# Online Appendix

## Wait-and-See or Step in? Dynamics of Interventions

D. Foarta and T. Sugaya

# 1 Proofs of Results from Main Text

## 1.1 Proof of Lemma 1

Our objective is to maximize the objective given in (7) subject to $\sigma_A \in \Sigma_A^* (\sigma_p)$. Notice that, for each $\sigma_P$, if $A$ is indifferent between two pure strategies (of the dynamic game), then we can arbitrarily pick the one better for $P$. In addition, the past effort levels or types do not directly affect $P$'s or $A$'s payoffs. Thus, for each $\sigma_A$ which depends on the past types or efforts, there exists another $\sigma'_A$ which (i) does not depend on those variables, (ii) solves (6), and (iii) brings $P$ the same payoff. Hence, without loss of generality, we assume that the agent takes pure strategies that do not depend on past effort levels or types. Now, the relaxed problem becomes (7). Therefore, it remains to show that the solution to (7) constitutes a PBE.

Given the agent's problem (6), the effort level by the $H$-type agent in period $t$ after $(h_t, z_t, \rho_t)$ depends only on the continuation payoff after each possible realization of $o_t$ conditional on $\theta_t = H$:

$$(w(\sigma_P, h_t, z_t, \rho_t, \iota_t, o_t))_{o_t} \equiv \left( \max_{\sigma_A} \mathbb{E} \left[ \sum_{\tau=t+1}^{\infty} \prod\nolimits_{j=t+1}^{\tau} \rho_j \delta^{\tau - t} u(e_\tau) \,|\sigma_P, \sigma_A, h_t, z_t, \rho_t, \iota_t, \{\theta_t = H\}, o_t \right] \right)_{o_t}. \quad (1)$$

By feasibility, all of them are included in $[0, 1/(1-\delta)]$.

For each $(h_t, z_t, \rho_t, \iota_t)$ and $(w_o)_o$ satisfying $w_o \in [0, 1/(1-\delta)]$ for each $o$, there exists $\sigma_P^*$ such that (i) $\sigma_P^*$ guarantees that the agent's payoff equals $(w_o)_o$ : $w(\sigma_P^*, h_t, z_t, \rho_t, \iota_t, o_t) = w_{o_t}$ after each $o_t$; and (ii) $\sigma_P^*$ guarantees that the principal's continuation payoff given each $h^{t+1}$ is no less than $\underline{v}_P$, where

$$\underline{v}_P \equiv \frac{1}{1-\delta} \cdot (-\Pr(b|0) \cdot l - \Pr((g, B)|0) \cdot L). \quad (2)$$

We construct such a $\sigma_P^*$ as follows: Given $w_o$, the principal calculates the probability $\alpha_o$ such that $\alpha_o \cdot 1/(1-\delta) = w_o$. Using the public randomization at the beginning of period $t+1$, the principal keeps the agent forever (regardless of the future outcomes) with probability $\alpha_o$ and replaces him with probability $1 - \alpha_o$ after $o_t = o$. In the latter case, the principal replaces the future agents after one period regardless of the outcome. The principal always intervenes. Given such $\sigma_P^*$, the $H$-type agent does not provide any effort after each history.

Hence, the principal's continuation payoff in (7) is no less than $\underline{v}_P$ after each history $h_t$.[1] It remains to show that there exists a punishment equilibrium such that the principal's payoff given $h_{t+1}$ is no more than $\underline{v}_P$.

Consider the following strategy profile: the principal replaces the agent and intervenes after each history, and the $H$-type agent does not provide any effort after each history. Clearly this strategy profile is a mutual best response. Moreover, the principal's payoff is no more than $\underline{v}_P$ after each history $h^t$, $(h^t, z_t)$, or $(h^t, z_t, \rho_t)$; and no more than $\mathbb{E}\left[u|e=0,\iota_t\right] + \delta \cdot \underline{v}_P \leq \underline{v}_P$ after each history $(h^t, z_t, \rho_t, \iota_t)$, as desired.

## 1.2 Proof of Lemma 2

**Monotonicity with respect to $\mu$.**

Suppose that $J(\mu) = J$ for some $\mu$. Then, for a higher value $\mu' > \mu$, we have $J(\mu') \geq J$. To see why, if $(\rho, \iota)$ is a feasible policy when the belief is $\mu$, then it is also feasible when the belief is $\mu'$. Recall that the instantaneous utility for $P$ given $\iota$ is

$$\Pr(s=g|\mu) \cdot u^P(0|\mu,e,s) + \Pr(s=b|\mu) \cdot u^P(\iota(s=b)|\mu,e,s)\,, \tag{3}$$

which is (weakly) increasing in $\mu$. Hence, by the standard arguments (Stokey, 1989) , $J(\mu)$ is (weakly) increasing in $\mu$.

**Convexity with respect to $\mu$.**

Let $J(\mu,\theta)$ be the payoff when $P$ follows the optimal strategy given $\mu$, and the current type is $\theta \in \{H, L\}$. Then,

$$J(\mu) = \mu \cdot J(\mu,H) + (1-\mu) \cdot J(\mu,L) = J(\mu,L) + \mu \cdot \left[J(\mu,H) - J(\mu,L)\right]. \tag{4}$$

Take $\mu, \mu_1, \mu_2$ and $\beta \in [0,1]$ such that $\mu = \beta \cdot \mu_1 + (1-\beta) \cdot \mu_2$. For $n \in \{1,2\}$, by taking the strategy given $\mu$ when the belief is $\mu_n$, $P$ obtains

$$J(\mu,L) + \mu_n \cdot \left[J(\mu,H) - J(\mu,L)\right] \leq J(\mu_n)\,. \tag{5}$$

Hence,

$$\begin{aligned}
&\beta \cdot J(\mu_1) + (1-\beta) \cdot J(\mu_2) \\
&\qquad \geq \beta \cdot J(\mu,L) + \beta \cdot \mu_1 \cdot \left[J(\mu,H) - J(\mu,L)\right] \\
&\qquad + (1-\beta) \cdot J(\mu,L) + (1-\beta) \cdot \mu_2 \cdot \left[J(\mu,H) - J(\mu,L)\right] \\
&\qquad = J(\mu,L) + \mu \cdot \left[J(\mu,H) - J(\mu,L)\right] = J(\mu)\,.
\end{aligned} \tag{6}$$

[1] Otherwise, replace the principal's continuation strategy from $h_t$ with $\sigma_P^*$; this change improves her continuation payoff from $h_t$ without affecting the agent's incentive before period $t$.

## 1.3 Proof of Lemma 4

After $s = b$, the principal does not intervene if

$$\frac{l}{C} \leq \Pr(H|\mu, s = b) \cdot \Pr(B|H, b) + \Pr(L|\mu, s = b) \cdot \Pr(B|L, b), \tag{7}$$

where $\theta = H$ implies $e = 1$ and $\theta = L$ implies $e = 0$. This inequality reduces to

$$\mu \geq \mu_S := \frac{\Pr(b, B|L) \cdot \left(1 - \frac{l}{C}\right)}{\left(\Pr(b, B|L) - \Pr(b, B|H)\right) \cdot \left(1 - \frac{l}{C}\right) + \frac{l}{C} \cdot \Pr(b, G|H)}. \tag{8}$$

## 1.4 Proof of Proposition 1

After $s = b$, by Bayes' rule, the belief is updated to

$$\mu_b = \frac{\mu}{\mu + (1 - \mu) \cdot \frac{\Pr(b,B|L)+\Pr(b,G|L)}{\Pr(b,B|H)+\Pr(b,G|H)}}. \tag{9}$$

Writing $u^\theta(\iota)$ as the expected payoff from $\iota \in \{0, 1\}$ if $s = b$ and the agent is of type $\theta$, the principal's problem given $s = b$ is

$$\begin{aligned} &\mu_b \cdot u^H(\iota) + (1 - \mu_b) \cdot u^L(\iota) \\ &\qquad + \delta \cdot \iota \cdot J(\mu_b) + \delta \cdot (1 - \iota) \cdot \left(\alpha \cdot \mu_b \cdot J(1) + (1 - \alpha \cdot \mu_b) \cdot J(\mu')\right), \end{aligned} \tag{10}$$

where $\mu'$ is the belief after $s = b$ and $y = B$:

$$\mu' = \frac{\mu \cdot \Pr(b, B|H)}{\mu \cdot \Pr(b, B|H) + (1 - \mu) \cdot \Pr(b, B|L)}. \tag{11}$$

Notice that when $\mu \geq \mu^S$, $\iota = 0$ is optimal by Lemma 4. When $\mu = \Delta$, where $\Delta \to 0$, we have $\mu' \to 0$. Thus, $\iota = 1$ is optimal for $\mu \to 0$. Therefore, we can establish that intervention is optimal for $\mu$ sufficiently small, and no intervention is optimal for $\mu$ sufficiently large.

