpha_0) = E(V_1 | S_1 = 1) - c > 0$ by assumption)

- Also note that it must fail at M sufficiently large (since as noted above, $v_\tau(m;\alpha_m) = v_\tau(m+1;\alpha_m)$ requires $\alpha_m - \alpha_{m-1} \geq \varepsilon$, and therefore when $M > 2 + \frac{1}{\varepsilon}$ the construction can't succeed)
- Finally, note that it must succeed at every $M \leq M^*$.
  For $M < M^*$, by construction, $\alpha^{(M)}_{M-m} = \alpha^{(M^*)}_{M^*-m}$; so $\alpha^{(M)}$ is well-defined, with $\alpha^{(M)}_1 = \alpha^{(M^*)}_{M^*-M+1}$ and $\alpha^{(M)}_0 = \alpha^{(M^*)}_{M^*-M}$.
  Since the construction works at M^*, $\alpha^*(\alpha^{(M^*)}_{M^*-M}, \alpha^{(M^*)}_{M^*-M+1}) = \alpha^{(M^*)}_{M^*-M-1} > 0$, and therefore $\Delta(0, \alpha^{(M^*)}_{M^*-M}, \alpha^{(M^*)}_{M^*-M+1}) > 0$, or $\Delta(0, \alpha^{(M)}_0, \alpha^{(M)}_1) > 0$; by part 2 of Lemma 6, then, $v_{\tau_M}(1;\alpha^{(M)}_0) > 0$.

Next, for a given $M \leq M^*$, do the following:

1. For $t \in [0,1]$, define $\alpha_M(t) = 1$ and $\alpha_{M-1}(t) = 1-t$.
2. For $m = M-2, M-3, \ldots, 0$, define $\alpha_m(t)$ recursively as $\alpha_m(t) = \alpha^*(\alpha_{m+1}(t), \alpha_{m+2}(t))$.
3. We will let v_α denote v_τ for the strategy τ defined by the thresholds α in the obvious way. Define $v(t) = v_{\alpha(t)}(1;\alpha_0(t))$.
4. Note that since α^* is continuous, each $\alpha_m(t)$ is continuous in t, at least until it hits 0
5. If we let $\bar{t}$ denote the value of t at which $\alpha_0(t)$ hits 0, then all of the $\alpha_m(t)$ are continuous on $[0,\bar{t}]$, and therefore $v(t)$ is continuous on $[0,\bar{t}]$ as well
6. Note that $v(0) > 0$.

   (If $M = M^*$, then $v(0) = v_{\tau_{M^*}}(1;\alpha^{(M^*)}_0) > 0$ by the original construction of M^*; if $M < M^*$, then in the original construction of M^*, $\alpha^*(\alpha_0(0), \alpha_1(0)) > 0$, or $\Delta(0, \alpha_0(0), \alpha_1(0)) > 0$, implying $v_{\alpha(0)}(1;\alpha_0(0)) > 0$ by the second part of Lemma 6.)
7. Note that $v(\bar{t}) < 0$.

   This is because $\alpha_0(\bar{t}) = 0$, and since $\Delta(\alpha_0(t), \alpha_1(t), \alpha_2(t)) = 0$, $\alpha_1(t) - \alpha_0(t) \geq \varepsilon$ for every t. So the first part of Lemma 6 holds at $\bar{t}$, meaning $v_{\alpha(\bar{t})}(1;\alpha_0(\bar{t})) < 0$.
8. Since $v(t)$ is continuous on $[0,\bar{t}]$, positive at 0, and negative at $\bar{t}$, it crosses 0; let t^* be the lowest value at which $v(t) = 0$, and define τ based on the thresholds $\alpha(t^*)$.
9. The recursive definition of $\alpha_m(t)$ ensures that $\Delta(\alpha_{m-1}(t^*), \alpha_m(t^*), \alpha_{m+1}(t^*)) = 0$ for $m = 1, 2, \ldots, M-1$, or that at τ defined by $\alpha(t^*)$, $v_\tau(m;\alpha_m(t^*)) = v_\tau(m+1;\alpha_m(t^*))$; the definition of t^* ensures $v_\tau(1;\alpha_0(t^*)) = 0$; so the indifference conditions all hold.
10. If $M = \overline{M}$, the second sufficient condition in Lemma 2 holds vacuously; so for $\overline{M} \leq M^*$, we have constructed an equilibrium when $M = \overline{M}$.
11. If $\overline{M} > M^*$, we need to show that at $M = M^*$, $v_\tau(M+1;1) \leq v_\tau(M;1)$. We do this below, completing the proof that we've found an equilibrium either way.