Since the prior $\mu$ and the interim belief $\mu_b$ have a monotone relationship, it suffices to show that there exists $\mu_b^* \in (0, 1)$ such that the intervention after $s = b$ is optimal if and only if $\mu_b \leq \mu_b^*$ for some $\mu_b^* \in (0, 1)$.

Notice also that $\mu'$ is increasing in $\mu$. Hence, $\mu' \leq \mu_H$ for all $\mu \leq \mu^S$ whenever the following condition is satisfied:

$$\frac{1 - \mu^S}{\mu^S} \geq \frac{1 - \mu_H}{\mu_H} \cdot \frac{\Pr(b, B|H)}{\Pr(b, B|L)}, \tag{12}$$

where $\mu^S$ is derived in (8). Thus, substituting for $\mu^S$, the above condition reduces to:

$$\frac{\Pr(b, B|H)}{\Pr(b, G|H)} \leq \frac{\frac{l}{C}}{1 - \frac{l}{C}} \cdot \mu_H. \tag{13}$$

Therefore, with the upper bound $l/C \cdot (1 - l/C)^{-1} \mu_H$ on $\Pr(b, B|H)/\Pr(b, G|H)$, we have $J(\mu') = J(\mu_H)$ for each $\mu \le \mu^S$. Hence, for each $\mu \le \mu^S$, in (10), the expression

$$\mu_b \cdot u^H(\iota) + (1 - \mu_b) \cdot u^L(\iota) + \delta \cdot (1 - \iota) \cdot (\alpha \cdot \mu_b \cdot J(1) + (1 - \alpha \cdot \mu_b) \cdot J(\mu')) \tag{14}$$

is linear in $\mu_b$, while $J(\mu_b)$ is convex in $\mu_b$. Together with the facts that (i) at $\mu \ge \mu^S$, no intervention is optimal and (ii) at $\mu_b = 0$, intervention is optimal, there exists a unique $\mu_b^*$ such that, conditional on $s = b$, intervention is optimal if and only if $\mu_b \le \mu_b^*$.

## 1.5 Proof of Lemma 5

We show that $J(V)$ is concave in $V$. Suppose $V = \beta \cdot V_1 + (1 - \beta) \cdot V_2$ for $V_1, V_2, \beta \in [0, 1]$; and let $\alpha[V_1]$ and $\alpha[V_2]$ be the optimal policies for $(V_1)$ and $(V_2)$, respectively. Suppose $P$ chooses $\alpha[V_1]$ with probability $\beta$ and $\alpha[V_2]$ with probability $1 - \beta$, according to the realization of the public randomization device.

1. Since $\alpha[V_1]$ delivers $V_1$ to the agent and $\alpha[V_2]$ delivers $V_2$, the agent's expected payoff is $V = \beta \cdot V_1 + (1 - \beta) \cdot V_2$. Hence, promise keeping is satisfied.

2. Conditional on the realization of the public randomization device, since both $\alpha[V_1]$ and $\alpha[V_2]$ are incentive compatible, the agent's incentive compatibility is satisfied.

3. With probability $\beta$, the principal achieves $J(V_1)$, and with probability $1 - \beta$, she achieves $J(V_2)$, since we fixed $\mu$. Hence she achieves $\beta \cdot J(V_1) + (1 - \beta) \cdot J(V_2)$.

Hence, the principal with $V$ achieves at least $\beta \cdot J(V_1) + (1 - \beta) \cdot J(V_2)$.

## 1.6 Proof of Proposition 2

Fix an equilibrium. Suppose there exist period $\bar{t}$ (of the current agent's tenure), history $\bar{h}^{\bar{t}}$, and public randomization $\bar{z}_{\bar{t}}$ such that (i) $e(\bar{h}^{\bar{t}}, \bar{z}_{\bar{t}}) = 0$ and (ii) there exists $\tau > \bar{t}$, history $h^\tau$ that is a continuation of $\left(h^{\bar{t}}, \bar{z}_{\bar{t}}\right)$, and public randomization $z_\tau$ satisfying $e(h^\tau, z_\tau) = 1$.

We show that there exists another equilibrium such that (i) it coincides with the original equilibrium up to period $\bar{t}$, and also after $\left(h^{\bar{t}}, z_{\bar{t}}\right) \neq \left(\bar{h}^{\bar{t}}, \bar{z}_{\bar{t}}\right)$, and (ii) after $\left(\bar{h}^{\bar{t}}, \bar{z}_{\bar{t}}\right)$, $P$ again draws a binary public randomization. After the first realization of the binary draw, the equilibrium is as if we skip period $\bar{t}$, and, after the other realization, the agent retires with probability one (i.e., he exerts $e = 0$ in all future periods). That is, we replace the continuation play after $\left(h^{\bar{t}}, \bar{z}_{\bar{t}}\right)$ with the following two paths: (a) the period of $e(h^\tau, z_\tau) = 1$ is front loaded by one period; and (b) the agent is allowed to retire. Recursively, we can create another equilibrium in which the agent takes $e = 1$ or he retires.

Let $V(h^t, z_t)$ be the agent's continuation payoff after $(h^t, z_t)$. Since $J(V)$ is concave in $V$, we have $V(h^t, z_t, \omega) = 1/\delta \cdot \{V(h^t, z_t) - 1\}$ for each $\omega$. Hence, the principal's payoff equals

$$J\left(V(h^t, z_t)\right) = u(0, 1) + \delta \cdot J\left(\frac{1}{\delta} \cdot \left\{V(h^t, z_t) - 1\right\}\right). \tag{15}$$

Suppose that, at $(h^t, z_t)$, the principal offers the relational contract to bring $1/\delta \cdot \{V(h^t, z_t) - 1\}$ with probability $\delta$ and the one to bring $1/(1-\delta)$ (that is, to let the agent retire) with probability $1-\delta$. Then, the agent still obtains the value $V(h^t, z_t)$, and the principal obtains

$$\delta \cdot J\left(\frac{1}{\delta}\left\{V(h^t, z_t) - 1\right\}\right) + (1-\delta) \cdot \frac{u(0,1)}{1-\delta} = J\left(V(h^t, z_t)\right). \tag{16}$$

Hence, the principal is (weakly) better off.

Note that the best PBE may not be unique since here we start from one equilibrium and create another with front loaded effort with the same equilibrium payoff for the principal.

## 1.7 Proof of Lemma 7

**Part 1. Concavity with respect to $V$.**

Since $\mu$ is fixed, the proof is the same as Lemma 5.

**Part 2. Convexity with respect to $\mu$.**

Since $V$ is fixed, the proof is the same as Lemma 2.

**Part 3. Monotonicity with respect to $\mu$.**

Suppose that $J(\mu, V) = J$ for some $\mu$ and $V$. Then, for a higher value $\mu' > \mu$ and the same promised utility $V$, we have $J(\mu', V) \geq J$. Since $V$ is fixed, the proof is the same as Lemma 2.

We now show it is strictly increasing for $V \in (0, 1/(1-\delta))$. Fix public history $h^t$ with $(\mu, V)$ with $V \in (0, 1/(1-\delta))$ arbitrarily, and let $\alpha[\mu]$ be the principal's optimal strategy from this history. Given the starting belief $\mu' > \mu$, suppose the principal in period $\tau \geq t$ takes the same strategy $\alpha[\mu'] = \alpha[\mu]$ as long as $e_{z_\tau} = 0$ for each $z_\tau$ given $\alpha[\mu]$. Then, as long as $e_{z_\tau} = 0$ for $z_\tau$ given $\alpha[\mu]$, the payoff is exactly the same between $\mu$ and $\mu'$ (and the belief stays the same unless replacement happens); and once the current agent exerts a positive effort (if he is of $H$ type), the principal's expected payoff is higher with $\mu'$ than with $\mu$. Hence, we have $J(\mu', V) > J(\mu, V)$ if there exist $\tilde{t} \geq t$ and $z_{\tilde{t}}$ such that, given $\alpha[\mu]$, (i) $h^{\tilde{t}}$ happens with a positive probability, (ii) the same agent stays until period $\tilde{t}$ given $h^{\tilde{t}}$, and (iii) $e_{z_{\tilde{t}}} > 0$.