All that's left is to show that at $M = M^*$ and τ defined by the thresholds $\alpha(t^*)$, $v_\tau(M+1;1) \leq v_\tau(M;1)$. Consider a bidder with type $S_i = 1$ who considers sending an unused message above M instead of M. If we calculate $v_\tau(M+1;1) - v_\tau(M;1)$, it's the same expression for the difference as before, but with $\alpha_m = \alpha_{m+1} = 1$, and therefore $y = z = 0$; as a result, $\Delta^{q,r} = 0$ for $q + r > 0$, and therefore

$$v_\tau(M+1;1) - v_\tau(M;1) \quad = \quad \sum_{p=n}^{N-1} \binom{N-1}{p} w^{N-1-p} x^p \left[1 - \frac{n}{p+1}\right] V^{0,0}$$

Thus, $v_\tau(M+1;1) - v_\tau(M;1)$ has the same sign as

$$V^{0,0} \quad = \quad E\left\{u_n(1, S_{-i,n}) \mid S_{-i,n} \in [\alpha_{M-1}, 1]^{n-1}\right\} - c \quad = \quad V(1, n-1, [\alpha_{M-1}, 1])$$

the payoff to a bidder with type 1 facing $n-1$ opponents with types in $[\alpha_{M-1}, 1]$. (In words: unless at least n of my opponents sent message M, I would have advanced for sure anyway by sending message M; so the only time message $M+1$ gives me a different outcome from message M, it's when I'll face an auction full of opponents who sent message M, and therefore have types above α_{M-1}.)

Now, if $V(1, n-1, [0,1]) \leq 0$, then by monotonicity, $V(1, n-1, [\alpha_{M-1}, 1]) \leq 0$, and therefore $v_\tau(M+1;1) \leq v_\tau(M;1)$ and we're done. So suppose $V(1, n-1, [0,1]) > 0$, and define $\hat{S}$ as the solution to

$$V(1, n-1, [\hat{S}, 1]) \quad = \quad 0$$

(Continuity, monotonicity, and small rents ensure that there's a unique solution, strictly less than 1.) Again by monotonicity, $v_\tau(M+1;1) \leq v_\tau(M;1)$ if and only if $\alpha_{M-1} \geq \hat{S}$. Also, since $\Delta(\hat{S}, 1, 1) = 0$ by definition, $\hat{S} = \alpha^*(1,1)$.

Now, recall that we defined our thresholds based on $\alpha_M(t) = 1$, $\alpha_{M-1}(t) = 1 - t$, and $\alpha_m(t) = \alpha^*(\alpha_{m+1}(t), \alpha_{m+2}(t))$, and defined t^* as the lowest value of t at which $v_\tau(1; \alpha_0(t)) = 0$; so for $t < t^*$, $v_\tau(1; \alpha_0(t)) > 0$. Define $\hat{t}$ as $1 - \hat{S}$, so that $\hat{S} = 1 - \hat{t} = \alpha_{M-1}(\hat{t})$. Noting that $\alpha_{M-1}(\hat{t}) = \alpha^*(1,1) = \alpha_{M-2}(0)$, it's easy to show inductively that $\alpha_m(\hat{t}) = \alpha_{m-1}(0)$.

We'll show that $t^* \leq \hat{t}$, which ensures that $\alpha_{M-1} \geq \hat{S}$, which in turn ensures that $V^{0,0} = V(1, n-1, [\alpha_{M-1}, 1]) \leq 0$, and therefore that $v_\tau(M+1;1) \leq v_\tau(M;1)$. Suppose (toward contradiction) that $t^* > \hat{t}$. Since $\alpha_0(t)$ is continuous at least until t^*, this means that $\alpha_0(\hat{t})$ exists, and that $v_\tau(1; \alpha_0(\hat{t})) > 0$. But $\alpha_0(\hat{t}) = \alpha^*(\alpha_1(\hat{t}), \alpha_2(\hat{t})) = \alpha^*(\alpha_0(0), \alpha_1(0))$; if $v_\tau(1; \alpha_0(\hat{t})) > 0$, then $v_\tau(1; \alpha^*(\alpha_0(0), \alpha_1(0))) > 0$, contradicting the definition of M^* as the largest M for which this is possible.

Thus, when $M = M^*$, $t^* \leq \hat{t}$, which means that $\alpha_{M-1} \geq \hat{S}$, and therefore that $v_\tau(M+1;1) \leq v_\tau(M;1)$, completing the proof. □

B.5 Proof $\Delta(\cdot, \alpha_m, \alpha_{m+1})$ is Strictly Single Crossing

Earlier, we deferred the proof of Lemma 5 part 4, i.e., that $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1})$ is strictly single-crossing from above in α_{m-1} on $[0, \alpha_m)$. We prove that now. We will do this by showing that $\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) = 0$ implies $\frac{\partial}{\partial \alpha_{m-1}} \Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) < 0$.

We begin by writing

$$\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) \quad = \quad \sum\sum_{q+r<n} C^{q,r} V^{q,r} + \sum\sum_{r<n, q+r \geq n} D^{q,r} V^{n-1-r,r}$$

where

$$C^{q,r} = \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1}{r}\binom{N-1-r}{q}\binom{N-1-q-r}{p} w^{N-1-p-q-r} x^p y^q z^r \left[1 - \frac{n-q-r}{p+1}\right]$$

$$D^{q,r} = \binom{N-1}{r}\binom{N-1-r}{q}(w+x)^{N-1-q-r} y^q z^r \left[\frac{n-r}{q+1}\right]$$

and $w = \alpha_{m-1}$, $x = \alpha_m - \alpha_{m-1}$, $y = \alpha_{m+1} - \alpha_m$, $z = 1 - \alpha_{m+1}$, and $V^{q,r}$ denotes the expected payoff in an auction given type α_m and r opponents with types above α_{m+1}, q opponents with types in $[\alpha_m, \alpha_{m+1}]$, and $n-1-q-r$ opponents with types in $[\alpha_{m-1}, \alpha_m]$.

Now, differentiate with respect to α_{m-1}, which we will indicate with subscript-α_{m-1}:

$$\begin{aligned}\Delta_{\alpha_{m-1}}(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) &= \sum_{q+r<n}\sum C^{q,r}_{\alpha_{m-1}} V^{q,r} + \sum_{q+r<n}\sum C^{q,r} V^{q,r}_{\alpha_{m-1}} \\ &\quad + \sum_{r<n, q+r\geq n}\sum D^{q,r}_{\alpha_{m-1}} V^{n-1-r,r} + \sum_{r<n, q+r\geq n}\sum D^{q,r} V^{n-1-r,r}_{\alpha_{m-1}}\end{aligned}$$

First, note that $D^{q,r}$ does not depend on α_{m-1} at all, because its only dependence on either w or x is through $w + x = 1 - \alpha_m$. Second, note that $V^{n-1-r,r}$ does not depend on α_{m-1} either, since it's the payoff in an auction where all the opponents have types in either $[\alpha_m, \alpha_{m+1}]$ or $[\alpha_{m+1}, 1]$, which doesn't depend on α_{m-1} at all. Third, note that for $q + r < n$, $V^{q,r}$ is weakly decreasing in α_{m-1}, since it is the payoff in an auction where $n - 1 - q - r$ opponents have types randomly drawn from $[\alpha_{m-1}, \alpha_m]$, and stronger opponents mean lower payoffs. This means that

$$\Delta_{\alpha_{m-1}}(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) \leq \sum_{q+r<n}\sum C^{q,r}_{\alpha_{m-1}} V^{q,r}$$

where $C^{q,r}$ depends on α_{m-1} through both w (positively) and x (negatively).

Since our interest is in signing this when $\Delta = 0$, we calculate

$$\begin{aligned}\Delta_{\alpha_{m-1}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\Delta &\leq \sum_{q+r<n}\sum C^{q,r}_{\alpha_{m-1}} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum_{q+r<n}\sum C^{q,r} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum_{r<n,\, q+r\geq n}\sum D^{q,r} V^{q,r} \\ &= \sum_{0<q+r<n}\sum \left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right] C^{q,r} V^{q,r} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}} \sum_{r<n,\, q+r\geq n}\sum D^{q,r} V^{q,r}\end{aligned}$$

By small rents, $V^{q,r} < 0$ for $q + r > 0$. Also note that $C^{q,r}$ and $D^{q,r}$ are all weakly positive, since they're basically sums of probabilities. We'll show that $C^{0,0}_{\alpha_{m-1}} \leq 0$, establishing that

$$\Delta_{\alpha_{m-1}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\Delta \leq \sum_{0<q+r<n}\sum \left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right] C^{q,r} V^{q,r}$$

and then we'll show that the difference in square brackets is positive, ensuring the right-hand side is negative; thus guaranteeing that when $\Delta = 0$, $\Delta_{\alpha_{m-1}} < 0$.