We now show that there exists such $(h^{\tilde{t}}, z_{\tilde{t}})$. Suppose otherwise. Then, the principal's payoff is equal to $J(\mu, V) = \alpha \cdot J(\mu, \tilde{V}) + (1-\alpha) \cdot \bar{J}$, where $1-\alpha$ is the probability of immediate replacement and the promise keeping constraint implies $V = \alpha \tilde{V}$. That is,

$$J(\mu, V) = \frac{V}{\tilde{V}} \cdot J(\mu, \tilde{V}) + \left(1 - \frac{V}{\tilde{V}}\right) \cdot \bar{J}. \tag{17}$$

Suppose $\tilde{V} = 1$. Then, since $e = 0$, we have $J(\mu, \tilde{V}) = (1-\delta) \cdot \underline{v}^P + \delta \cdot \bar{J}$, and so $\bar{J} - J(\mu, \tilde{V}) = (1-\delta) \cdot \left(\bar{J} - \underline{v}^P\right)$. Suppose next that $\tilde{V} = 1 + \Delta$. Then, the principal can implement $e = 0$ in period $t$, which makes the next-period promised value equal to $(\tilde{V} - 1)/\delta = \Delta/\delta$. Hence, the principal can achieve the payoff at least

$$(1-\delta) \cdot \underline{v}^P + \delta \cdot \left(\frac{\Delta}{\delta} \cdot \left((1-\delta) \cdot \underline{v}^P + \delta \cdot \bar{J}\right) + \left(1 - \frac{\Delta}{\delta}\right) \cdot \bar{J}\right). \tag{18}$$

Hence,

$$\begin{aligned}
&\frac{J(\mu, \tilde{V} + \Delta) - J(\mu, \tilde{V})}{\Delta} \\
&\quad \geq \frac{(1-\delta) \cdot \underline{v}^P + \delta \cdot \left(\frac{\Delta}{\delta} \cdot \left((1-\delta) \cdot \underline{v}^P + \delta \cdot \bar{J}\right) + \left(1 - \frac{\Delta}{\delta}\right) \cdot \bar{J}\right) - (1-\delta) \cdot \underline{v}^P - \delta \cdot \bar{J}}{\Delta} \\
&\qquad \geq -(1-\delta) \cdot \left(\bar{J} - \underline{v}^P\right).
\end{aligned} \tag{19}$$

In total,

$$\frac{d}{d\tilde{V}} \left[\frac{V}{\tilde{V}} \cdot J(\mu, \tilde{V}) + \left(1 - \frac{V}{\tilde{V}}\right) \cdot \bar{J}\right]\Bigg|_{\tilde{V}=1} \geq 0. \tag{20}$$

Hence, the first order effect of increasing $\tilde{V}$ by $\Delta$ keeping $e$ fixed is no less than 0. Suppose that the principal increases $V'_{gG}$ in the problem to maximize $J(\mu, \tilde{V})$, keeping all the other continuation payoffs fixed. This increases $e$ and $\tilde{V}$. Since the first order effect of changing $\tilde{V}$ given $e$ is 0, the principal is strictly better off by implementing $e > 0$, as desired.

## 1.8 Proof of Lemma 8

We have $J(\mu, 0) = \bar{J}$ for each $\mu$ since $P$ has to replace $A$ right away. Hence we are left to prove the other four properties:

**Part 1. There exists $V(\mu)$ such that $J(\mu, V)$ is linear for $V \in [0, V(\mu)]$.**

Suppose such $V(\mu)$ does not exist. By Lemma 7, this means that $J(\mu, V)$ is strictly concave near $V = 0$.

Take $V \in (0, 1)$. This means that $P$ needs to stochastically replace $A$, since otherwise $A$ receives 1 by not working. Let $\beta$ be the probability of a replacement. The promise keeping condition implies

$$\beta \cdot 0 + (1-\beta) \cdot \hat{V} = V, \tag{21}$$

where $\hat{V} \geq 1$ is the promised utility conditional on $A$ not being replaced.

$P$ maximizes

$$\max_{\beta \in [0,1], \hat{V} \in \left[0, \frac{1}{1-\delta}\right]} \beta \cdot J(\mu, 0) + (1-\beta) \cdot J(\mu, \hat{V}) \tag{22}$$

subject to

$$\beta \cdot 0 + (1-\beta) \cdot \hat{V} = V \text{ and } \hat{V} \geq 1. \tag{23}$$

Substituting the constraint, $P$'s payoff is

$$J(\mu, 0) + \frac{V}{\hat{V}} \cdot \left[ J(\mu, \hat{V}) - J(\mu, 0) \right]. \tag{24}$$

Taking the derivative with respect to $\hat{V}$ (the differentiability of $J(\mu, \hat{V})$ follows from the Envelope Theorem), we obtain

$$V \cdot \frac{J(\mu, 0) + \left[ J_2(\mu, \hat{V}) \cdot \hat{V} - J(\mu, \hat{V}) \right]}{\hat{V}^2}, \tag{25}$$

where $J_n$ is the derivative of $J$ with respect to its $n^{\text{th}}$ argument.

We show that the numerator is always negative for each $\hat{V} \geq 0$. With $\hat{V} = 0$, the numerator is 0. Taking the derivative of the numerator,

$$\frac{d}{d\hat{V}} \cdot \left\{ J(\mu, 0) + \left[ J_2(\mu, \hat{V})\hat{V} - J(\mu, \hat{V}) \right] \right\} = \hat{V} \cdot \frac{d^2}{d\hat{V}^2} \cdot J(\mu, \hat{V}). \tag{26}$$

Since we assumed $J(\mu, \cdot)$ is strictly concave, this is negative for each $\hat{V} \geq 0$. Therefore, the numerator is globally negative.

Hence, the smallest $\hat{V} = 1$ is optimal. Given $\hat{V} = 1$, by (24),

$$J(\mu, V) = J(\mu, 0) + V \cdot [J(\mu, 1) - J(\mu, 0)], \tag{27}$$

for $V \in [0, 1]$, which is linear in $V$.

**Part 2. For $\mu \geq \mu_H$ , we have $V(\mu) > 1$.**

Suppose $\mu \geq \mu_H$. For the sake of contradiction, assume that $V \leq 1$ for each $V \in \arg\max_V J(\mu, V)$. Then, in the above problem (22), $\hat{V} = 1$ — the smallest continuation payoff without immediate replacement — is the unique optimum. Recall that $\beta$ is defined as the probability of immediate replacement in (21). Hence $P$ cannot replace $A$ in the current period after $P$ picks $\hat{V}$ with probability $1-\beta$. If $P$ promised a positive continuation payoff from the next period, then since $c(0) = \lim_{e \to 0} c'(e) = 0$, $A$ could obtain a payoff greater than 1 with providing a sufficiently small $e$. We therefore have to make sure that $V_z'[\hat{V}](\omega) = 0$ for each $z$ and $\omega$, and so $e_z = 0$ for each $z$. Therefore, the effort has to be equal to 0. Then, $P$'s instantaneous payoff is $(1-\delta) \cdot \underline{v}^P$. Moreover, since $V_z'[\hat{V}](\omega) = 0$ for each $z$ and $\omega$, the agent will be replaced in the next period with probability one. Hence, the continuation payoff is $\delta \cdot \bar{J}$. Since $\beta = 0$ if the current promised value is 1 and $\hat{V} = 1$,

$$J(\mu, 1) = (1-\delta) \cdot \underline{v}^P + \delta \cdot \bar{J}. \tag{28}$$

Recall that $\underline{v}^P$ is defined as the principal's dynamic game payoff when no effort is provided and $P$ intervenes every period.

It will be useful to verify that the payoff at the arrival of a new agent is higher than $\underline{v}^P$. To see why, the principal can improve upon $\underline{v}^P$ as follows: For each $z$, the principal always takes $\iota_z = 1$ as in the no effort equilibrium. If $\omega = (g, G)$, then $P$ keeps the agent forever. Otherwise, $P$ replaces the agent (and goes back to the no effort equilibrium). That is, $P$ rewards the agent after a good outcome in the first period, which incentivizes the high-type agent to supply a positive effort. Hence, the principal can obtain a payoff greater than $\underline{v}^P$ in the first period, and then obtain the continuation payoff of $\delta \cdot \underline{v}^P$. In total, we have $\bar{J} > \underline{v}^P$. Given $\bar{J} > \underline{v}^P$, for each $(\mu, V)$ with $V \in (0, 1/(1-\delta))$, by concavity of $J(\mu, \cdot)$,

$$J(\mu, V) \geq \frac{\frac{1}{1-\delta} - V}{\frac{1}{1-\delta}} \cdot \bar{J} + \frac{V}{\frac{1}{1-\delta}} \cdot J\left(\mu, \frac{1}{1-\delta}\right) > \underline{v}^P. \tag{29}$$

For $\mu = \mu_H$, (28) together with (29) implies that $J(\mu_H, 0) = \bar{J}$ and $J(\mu_H, V)$ is linear and less than $\bar{J}$ for each $V \in (0, 1]$. By concavity, this means that $J(\mu_H, V) < \bar{J}$ for each $V > 0$. Thus, $\arg\max_V J(\mu_H, V) = 0$. This means that $\bar{J}$ is uniquely obtained by always replacing $A$; however, this implies that $A$ exerts no effort, which is a contradiction. Hence, $V(\mu_H) > 1$. Moreover, since $\bar{J} = \max_V J(\mu_H, V)$, it follows that

$$J(\mu_H, V) = \bar{J} \text{ for } V \in [0, V(\mu_H)]. \tag{30}$$

For $\mu > \mu_H$, by Lemma 7, we have $J(\mu, 1) > J(\mu_H, 1) \geq \bar{J}$, which contradicts (28). Hence, $V(\mu) > 1$ as well.