Define $Z(q,r) = \binom{N-1}{r}\binom{N-1-r}{q} y^q z^r$, and write $C^{q,r}$ as

$$C^{q,r} \quad = \quad Z(q,r) \sum_{p=n-q-r}^{N-1-q-r} \binom{N-1-q-r}{p} w^{N-1-p-q-r} x^p \left[1 - \frac{n-q-r}{p+1}\right]$$

Note that $Z(q,r)$ does not depend on α_{m-1}, and $C^{q,r}$ depends on q and r only through $q+r$; letting $e \equiv q+r$, then,

$$C^{q,r} \quad = \quad Z(q,r) \sum_{p=n-e}^{N-1-e} \binom{N-1-e}{p} w^{N-1-e-p} x^p \left[1 - \frac{n-e}{p+1}\right]$$

Now, α_{m-1} effects $C^{q,r}$ through both w and x: specifically, since $w = \alpha_{m-1}$ and $x = \alpha_m - \alpha_{m-1}$, $dw = d\alpha_{m-1}$ and $dx = -d\alpha_{m-1}$, so

$$\begin{aligned} C^{q,r}_{\alpha_{m-1}} \quad = \quad & Z(q,r) \sum_{p=n-e}^{N-2-e} \frac{(N-1-e)!}{(N-1-e-p)!p!}(N-1-e-p) w^{N-2-e-p} x^p \left[1 - \frac{n-e}{p+1}\right] \\ & -Z(q,r) \sum_{p=n-e}^{N-1-e} \frac{(N-1-e)!}{(N-1-e-p)!p!} w^{N-1-e-p} p x^{p-1} \left[1 - \frac{n-e}{p+1}\right] \end{aligned}$$

(Note that the first sum only goes up to $N-2-e$, because the $p = N-1-e$ term had no w and therefore vanishes when we differentiate; on the other hand, since $e = q+r < n$, every term had a positive x power, so the entire sum survives in the second row.) Next, cancelling $N-1-e-p$ from the numerator and denominator in the first row, cancelling p from the numerator and denominator in the second row, and reindexing the sum by $p' = p-1$,

$$\begin{aligned} C^{q,r}_{\alpha_{m-1}} \quad = \quad & Z(q,r) \sum_{p=n-e}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p)!p!} w^{N-2-e-p} x^p \left[1 - \frac{n-e}{p+1}\right] \\ & -Z(q,r) \sum_{p'=n-e-1}^{N-2-e} \frac{(N-1-e)!}{(N-2-e-p')!p'!} w^{N-2-e-p'} x^{p'} \left[1 - \frac{n-e}{p'+2}\right] \end{aligned}$$

If we separate the $n-e-1$ term from the second sum, and combine corresponding terms otherwise,

$$
\begin{aligned}
C^{q,r}_{\alpha_{m-1}} &= Z(q,r)\sum_{p=n-e}^{N-2-e}\frac{(N-1-e)!}{(N-2-e-p)!p!}w^{N-2-e-p}x^p\left[\frac{n-e}{p+2}-\frac{n-e}{p+1}\right] \\
&\quad -Z(q,r)\sum_{p=n-e-1}^{n-e-1}\frac{(N-1-e)!}{(N-2-e-p)!p!}w^{N-2-e-p}x^p\left[1-\frac{n-e}{p+2}\right] \\
&= Z(q,r)\sum_{p=n-e-1}^{N-2-e}\frac{(N-1-e)!}{(N-2-e-p)!p!}w^{N-2-e-p}x^p\left[\frac{n-e}{p+2}-\frac{n-e}{p+1}\right] \\
&= Z(q,r)\sum_{p=n-e-1}^{N-2-e}\frac{(N-1-e)!}{(N-2-e-p)!p!}w^{N-2-e-p}x^p\left[-\frac{n-e}{(p+1)(p+2)}\right] \\
&= -Z(q,r)\frac{n-e}{N-e}\sum_{p=n-e-1}^{N-2-e}\frac{(N-e)!}{(N-2-e-p)!(p+2)!}w^{N-2-e-p}x^p \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e-1}^{N-2-e}\frac{(N-e)!}{(N-2-e-p)!(p+2)!}w^{N-2-e-p}x^{p+2} \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e-1}^{N-2-e}\binom{N-e}{p+2}w^{N-2-e-p}x^{p+2} \\
&= -Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e+1}^{N-e}\binom{N-e}{p}w^{N-e-p}x^p
\end{aligned}
$$

Note that this is negative for any $e < n$, including $e = q + r = 0$, so $C^{0,0}_{\alpha_{m-1}} < 0$, as noted above.