**Part 3. The Slope of the Linear Part.**

Since $J(\mu, V)$ is strictly increasing in $\mu \in (0, 1)$, and $J(\mu, 0) = \bar{J}$ for each $\mu$, (30) implies the slope of the linear part is negative for $\mu < \mu_H$ and positive for $\mu > \mu_H$.

**Part 4. Property of** $V \in \arg\max_{\hat{V}} J(\mu, \hat{V})$.

Define

$$u^P(\iota_z|\mu, e_z) \equiv \sum_s \Pr(s|\mu, e_z) \cdot u^P(\iota_z|\mu, e_z, s). \tag{31}$$

Without loss of generality, we can take $V \in \arg\max_{\hat{V}} J(\mu, \hat{V})$ such that $V$ is the extreme point of the graph $\{\hat{V}, J(\mu, \hat{V})\}_{\hat{V}}$. This means that no mixture can implement $(V, J(\mu, V))$. Hence, $P$'s payoff $J(\mu, V)$ at $V \in \arg\max_{\hat{V}} J(\mu, \hat{V})$, denoted by $J(\mu)$, is determined by the dynamic program without mixture:

$$J(\mu) = \max_{(e, \iota, V')} \{u^P(\iota|\mu, e) + \delta \cdot \sum_\omega \Pr(\omega|\mu, e, \iota) \cdot J(\mu'(\mu, e, \omega), V'(\omega))\}, \tag{32}$$

subject to incentive compatibility constraint:

$$e \in \arg\max \left[ 1 - c\left(e\right) + \delta \cdot \sum_{\omega} \Pr\left(\omega | e, \iota\right) \cdot V'\left(\omega\right) \right]. \tag{33}$$

Note that we do not impose the promise keeping constraint since we are free to choose $\hat{V}$ to maximize $J(\mu, \hat{V})$. Moreover, since the first-order condition for $e$ is always necessary and sufficient by the assumption of the cost function $c$, we can see the above dynamic program as deciding $\left(V'\left(\omega\right)\right)_{\omega}$, and then $e$ is determined by the first-order condition.

In this problem, we first show that $V'\left(\omega\right) \leq \arg\max_{\hat{V}} J(\mu'\left(\mu, e, \omega\right), \hat{V})$ after $\mu'\left(\mu, e, \omega\right) \leq \mu$. Suppose otherwise: There exists $\bar{\omega}$ such that $V'\left(\bar{\omega}\right) > \arg\max_{\hat{V}} J(\mu'\left(\mu, e, \bar{\omega}\right), \hat{V})$ after $\mu'\left(\mu, e, \bar{\omega}\right) \leq \mu$.

Since

$$\mu'\left(\mu, e, \bar{\omega}\right) = \frac{\mu \cdot \Pr\left(\bar{\omega} | e, \iota\right)}{\mu \cdot \Pr\left(\bar{\omega} | e, \iota\right) + (1 - \mu) \cdot \Pr\left(\bar{\omega} | 0, \iota\right)} \leq \mu, \tag{34}$$

we have $\Pr\left(\bar{\omega} | 0, \iota\right) \geq \Pr\left(\bar{\omega} | e, \iota\right)$. We assume $\Pr\left(\omega | e, \iota\right)$ is monotone in $e$ for each $\omega$ and $\iota$, so the probability $\Pr\left(\bar{\omega} | e, \iota\right)$ is decreasing in $e$.

Then, the first-order condition for the optimality of $V'\left(\bar{\omega}\right)$ is

$$\begin{aligned} 0 = \frac{d}{dV'\left(\bar{\omega}\right)} & \{u^P\left(\iota | \mu, e\right) + \delta \cdot \sum_{\omega} \Pr\left(\omega | \mu, e, \iota\right) \cdot J\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right)\} \\ &= \{u_e^P\left(\iota | \mu, e\right) + \delta \cdot \sum_{\omega} \Pr_e\left(\omega | \mu, e, \iota\right) \cdot J\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \\ &+ \delta \cdot \sum_{\omega} \Pr\left(\omega | \mu, e, \iota\right) \cdot J_1\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \cdot \mu'_e\left(\mu, e, \omega\right)\} \cdot \frac{de}{dV'\left(\bar{\omega}\right)} \\ &+ \delta \cdot \Pr\left(\bar{\omega} | \mu, e, \iota\right) \cdot J_2\left(\mu'\left(\mu, e, \bar{\omega}\right), V'\left(\bar{\omega}\right)\right), \end{aligned} \tag{35}$$

where $J_n$ is the derivative of $J$ with respect to its $n^{\text{th}}$ argument; and $u_e^P \geq 0$, $\Pr_e$, and $\mu'_e$ are the derivatives of $u^P$, Pr, and $\mu'$ with respect to $e$, respectively. Since $\Pr\left(\bar{\omega} | e, \iota\right)$ is decreasing in $e$, it follows that $de/dV'\left(\bar{\omega}\right) < 0$. Moreover, $J_2\left(\mu'\left(\mu, e, \bar{\omega}\right), V'\left(\bar{\omega}\right)\right) < 0$, since $V'\left(\bar{\omega}\right) > \arg\max_{\hat{V}} J(\mu'\left(\mu, e, \bar{\omega}\right), \hat{V})$ and $J$ is concave. Hence,

$$\begin{aligned} & u_e^P\left(\iota | \mu, e\right) + \delta \cdot \sum_{\omega} \Pr_e\left(\omega | \mu, e, \iota\right) \cdot J\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \\ & \qquad + \delta \cdot \sum_{\omega} \Pr\left(\omega | \mu, e, \iota\right) \cdot J_1\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \cdot \mu'_e\left(\mu, e, \omega\right) < 0. \end{aligned} \tag{36}$$

Similarly, if there exists $\hat{\omega}$ such that $\Pr\left(\hat{\omega} | e, \iota\right)$ is decreasing in $e$ but $V'\left(\hat{\omega}\right) \leq \arg\max_{\hat{V}} J(\mu'\left(\mu, e, \hat{\omega}\right), \hat{V})$, then the symmetric argument implies that

$$\begin{aligned} & \{u_e^P\left(\iota | \mu, e\right) + \delta \cdot \sum_{\omega} \Pr_e\left(\omega | \mu, e, \iota\right) \cdot J\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \\ & \qquad + \delta \cdot \sum_{\omega} \Pr\left(\omega | \mu, e, \iota\right) \cdot J_1\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) \cdot \mu'_e\left(\mu, e, \omega\right)\} \geq 0, \end{aligned} \tag{37}$$

which is a contradiction.

Therefore, letting $\Omega_-$ be the set of signal-outcome pairs $\omega$ such that $\Pr(\omega|e,\iota)$ is decreasing in $e$, for each $\omega \in \Omega_-$, we have $V'(\omega) > \arg\max_{\hat{V}} J(\mu'(\mu,e,\omega), \hat{V})$. Symmetrically, letting $\Omega_+$ be the set of $\omega$ such that $\Pr(\omega|e,\iota)$ is increasing in $e$, for each $\omega \in \Omega_+$, we have $V'(\omega) < \arg\max_{\hat{V}} J(\mu'(\mu,e,\omega), \hat{V})$.

Now we set $V^*(\omega) = \arg\max_{\hat{V}} J(\mu'(\mu,e,\omega), \hat{V})$ for each $\omega$, and let $e^*$ be the new optimal effort (fixing $\iota$ throughout). Since $V^*(\omega) < V'(\omega)$ for $\omega \in \Omega_-$ and $V^*(\omega) > V'(\omega)$ for $\omega \in \Omega_+$, we have $e^* > e$ (here, $e$ is the original effort). Hence,

$$u^P(\iota|\mu,e^*) > u^P(\iota|\mu,e). \tag{38}$$

In addition, we adjust $V^*(\omega)$ so that the continuation payoff increases with fixed $e$:

$$\sum_{\omega} \Pr(\omega|\mu,e,\iota) \cdot J(\mu'(\mu,e,\omega), V'(\omega)) < \sum_{\omega} \Pr(\omega|\mu,e,\iota) \cdot J(\mu'(\mu,e,\omega), V^*(\omega)). \tag{39}$$

Moreover, since $\max_{\hat{V}} J(\mu', \hat{V})$ is increasing in $\mu'$,

$$J(\mu'(\mu,e,\omega), V^*(\omega)) < J(\mu'(\mu,e,\hat{\omega}), V^*(\hat{\omega})) \tag{40}$$

for each $\omega \in \Omega_-$ and $\hat{\omega} \in \Omega_+$. Since increase in $e$ increases the probability of event $\omega$ if and only if $\omega \in \Omega_+$,

$$\sum_{\omega} \Pr(\omega|\mu,e,\iota) \cdot J(\mu'(\mu,e,\omega), V^*(\omega)) < \sum_{\omega} \Pr(\omega|\mu,e^*,\iota) \cdot J(\mu'(\mu,e,\omega), V^*(\omega)). \tag{41}$$

Finally, learning (the difference between $\mu'(\mu,e,\omega)$ and $\mu'(\mu,e^*,\omega)$) further increases the continuation payoff. To show this, we first make the following claim:

**Claim 1** *For $\mu_1 < \mu_2$, $V^*(\mu_1) \in \arg\max_{\hat{V}} J(\mu_1, \hat{V})$ and $V^*(\mu_2) \in \arg\max_{\hat{V}} J(\mu_2, \hat{V})$, we have $J_1(\mu_1, V^*(\mu_1)) \leq J_1(\mu_2, V^*(\mu_2))$.*

**Proof.** $J$ is convex in $\mu$, so

$$J(\mu_1, V^*(\mu_1)) + J_1(\mu_1, V^*(\mu_1))[\mu_2 - \mu_1] \leq J(\mu_2, V^*(\mu_1)). \tag{42}$$

$V^*(\mu_2)$ maximizes $J(\mu_2, V)$ at $\mu_2$, so

$$J(\mu_1, V^*(\mu_1)) + J_1(\mu_1, V^*(\mu_1))[\mu_2 - \mu_1] \leq J(\mu_2, V^*(\mu_2)). \tag{43}$$

Also,

$$J(\mu_1, V^*(\mu_1)) \geq J(\mu_2, V^*(\mu_2)) - J_1(\mu_2, V^*(\mu_2))[\mu_2 - \mu_1], \tag{44}$$

since $J$ is convex in $\mu$. From the first inequality of the proof,

$$\begin{aligned} J(\mu_1, V^*(\mu_1)) \geq{}& J(\mu_1, V^*(\mu_1)) + J_1(\mu_1, V^*(\mu_1))[\mu_2 - \mu_1] \\ & - J_1(\mu_2, V^*(\mu_2))[\mu_2 - \mu_1]. \end{aligned} \tag{45}$$

Hence,

$$0 \geq \left[J_1\left(\mu_1, V^*\left(\mu_1\right)\right) - J_1\left(\mu_2, V^*\left(\mu_2\right)\right)\right]\left(\mu_2 - \mu_1\right). \tag{46}$$

■

Given this claim, $J_1\left(\mu'\left(\mu, e, \omega\right), V^*\left(\omega\right)\right)$ is larger for $\omega$ with $\mu'\left(\mu, e, \omega\right) > \mu$ than for $\omega$ with $\mu'\left(\mu, e, \omega\right) < \mu$. Since the distribution of $\left\{\mu'\left(\mu, e^*, \omega\right)\right\}_\omega$ given $e^*$ is the mean-preserving spread of the distribution of $\left\{\mu'\left(\mu, e, \omega\right)\right\}_\omega$ given $e$ and we have $\mu'\left(\mu, e^*, \omega\right) \geq \mu'\left(\mu, e, \omega\right)$ if and only if $\omega$ satisfies $\mu'\left(\mu, e, \omega\right) \geq \mu$, faster learning increases the continuation payoff. Together with (39) and (41), this leads to

$$\begin{aligned}&\sum_\omega \Pr\left(\omega|\mu, e, \iota\right) \cdot J\left(\mu'\left(\mu, e, \omega\right), V'\left(\omega\right)\right) < \\ &\qquad\qquad\qquad \sum_\omega \Pr\left(\omega|\mu, e, \iota^*\right) \cdot J\left(\mu'\left(\mu, e^*, \omega\right), V^*\left(\omega\right)\right).\end{aligned} \tag{47}$$

Together with (38), we have proven that $P$'s payoff increases.

The proof for $V'\left(\omega\right) \geq \arg\max_{\hat{V}} J(\mu'\left(\mu, e, \omega\right), \hat{V})$ after $\mu'\left(\mu, e, \omega\right) \geq \mu$ is completely symmetric, and so it is omitted.

## 1.9 Proof of Lemma 9

Recall that we refer to intervention as the intervention decision after signal $s = b$, since $P$ never intervenes after $s = g$. Given $s = g$, the principal observes the same information regardless of $\iota(s = b)$. Given $s = b$, she can observe $o \in \{G, B\}$ after $s = b$ without intervention while she can only observe $o = I$ with intervention. Hence, intervention is more informative in the Blackwell sense, and, given $e$, the distribution of the updated beliefs $\left(\mu'\left(\mu, e, \omega\right)\right)_\omega$ after no intervention is a mean-preserving spread of that after intervention.

In particular, the belief update is given by

$$\mu'(\mu, e, b, I) = \frac{\mu \cdot \Pr\left(b|e\right)}{\mu \cdot \Pr\left(b|e\right) + (1-\mu) \cdot \Pr\left(b|0\right)}, \tag{48}$$

$$\mu'(\mu, e, b, G) = \frac{\mu \cdot \Pr\left(b, G|e\right)}{\mu \cdot \Pr\left(b, G|e\right) + (1-\mu) \cdot \Pr\left(b, G|0\right)}, \tag{49}$$

$$\mu'(\mu, e, b, B) = \frac{\mu \cdot \Pr\left(b, B|e\right)}{\mu \cdot \Pr\left(b, B|e\right) + (1-\mu) \cdot \Pr\left(b, B|e\right)}. \tag{50}$$

Hence, the difference in the variance of $\mu'(\mu, e, \omega)$ is given by

$$\begin{aligned} d\left(\mu\right) &:= \sum_\omega \Pr\left(\omega|\mu, e, \iota = 0\right) \cdot \left(\mu'(\mu, e, \omega) - \mu\right)^2 - \sum_\omega \Pr\left(\omega|\mu, e, \iota = 1\right) \cdot \left(\mu'(\mu, e, \omega) - \mu\right)^2 \\ &= \sum_{y \in \{G, B\}} \frac{\mu^2 \cdot (1-\mu)^2 \cdot \left(\Pr\left(b, y|e\right) - \Pr\left(b, y|0\right)\right)^2}{\mu \cdot \Pr\left(b, y|e\right) + (1-\mu) \cdot \Pr\left(b, y|0\right)} \\ &\qquad - \frac{\mu^2 \cdot (1-\mu)^2 \cdot \left(\Pr\left(b|e\right) - \Pr\left(b|0\right)\right)^2}{\mu \cdot \Pr\left(b|e\right) + (1-\mu) \cdot \Pr\left(b|0\right)}. \end{aligned} \tag{51}$$

Note that this difference is 0 with $\mu = 0$ and $\mu = 1$. Moreover, taking the second derivative of $d(\mu)$ with respect to $\mu$ yields

$$\sum_{y \in \{G,B\}} \frac{\Pr(b, y|e)^{2 \cdot} \Pr(b, y|0)^2}{(\mu \cdot \Pr(b, y|e) + (1-\mu) \cdot \Pr(b, y|0))^3} - \frac{\Pr(b|e)^2 \cdot \Pr(b|0)^2}{(\mu \cdot \Pr(b|e) + (1-\mu) \cdot \Pr(b|0))^3}. \tag{52}$$

The function $f(x, y) := x^2 y^2 (\mu x + (1-\mu) y)^{-3}$ is convex since, for each $(a, b) \in \mathbb{R}^2$,

$$(a, b) \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{2(b^2x^2 - abxy + a^2y^2)(\mu^2 x^2 + 4\mu(1-\mu)xy + y^2)}{(\mu x + (1-\mu) y)^5} \geq 0, \tag{53}$$

as $b^2x^2 - abxy + a^2y^2 = (bx + ay)^2 - abxy = (bx - ay)^2 + abxy$. Given $\Pr(b|e) = \Pr(b, G|e) + \Pr(b, B|e)$, we thus have $d''(\mu) \leq 0$.