What's left to show is that $\left[\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}\right]$ is positive. Before we do that, rewrite

$$
\begin{aligned}
C^{q,r} &= Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p\left[1-\frac{n-e}{p+1}\right] \\
&= Z(q,r)\left[\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p - \frac{n-e}{N-e}\sum_{p=n-e}^{N-1-e}\binom{N-e}{p+1}w^{N-1-e-p}x^p\right] \\
&= Z(q,r)\left[\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p - \frac{n-e}{N-e}\frac{1}{x}\sum_{p'=n-e+1}^{N-e}\binom{N-e}{p'}w^{N-e-p'}x^{p'}\right] \\
&= Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p + xC^{q,r}_{\alpha_{m-1}}
\end{aligned}
$$

Then, calculate

$$
\begin{aligned}
&C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}} \\
&= \left(Z(0,0)\sum_{p=n}^{N-1}\binom{N-1}{p}w^{N-1-p}x^p\right)\left(-Z(q,r)\frac{n-e}{N-e}\frac{1}{x^2}\sum_{p=n-e+1}^{N-e}\binom{N-e}{p}w^{N-e-p}x^p\right) \\
&\quad -xC^{0,0}_{\alpha_{m-1}}C^{q,r}_{\alpha_{m-1}} \\
&\quad -\left(Z(q,r)\sum_{p=n-e}^{N-1-e}\binom{N-1-e}{p}w^{N-1-e-p}x^p\right)\left(-Z(0,0)\frac{n}{N}\frac{1}{x^2}\sum_{p=n+1}^{N}\binom{N}{p}w^{N-p}x^p\right) \\
&\quad +xC^{q,r}_{\alpha_{m-1}}C^{0,0}_{\alpha_{m-1}} \\
&= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\left(\sum_{i=n}^{N-1}\binom{N-1}{i}w^{N-1-i}x^i\right)\left(\frac{n-e}{N-e}\sum_{j=n-e+1}^{N-e}\binom{N-e}{j}w^{N-e-j}x^j\right)\right. \\
&\quad \left. -\left(\sum_{j=n-e}^{N-1-e}\binom{N-1-e}{j}w^{N-1-e-j}x^j\right)\left(\frac{n}{N}\sum_{i=n+1}^{N}\binom{N}{i}w^{N-i}x^i\right)\right]
\end{aligned}
$$

Reindexing the last three sums,

$$C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$$

$$= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\left(\sum_{i=n}^{N-1}\binom{N-1}{i}w^{N-1-i}x^i\right)\left(\frac{n-e}{N-e}\sum_{j'=n}^{N-1}\binom{N-e}{j'+1-e}w^{N-1-j'}x^{j'+1-e}\right)\right.$$

$$\left.-\left(\sum_{j'=n}^{N-1}\binom{N-1-e}{j'-e}w^{N-1-j'}x^{j'-e}\right)\left(\frac{n}{N}\sum_{i'=n}^{N-1}\binom{N}{i'+1}w^{N-1-i'}x^{i'+1}\right)\right]$$

$$= \frac{-Z(0,0)Z(q,r)}{x^2}\left[\frac{1}{x^{e-1}}\frac{n-e}{N-e}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\binom{N-1}{i}\binom{N-e}{j+1-e}w^{N-1-i}x^iw^{N-1-j}x^j\right.$$

$$\left.-\frac{1}{x^{e-1}}\frac{n}{N}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\binom{N}{i+1}\binom{N-1-e}{j-e}w^{N-1-i}x^iw^{N-1-j}x^j\right]$$

$$= \frac{-Z(0,0)Z(q,r)}{x^{e+1}}\sum_{i=n}^{N-1}\sum_{j=n}^{N-1}\left[\frac{n-e}{N-e}\binom{N-1}{i}\binom{N-e}{j+1-e}-\frac{n}{N}\binom{N}{i+1}\binom{N-1-e}{j-e}\right]w^{2N-2-i-j}x^{i+j}$$

Next, we rewrite the double-sum, pairing the (i,j) term with the (j,i) term and separating the "diagonal" $(i=j)$ terms:

$$C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$$

$$= \frac{-Z(0,0)Z(q,r)}{x^{e+1}}\left(\sum_{i=n}^{N-1}\left[\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{i+1-e}-\tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{i-e}\right]w^{2N-2-2i}x^{2i}\right.$$

$$\left.+\sum\sum_{i<j}\left[\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e}+\tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e}-\tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e}-\tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e}\right]w^{2N-2-i-j}x^{i+j}\right)$$