## 1.10 Proof of Proposition 3

The proof consists of the three steps: (1) proving that intervention is optimal in the initial period, (2) intervention is optimal for a sufficiently large $T$, and (3) in some period $t \geq 2$, no intervention is optimal.

**Intervention is Optimal in the Initial Period.**

**Lemma 1** *There exist $\bar{\mu}_H \in (0, 1)$ and $\bar{q} \in (0, 1)$ such that, for each $\mu_H \leq \bar{\mu}_H$ and $\Pr(G|0) \leq \bar{q}$, it is optimal to intervene after $s = b$ in the initial period of an agent's appointment.*

**Proof.** In period 1 of the agent's appointment, after an $s = b$, the belief is no more than $\mu_H$. Hence, the instantaneous cost of non-intervention is no less than

$$-l - [\mu_H \cdot \Pr(G|e, b) \cdot 0 + (1-\mu_H) \cdot (-C)] \geq C - l - \mu_H \cdot C. \tag{54}$$

On the other hand, the gain in the continuation payoff of no intervention is at most

$$\delta \cdot \left[\mu_H \cdot 0 + (1-\mu_H) \cdot \max_V J(\mu_H, V)\right] - \delta \cdot J(\mu_b, V_b). \tag{55}$$

We now drive an upper bound for $\max_V J(\mu_H, V)$:

$$\max_V J(\mu_H, V) \leq \mu_H \cdot 0 + (1-\mu_H) \cdot \Big(\Pr(y = B|0) \cdot (-l) + \delta \cdot \max_V J(\mu_H, V)\Big). \tag{56}$$

Here, the $H$-type would deliver the best outcome, the $L$-type would be replaced immediately after period 1, and we allow $P$ to intervene if and only if the outcome is bad, so that we derive an upper bound. Rearranging,

$$\max_V J(\mu_H, V) \leq \frac{-(1-\mu_H) \cdot (1-\bar{q}) \cdot l}{1 - (1-\mu_H) \cdot \delta}. \tag{57}$$

By contrast, $J(\mu_b, V_b) \geq -l/(1-\delta)$ since the principal can always intervene. Hence, the continuation payoff gain is bounded by

$$\delta \cdot \left( (1-\mu_H) \cdot \frac{-(1-\mu_H) \cdot (1-\bar{q}) \cdot l}{1-(1-\mu_H) \cdot \delta} - \frac{-l}{1-\delta} \right). \tag{58}$$

Hence, if

$$C - l - \mu_H \cdot C > \delta \cdot \left( (1-\mu_H) \cdot \frac{-(1-\mu_H) \cdot (1-\bar{q}) \cdot l}{1-(1-\mu_H) \cdot \delta} - \frac{-l}{1-\delta} \right), \tag{59}$$

then intervention is uniquely optimal. At $\mu_H = 0$ and $\bar{q} = 0$, (59) holds since $C - l > 0$. Therefore, there exist $\bar{\mu}_H > 0$ and $\bar{q} > 0$ such that, for $\mu_H \leq \bar{\mu}_H$ and $\Pr(G|0) \leq \bar{q}$, we have (59). ■

**Intervention at the Limit.**

**Lemma 2** *For any parameter values, if we start from $\mu = \mu_H$ and $V = \arg\max_{\tilde{V}} J(\mu_H, \tilde{V})$, then after $\omega$ with $\Pr_e(\omega|e) < 0$, we have $J_2(\mu'(\mu, e_z, \omega), V'_z(\omega)) = 0$.*[2]

**Proof.** From Lemma 2, we have $J(\mu'(\mu, e_z, \omega), V'_z(\omega)) = \max_V J(\mu_H, V)$. Hence, Lemma 8 implies the result. ■

We form the Lagrangian

$$\begin{aligned} J(\mu, V) = \int_z & (1-\rho_z) \cdot \bar{J} \\ & + \rho_z \cdot \left[ u^P(\iota_z|\mu, e_z) + \delta \cdot \sum_\omega \Pr(\omega|\mu, e_z, \iota_z) \cdot J(\mu'(\mu, e_z, \omega), V'_z(\omega)) \, dz \right] \\ & + \lambda \cdot \left( V - \int_z \rho_z \cdot \left\{ 1 - c(e_z) + \delta \cdot \sum_\omega \Pr(\omega|e_z, \iota_z) \cdot V'_z(\omega) \right\} dz \right) \\ & + \int_z \rho_z \cdot \eta_z \cdot \left( \delta \cdot \sum_\omega \Pr_e(\omega|e_z, \iota_z) \cdot V'_z(\omega) - c'(e_z) \right) dz \end{aligned} \tag{60}$$

with $\eta_z \geq 0$ (higher effort is beneficial). Recall that $\Pr(\omega|e_z, \iota_z) = \Pr(\omega|\mu = 1, e_z, \iota_z)$. By the Envelope theorem, $J_2(\mu, V) = \lambda$. Taking the first order conditions and substituting

[2] $J_n$ is the derivative of $J$ with respect to its $n^{\text{th}}$ argument.

$J_2(\mu, V) = \lambda$, we obtain

$$
\begin{aligned}
e_z &: -J_2(\mu, V) \cdot c'(e_z) + \eta_z \cdot c''(e_z) \\
&= u_e^P(\iota_z|\mu, e_z) + \delta \cdot \sum_{\omega} \Pr_e(\omega|\mu, e_z, \iota_z) \cdot J(\mu'(\mu, e_z, \omega), V_z'(\omega)) \\
&\quad + \delta \cdot \sum_{\omega} \Pr(\omega|\mu, e_z, \iota_z) \cdot J_1(\mu'(\mu, e_z, \omega), V_z'(\omega)) \cdot \mu_e'(\mu, e_z, \omega) \\
&\quad - \delta \cdot \sum_{\omega} \Pr_e(\omega|e_z, \iota_z) \cdot V_z'(\omega) \cdot J_2(\mu, V) \\
&\quad + \delta \cdot \eta_z \cdot \sum_{\omega} \Pr_{ee}(\omega|e_z, \iota_z) \cdot V_z'(\omega), \qquad (61)
\end{aligned}
$$

and

$$
V_z'(\omega) : J_2(\mu'(\mu, e_z, \omega), V_z'(\omega)) = \frac{\Pr(\omega|e_z, \iota_z)}{\Pr(\omega|\mu, e_z, \iota_z)} \cdot J_2(\mu, V) - \eta_z \cdot \frac{\Pr_e(\omega|e_z, \iota_z)}{\Pr(\omega|\mu, e_z, \iota_z)}. \qquad (62)
$$

Using these two first order conditions, we will show that the effort level converges to 0.

**Lemma 3** *On the equilibrium path, given a history $h$ such that the belief updates positively, $\mu(h^t) \geq \mu(h^{t-1})$ for each $t$, effort converges to* 0.

**Proof.** Fix $(z_t, \omega_t)_{t=1}^{\infty}$ to satisfy $\mu(h^t) \geq \mu(h^{t-1})$ for each $t$, and let $(\iota_t, e_t)_{t=1}^{\infty}$ be the implemented intervention decisions and effort levels along the history. For notational simplicity, we omit $(z_t)_{t=1}^{\infty}$ since the argument holds conditional on $(z_t)_{t=1}^{\infty}$.

On such a history, we have $J_2(\mu'(\mu, e_1, \omega_1), V'(\omega_1)) < 0$. To see why, since $\Pr_e(\omega_1|e_1, \iota_1) > 0$ given $\mu(h^t) \geq \mu(h^{t-1})$ and $J_2(\mu_1, V_1) = 0$ in the initial period, given (62), it suffices to show that $\eta > 0$. If $\eta = 0$, since $J_2(\mu, V) = 0$ in the initial period, Lemma 2 and (61) yield

$$
\begin{aligned}
0 &= u_e^P(\iota_1|\mu_1, e_1) + \delta \cdot \sum_{\tilde{s}_1, \tilde{o}_1} \Pr_e(\tilde{\omega}_1|\mu_1, e_1, \iota_1) \cdot J(\mu'(\mu_1, e_1, \tilde{\omega}_1), V'(\tilde{\omega}_1)) \\
&\quad + \delta \cdot \sum_{\tilde{\omega}_1 : \mu_e'(\mu_1, e_1, \tilde{\omega}_1) > 0} \Pr(\tilde{\omega}_1|\mu_1, e_1, \iota_1) \cdot J_1(\mu'(\mu_1, e_1\tilde{\omega}_1), V'(\tilde{\omega}_1)) \\
&\quad \cdot \mu_e'(\mu_1, e_1, \tilde{\omega}_1). \qquad (63)
\end{aligned}
$$

The first two terms of the right hand side is the benefit of increasing $e_1$ to the principal's value fixing $\iota_1$ and $V'(\tilde{\omega}_1)$; and the last term is non-negative given $J_1(\mu, V) \geq 0$. Hence, the right hand side is positive.[3] This is a contradiction.