Next, we show the summand in the first sum is negative:

$$\frac{n-e}{N-e}\binom{N-1}{i}\binom{N-e}{i+1-e} - \frac{n}{N}\binom{N}{i+1}\binom{N-1-e}{i-e}$$

$$\begin{aligned}
&= \frac{n-e}{N-e}\frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e)!}{(N-i-1)!(i+1-e)!} - \frac{n}{N}\frac{N!}{(N-i-1)!(i+1)!}\frac{(N-1-e)!}{(N-1-i)!(i-e)!} \\
&= \frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e-1)!}{(N-i-1)!(i-e)!}\left[(n-e)\frac{1}{i+1-e} - n\frac{1}{i+1}\right] \\
&\propto (n-e)(i+1) - n(i+1-e) \\
&= ni - ei + n - e - ni - n + ne \\
&= -ei - e + ne
\end{aligned}$$

which is negative since $i \geq n$ (the sum runs from $i = n$ to $i = N-1$).

Next, we show the summand in the second sum is negative, which takes a bit more work:

$$\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e} + \tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e} - \tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e} - \tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e}$$

$$\begin{aligned}
&= \frac{n-e}{N-e}\frac{(N-1)!}{(N-1-i)!i!}\frac{(N-e)!}{(N-1-j)!(j+1-e)!} + \frac{n-e}{N-e}\frac{(N-1)!}{(N-1-j)!j!}\frac{(N-e)!}{(N-1-i)!(i+1-e)!} \\
&\quad -\frac{n}{N}\frac{N!}{(N-i-1)!(i+1)!}\frac{(N-1-e)!}{(N-1-j)!(j-e)!} - \frac{n}{N}\frac{N!}{(N-j-1)!(j+1)!}\frac{(N-1-e)!}{(N-1-i)!(i-e)!} \\
&= \frac{(N-1)!}{(N-1-i)!}\frac{(N-1-e)!}{(N-1-j)!}\left[(n-e)\frac{1}{i!}\frac{1}{(j+1-e)!} + (n-e)\frac{1}{j!}\frac{1}{(i+1-e)!}\right. \\
&\quad \left. -n\frac{1}{(i+1)!}\frac{1}{(j-e)!} - n\frac{1}{(j+1)!}\frac{1}{(i-e)!}\right]
\end{aligned}$$

Dividing by $\frac{(N-1)!}{(N-1-i)!}\frac{(N-1-e)!}{(N-1-j)!}$ and multiplying by $(i+1)!(j+1)!$,

$$
\begin{aligned}
&\tfrac{n-e}{N-e}\tbinom{N-1}{i}\tbinom{N-e}{j+1-e}+\tfrac{n-e}{N-e}\tbinom{N-1}{j}\tbinom{N-e}{i+1-e}-\tfrac{n}{N}\tbinom{N}{i+1}\tbinom{N-1-e}{j-e}-\tfrac{n}{N}\tbinom{N}{j+1}\tbinom{N-1-e}{i-e} \\
\propto\quad & (n-e)\frac{(i+1)!}{i!}\frac{(j+1)!}{(j+1-e)!}+(n-e)\frac{(j+1)!}{j!}\frac{(i+1)!}{(i+1-e)!}-n\frac{(j+1)!}{(j-e)!}-n\frac{(i+1)!}{(i-e)!} \\
=\quad & (n-e)(i+1)\frac{(j+1)!}{(j+1-e)!}-n\frac{(j+1)!}{(j-e)!}+(n-e)(j+1)\frac{(i+1)!}{(i+1-e)!}-n\frac{(i+1)!}{(i-e)!} \\
=\quad & \frac{(j+1)!}{(j-e+1)!}\left[(n-e)(i+1)-n(j-e+1)\right]+\frac{(i+1)!}{(i+1-e)!}\left[(n-e)(j+1)-n(i-e+1)\right] \\
=\quad & \frac{(j+1)!}{(j-e+1)!}\left[ni-ei+n-e-nj+ne-n\right]+\frac{(i+1)!}{(i+1-e)!}\left[nj-ej+n-e-ni+ne-n\right] \\
=\quad & \frac{(j+1)!}{(j-e+1)!}\left[n(i-j)-e(i+1-n)\right]+\frac{(i+1)!}{(i+1-e)!}\left[n(j-i)-e(j+1-n)\right] \\
=\quad & n(i-j)\left(\frac{(j+1)!}{(j-e+1)!}-\frac{(i+1)!}{(i+1-e)!}\right)-\frac{(j+1)!}{(j-e+1)!}e(i+1-n)-\frac{(i+1)!}{(i+1-e)!}e(j+1-n)
\end{aligned}
$$

This, it turns out, is the sum of three negative terms. If $i > j$, then $\frac{(j+1)!}{(j-e+1)!} < \frac{(i+1)!}{(i+1-e)!}$, and if $i < j$, then $\frac{(j+1)!}{(j-e+1)!} > \frac{(i+1)!}{(i+1-e)!}$; either way, the first term is negative. And since both sums run from $i, j = n$ upwards, the last two terms are both negative as well.

So all the summands in both sums are negative, so the big double sum is negative; so $C^{0,0}C^{q,r}_{\alpha_{m-1}} - C^{q,r}C^{0,0}_{\alpha_{m-1}}$, which is $\frac{-Z(0,0)Z(q,r)}{x^{e+1}}$ times that big double sum, is positive. That means $\frac{C^{q,r}_{\alpha_{m-1}}}{C^{q,r}} - \frac{C^{0,0}_{\alpha_{m-1}}}{C^{0,0}}$ is positive, which was all that was left in proving $\Delta_{\alpha_{m-1}} < 0$ when $\Delta = 0$. Thus, Δ is strictly single-crossing from above in its first term, as claimed. □

B.6 Essential Uniqueness of Symmetric Equilibrium

We note in the text two special cases in which symmetric equilibrium is essentially unique:

1. $V_i = u(S_i)$, with u increasing and weakly convex, and
2. $n = 2$ and $V_i = \beta S_i + T_i$, with $\{T_i\}$ independent of $\{S_i\}$

(The former is isomorphic to a model where $V_i = S_i$ but $\{S_i\}$ are drawn independently from a general distribution F rather than the uniform distribution, which is how the model was presented in earlier versions of this paper. The restriction here that u is convex corresponds to the restriction there that the distribution F has a decreasing density function.)

Uniqueness in these two cases was proved in earlier versions of this paper. The key step of the proof is to apply a change of variables to

$$\Delta(\alpha_{m-1}, \alpha_m, \alpha_{m+1}) \quad = \quad v_\tau(m+1; \alpha_m) - v_\tau(m; \alpha_m)$$

and write it as a function of $x = \alpha_m - \alpha_{m-1}$, $y = \alpha_{m+1} - \alpha_m$, and $z = 1 - \alpha_{m+1}$. In the two special cases, we can show that written in this way, Δ is not only strictly single-crossing from below in x, but also strictly single-crossing from above in both y and z. This in turn implies that when we define $\alpha^*(\alpha_m, \alpha_{m+1})$ as the unique value of α_{m-1} satisfying $v_\tau(m+1;\alpha_m) = v_\tau(m;\alpha_m)$, that $\alpha_m - \alpha^*$ increases monotonically as $\alpha_{m+1} - \alpha_m$ and $1 - \alpha_{m+1}$ increase. In turn, this implies that as t increases in the construction of equilibrium thresholds, each threshold $\alpha_m(t)$ monotonically falls, and each interval $\alpha_{m+1}(t) - \alpha_m(t)$ monotonically gets wider. A similar single-crossing property of $v_\tau(1;\alpha_0)$, written as a function of $\alpha_1 - \alpha_0$ and $1 - \alpha_1$, then ensures that the value of t satisfying $v(t) = 0$ in the equilibrium construction is unique – or that given M, the thresholds satisfying the M indifference conditions are unique.

The same single-crossing properties can also be used to show that the "no-incentive-to-separate" condition $v_\tau(M+1;1) \leq v_\tau(M;1)$ cannot hold when $M < M^*$, and therefore that no symmetric equilibrium can exist when $M < \min\{M^*, \overline{M}\}$. Finally, the same conditions can be used to show that no series of thresholds α can satisfy the M indifference conditions when $M > M^*$. Thus, in the two special cases, the number of messages in the support of τ and the equilibrium partition are uniquely determined; since these determine equilibrium strategies up to the messages sent by threshold types, symmetric equilibrium is essentially unique.

B.7 Proof of Bidder Surplus Results

Finally, in the text, we mentioned (footnote 27) that when $V_i = u(S_i) + T_i$ with u differentiable and N is sufficiently large, bidder surplus is higher with indicative bidding than without, decreasing in n, and decreases when a reserve price is used.