[3]Otherwise, the principal should have implemented $e_1 = 0$ and $V'(\tilde{\omega}_1) = V'(\tilde{\omega}_1')$ for each $\tilde{\omega}_1, \tilde{\omega}_1'$ given concavity of $J(\mu, V)$. However, (i) the first order condition for $e$ (this is necessary and sufficient given our assumption), (ii) Lemma 2, and (iii) parts 2 and 3 of the proof to Lemma 8(omitting $z$ for notational simplicity) imply

$$
c'(e_1) = \delta \sum_{\omega_1 : \Pr_e(\omega_1|e_1) > 0} \Pr_e(\omega_1|\iota_1, e_1),
$$

which means $e_1 > 0$. This is a contradiction.

In addition, on such a history, we have $\Pr_e(\omega_t|e_t,\iota_t) \geq 0$ and $\Pr(\omega_t|e_t,\iota_t) \geq \Pr(\omega_t|\mu_t,e_t,\iota_t)$ for each $t$. Hence, recursively applying to (62),

$$J_2\left(\mu'\left(\mu_t,e_t,\omega_t\right),V_{t+1}\left(\omega_t\right)\right) \;\leq\; \frac{\Pr\left(\omega_t|e_t,\iota_t\right)}{\Pr\left(\omega_t|\mu_t,e_t,\iota_t\right)} \cdot J_2\left(\mu_t,V_t\right) - \eta_t \cdot \frac{\Pr_e\left(\omega_t|e_t,\iota_t\right)}{\Pr\left(\omega_t|\mu_t,e_t,\iota_t\right)}, \quad (64)$$

so it is monotonically decreasing. If $e_t$ does not converge to 0, then $\mu_t$ converges to 1 and $\eta_t \geq 0$ converges to 0, since otherwise $J_2$ diverges to $-\infty$.

Suppose $\mu_t$ converge to 1 and $\eta_t$ converges to 0. At this limit, (61) converges to

$$\begin{aligned} -J_2(1,V)\,c'(e) = u_e^P(1,e,\iota) &+ \delta\cdot\sum_{\omega}\Pr_e(\omega|1,e,\iota)\cdot J\left(\mu'(1,e,\omega),V'(\omega)\right) \\ &+ \delta\cdot\sum_{\omega}\Pr(\omega|e,\iota)\cdot J_1\left(1,V'(\omega)\right)\cdot\mu_e'(1,e,\omega) \\ &- \delta\cdot\sum_{\omega}\Pr_e(\omega|e,\iota)\cdot V'(\omega)\cdot J_2(1,V). \quad (65)\end{aligned}$$

Since

$$\mu_e'(1,e,\omega) = \lim_{\mu\to 1}\left(\frac{d}{de}\cdot\frac{\mu\cdot\Pr(\omega|e)}{\mu\cdot\Pr(\omega|e)+(1-\mu)\cdot\Pr(\omega|0)}\right) = 0 \quad (66)$$

for each $\Pr(\omega|e)$ with $e>0$ (recall that we assumed that $e>0$ for the sake of a contradiction) and $c'(e) = \delta\cdot\sum_{\omega}\Pr(\omega|e,\iota)\cdot V'(\omega)$ from (15),

$$0 = u_e^P(\iota|1,e) + \delta\cdot\sum_{\omega}\Pr_e(\omega|e,\iota)\cdot J\left(1,V'(\omega)\right). \quad (67)$$

This means that the benefit of increasing $e$ to the principal's value fixing $V'(\omega)$, i.e.,

$$\frac{d}{de}[u^P(\iota|1,e) + \delta\cdot\sum_{\omega}\Pr(\omega|e,\iota)\cdot J\left(1,V'(\omega)\right)], \quad (68)$$

is 0. This in turn implies that $e$ is equal to 0. Therefore, $e_t$ converges to 0. ■

Given that $e$ converges to 0, intervention is optimal at the limit:

**Lemma 4** *There exists $\hat{e}\in(0,1)$ such that, for any belief $\mu\in[0,1]$ and promised value $V$, if the principal implements $e\leq\hat{e}$, then $\iota=1$ is optimal.*

**Proof.** With discounting, $e\in[0,1]$, and $V\in[0,1/(1-\delta)]$, the principal's payoff is continuous in $e$. Hence, it suffices to show that it is uniquely optimal for the principal to choose $\iota=1$ for effort $e=0$. With $e=0$, we have $\mu'(\mu,e,\omega)=\mu$. Since $J(\mu,V)$ is concave in $V$, it is optimal to choose $V'(\omega|\iota)=V'(\omega'|\iota)$ for each $\omega,\omega'$. Hence, the continuation payoff is fixed regardless of $\iota$. Since $\iota=1$ maximizes the instantaneous utility $u^P(\iota|\mu,e,s)$ after $s=b$ given $e=0$, intervention $\iota(s=b)=1$ is uniquely optimal. ■

**No intervention is Optimal in a Period after the Initial Period.**

The following lemma ensures that $e_1$ is bounded below:

**Lemma 5** *For sufficiently small $\bar{q} > 0$, if $c'(\bar{e}) \leq \Pr_e(g, G|\bar{e}) \cdot \bar{q}$, then the initial effort level $e(\emptyset)$ is no less than $\bar{e}$.*

**Proof.** From (i) the first order condition for $e$ (this is necessary and sufficient given our assumption), (ii) Lemma 2, and (iii) parts 2 and 3 of Lemma 8 (omitting $z$ for notational simplicity), we have

$$c'(e_1) = \delta \cdot \sum_{\omega_1 : \Pr_e(\omega_1|e_1)>0} \Pr_e(\omega_1|e_1, \iota_1) \geq \delta \cdot \Pr_e(g, G|e_1, \iota_1) = \delta \cdot \Pr_e(g, G|e_1 \tag{69}$$

Hence, $e_1 \geq \bar{e}$. ■

**Lemma 6** *For each $\mu_H$ and $(\Pr(b|e))_{e\in[0,1]}$, there exists $\bar{q} > 0$ such that, if the effort provision condition holds given $\bar{q}$ and $\Pr(G|0) \leq \bar{q}$, then there exists $t \geq 2$ such that no intervention is optimal in period $t$.*

**Proof.** It suffices to show that there exists $t \geq 2$ with $e \geq \bar{e}$, and $\mu'(h^t)$ is sufficiently close to 1 since then no intervention is statically optimal. Note that we first fix $(\Pr(b|e))_{e\in[0,1]}$. Hence, if $\mu'(h^t)$ is sufficiently close to one, $\mu'(h^t, b)$ is also close to one.

On the one hand, if there is no period $t \geq 2$ such that no intervention is optimal along the path of repeated $(g, G)$. Then, the payoff is bounded by

$$u^P(\iota_1|\mu_H, \bar{e}) + \delta \cdot \max\left\{\max_V J(\mu_H, V), \frac{1}{1-\delta} \cdot \Pr(s = b|\bar{e}) \cdot (-l)\right\}. \tag{70}$$

Here, to obtain an upper bound, we allow $P$ to replace the $L$-type at the end of period 1, and she learns that the agent is an $H$-type at the end of period 1 (we then take the maximum of these two continuation payoffs). In the latter event, intervention is optimal after $s = b$ since (i) there is no learning benefit if $P$ learned the type and (ii) if the belief were sufficiently high for no intervention to be statically optimal after some history, then it would get sufficiently high along the path of repeated $(g, G)$.