To show this, first recall that as in the earlier calculation of bidder surplus, when $M = 1$ and N gets large, we can write $v_\tau(1;s_i)$ as

$$\begin{aligned} v_\tau(1;s_i) &\approx \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \left(V(s_i,k,[\alpha_0,1]) - V(\alpha_0,k,[\alpha_0,1])\right) \\ &\quad + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \left(V(s_i,n-1,[\alpha_0,1]) - V(\alpha_0,n-1,[\alpha_0,1])\right) \\ &= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \int_{\alpha_0}^{s_i} V'(s,k,[\alpha_0,1])ds + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \int_{\alpha_0}^{s_i} V'(s,n-1,[\alpha_0,1])ds \end{aligned}$$

where $V(s_i,k,[\alpha_0,1])$ is the expected payoff to a bidder with type s_i in an auction with k opponents with types uniformly distributed on $[\alpha_0,1]$.

Next, we calculate the derivative of $V(s_i,k,[\alpha_0,1])$ with respect to s_i. Now,

$$V(s_i,k,[\alpha_0,1]) \quad = \quad E_{T_i,\{T_j\},\{S_j\}|\{S_j\}\in[\alpha_0,1]} \max\{0, u(s_i) + T_i - \max\{u(S_j) + T_j\}\} \; - \; c$$

For given realizations of $\{S_j\}$, T_i, and $\{T_j\}$, this maximum has derivative $u'(s_i)$ if $u(s_i) + T_i \geq \max_{j \neq i}\{u(S_j) + T_j\}$, and derivative 0 if not; so taking the expectation over $\{S_j\}$ and all the T, V' is exactly $u'(s_i)$ times the probability, given s_i, that bidder i wins the auction. If the joint distribution of all the T is nondegenerate (in the sense of having continuous, bounded density) then as $\alpha_0 \to 1$, the variation in $\{T\}$ will swamp differences among $\{S\}$, and by symmetry, this probability will simply be $\frac{1}{k+1}$. We can therefore further simplify

$$\begin{aligned}
v_\tau(1; s_i) &\approx \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{k!} \int_{\alpha_0}^{s_i} \frac{u'(s)}{k+1} ds + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \frac{n}{k+1} \int_{\alpha_0}^{s_i} \frac{u'(s)}{n} ds \\
&= \sum_{k=0}^{n-1} \frac{\lambda^k e^{-\lambda}}{(k+1)!}(u(s_i) - u(\alpha_0)) + \sum_{k=n}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k+1)!}(u(s_i) - u(\alpha_0)) \\
&= \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1} e^{-\lambda}}{(k+1)!}(u(s_i) - u(\alpha_0)) \\
&= \frac{1}{\lambda}(1 - e^{-\lambda})(u(s_i) - u(\alpha_0))
\end{aligned}$$

Combined ex ante surplus of all the bidders, then, is

$$N \cdot E_{S_i} \max\{0, v_\tau(1; S_i)\} = N \int_{\alpha_0}^{1} \frac{1}{\lambda}(1 - e^{-\lambda})(u(s_i) - u(\alpha_0)) ds_i$$

As $\alpha_0 \to 1$, $u(s_i) - u(\alpha_0) \approx (s_i - \alpha_0)u'(s_i) \approx (s_i - \alpha_0)u'(1)$, so

$$\begin{aligned}
N \cdot E_{S_i} \max\{0, v_\tau(1; S_i)\} &\approx N \int_{\alpha_0}^{1} \frac{1}{\lambda}(1 - e^{-\lambda})(s_i - \alpha_0)u'(1) ds_i \\
&= \frac{N}{\lambda}(1 - e^{-\lambda})\frac{1}{2}(1 - \alpha_0)^2 u'(1) \\
&\approx \frac{N}{\lambda}(1 - e^{-\lambda})\frac{1}{2}\left(\frac{\lambda}{N}\right)^2 u'(1) \\
&= \frac{\lambda}{2N}(1 - e^{-\lambda})u'(1)
\end{aligned}$$

We already knew this was 0 in the limit; but for finite but large N (such that the approximation is valid but the term is not yet 0), this is strictly increasing in λ, and independent of n other than through λ. Since λ decreases with n, this means that bidder surplus is strictly decreasing in n, and higher with indicative bidding than without, for large finite N; similarly, λ decreases with a reserve price, and increases with a bidder subsidy, and therefore bidder surplus does as well.

(The proof is different when $V_i = u(S_i)$ or $\{T_i\}$ are perfectly correlated, since without a nondegenerate distribution of $\{T_i\}$, $V'(\cdot, k, \cdot)$ is $u'(s_i)$ when $s_i \geq \max\{S_j\}$ and 0 otherwise, rather than being approximately $\frac{u'(s_i)}{k+1}$ either way. The proof for that case was in a previous version of this paper, the result is the same.)