On the other hand, if $P$ implements $e_t = \bar{e}$ without replacement for each $t = 1, ..., T$ as long as $\omega = (g, G)$, then she obtains a payoff of at least

$$\begin{aligned} u^P(\iota_1|\mu_H, \bar{e}) + \sum_{t=1}^{T} \delta^{t-1} \cdot \left\{\prod\nolimits_{\tau=1}^{t-1} \Pr(\omega_\tau = (g, G))\right\} \cdot \Pr(\omega_t \neq (g, G)) \\ \cdot \left(-C + \delta \cdot \max_V \cdot J(\mu_H, V)\right) + \delta^{T-1} \cdot \prod\nolimits_{\tau=1}^{T} \Pr(\omega_\tau = (g, G)) \cdot \frac{-l}{1-\delta}, \end{aligned} \tag{71}$$

where the probability is determined by the initial belief $\mu_H$ and the $H$-type agent taking $\bar{e}$. The second line says that, until $\omega_t \neq (g, G)$ is first observed, no cost is incurred, and once

$\omega_t \neq (g, G)$ happens, the principal pays $C$ and replaces the agent. The last line says that, if $\omega_t \neq (g, G)$ never happens until period $T$, then $s = b$ happens all the time and the principal always intervenes for $t = T + 1, ....$

For each $\mu_H$, for sufficiently large $\Pr(g, G|\bar{e})$ and sufficiently small $\Pr(g, G|0)$, the belief $\mu'(\mu_H, g, G)$ is sufficiently close to 1 and $u^P(\iota_1|\mu_H, 1)$ and $u^P(\iota_1|\mu_H, \bar{e})$ are close to each other. Hence, at $T = \infty$ (namely, $\delta^{T-1} = 0$), the latter is larger.

For each $T$, for sufficiently small $\bar{q}$, it is possible to implement $e_t \geq \bar{e}$ for each $t = 1, ..., T$ by keeping the agent if and only if he generates the outcome $(g, G)$. Hence, $\lim_{\bar{q}\to 0} T = \infty$. Therefore, for sufficiently small $\bar{q}$, there exists $t \geq 2$ with $e \geq \bar{e}$, and $\mu'(h^t)$ sufficiently close to 1. ■

# 2 Three-Period Model for Comparative Statics

In this appendix, we introduce the three-period model to theoretically derive comparative statics. As mentioned in the main text, effort choice is binary, $e \in \{0, 1\}$ with $c(0) = 0$ and $c(1) = \gamma$. We write $\Pr(\omega|e) = \alpha_\omega + \beta_\omega e$ for each $\omega$. We also write $\beta_G = \beta_{gG} + \beta_{bG}$, $\beta_b = \beta_{bG} + \beta_{bB}$, and so on. We keep Assumption 2: $\beta_{gG}, \beta_{bG} > 0$, $\beta_{gB}, \beta_{bB} < 0$, and $\alpha_{bG} = 0$. We also assume that $\beta_{bB}/\alpha_{bB} < \beta_{gB}/\alpha_{gB}$ (hazard rate is lower for $bB$). Given this assumption, starting from a fixed $\mu$ and $e = 1$, the belief update $\mu_\omega = \frac{\alpha_\omega + \beta_\omega}{\alpha_\omega + \mu\beta_\omega}$ satisfies

$$\mu_{bB} < \mu_{gB} < \mu < \mu_{gG} < \mu_{bG} = 1. \tag{72}$$

We assume that keeping the agent for another term after $\omega \in \{gG, bG\}$ or keeping him for another two terms after $\omega \in \{gG\}$ is sufficient to incentivize the effort: $\gamma < \min\left\{\delta\beta_G, \delta\beta_{gG}\left(1 + \delta - \frac{1-\alpha_G}{\beta_G}\gamma\right)\right\}$, where the term $-\frac{1-\alpha_G}{\beta_G}\gamma$ represents the cost of effort that the agent pays to exert effort in the second period.

We assume that there are only three periods. In period 1, the principal has initial belief $\mu_1 \in [0, 1]$ and starts with the initial promised value $V_1 \in [1, 1 + \delta + \delta^2]$, and the principal cannot replace the agent in period 1 (that is, $V_1$ is the promised value conditional on the realization of the public randomization in the language of the infinitely repeated game). Although the game formally starts from period 1, by endowing the principal with $(\mu_1, V_1)$, we can measure the effect of the state variable $(\mu_1, V_1)$.

In periods 2 and 3, for simplicity, we assume that the principal cannot observe a signal $s$. Hence, the principal's payoff is $-C$ if $y = B$ and 0 otherwise.[4] As in the main text, in periods 2 and 3, the newly arriving agent is an $H$-type with probability $\mu_H$.

## 2.1 Backward Induction

We now specify the principal's optimal strategy by backward induction.

[4]This assumption keeps the derivation of the continuation payoff simple and allows us to talk about the main trade-off faced by the principal in a clear way. If we allowed her to observe a warning, then the principal could tailor the intervention decision based on her belief and promised value in period 2. This additional effect would increase the principal's incentive to learn the agent's type, so not to intervene in period 1.

**Period 3** In the third period, since there is no future period, no agent exerts effort. Hence, regardless of $(\mu, V)$, the principal's payoff is $J_3(\mu, V) = \alpha_B(-C)$.

**Period 2** If the public randomization tells her to keep the agent and provide him with value $V$, the principal solves (ignoring the continuation payoff since it is constant)

$$J_2^*(\mu, V) = \max_{e\in\{0,1\}, V_\omega\in[0,\delta]} \mu_H u^P(e) + (1-\mu_H)u^P(0) \tag{73}$$

subject to

$$\gamma \le \delta \sum\nolimits_\omega \beta_\omega V_\omega \text{ if } e = 1, \tag{74}$$

$$V = 1 - \gamma \cdot 1_{\{e=1\}} + \sum_\omega (\alpha_\omega + \beta_\omega e) V_\omega. \tag{PK}$$

Suppose the promise keeping (PK) is not binding. Since $u^P(e)$ is increasing in $e$ and $e = 1$ is implementable, we have

$$J_2^*(\mu) = -(\alpha_B + \mu\beta_B) C. \tag{75}$$

We now derive the range of $V$ in which (PK) is not binding. Effort $e = 1$ is implementable if and only if $\beta_G \delta V_G + \beta_B \delta V_B \ge \gamma$, and given $e = 1$, the agent obtains $1 - \gamma + (\alpha_G + \beta_G)\delta V_G + (\alpha_B + \beta_B)\delta V_B$. Given $\gamma < \delta\beta_G$, the lowest payoff that the agent obtains given $e = 1$ is $\underline{V}_2 := 1 + \gamma\alpha_G/\beta_G$ and the highest payoff is

$$\bar{V}_2 := 1 - \gamma + (\alpha_G + \beta_G)\delta + (\alpha_B + \beta_B)\frac{\gamma - \delta\beta_G}{\beta_B} = 1 + \delta - \frac{1-\alpha_G}{\beta_G}\gamma, \tag{76}$$

where the second equality follows from $\alpha_G + \alpha_B = 1$ and $\beta_G + \beta_B = 0$.

Given this observation, we solve for $J_2^*(\mu, V)$. If

$$\frac{J_2^*(\mu_H) + \alpha_B C}{1+\delta} \ge \frac{J_2^*(\mu) + \alpha_B C}{1 + \delta - \bar{V}_2} \Leftrightarrow \mu \le \underline{\mu} := \mu_H \frac{1}{1+\delta}\frac{1-\alpha_G}{\beta_G}\gamma, \tag{77}$$

then the principal maximizes the probability of replacing the current agent by mixing $V = 0$ and $V = 1 + \delta$. Hence,

$$J_2^*(\mu, V) = \frac{1+\delta-V}{1+\delta} J^*(\mu_H) - \frac{V}{1+\delta}\alpha_B C \text{ for each } V \in [0, 1+\delta]. \tag{78}$$

If $\mu \in [\underline{\mu}, \mu_H]$, then since the problem is linear, it is optimal to mix 0 and $\bar{V}_2$ for $V \le \bar{V}_2$ and to mix $\bar{V}_2$ and $1 + \delta$ otherwise. Hence, we have

$$J_2^*(\mu, V) = \begin{cases} \frac{\bar{V}_2 - V}{\bar{V}_2} J_2^*(\mu_H) + \frac{V}{\bar{V}_2} J_2^*(\mu) & \text{for } V \le \bar{V}_2, \\ \frac{1+\delta-V}{1+\delta-\bar{V}_2} J^*(\mu) - \frac{V-\bar{V}_2}{1+\delta-\bar{V}_2}\alpha_B C & \text{for } V > \bar{V}_2. \end{cases} \tag{79}$$

Finally, for $\mu \ge \mu_H$, for $V < \underline{V}_2$, it is optimal to mix 0 and $\underline{V}_2$; and for $V > \bar{V}_2$, it is optimal to mix $\bar{V}_2$ and $1 + \delta$. Hence,

$$J_2^*(\mu, V) = \begin{cases} \frac{\underline{V}_2 - V}{\underline{V}_2} J_2^*(\mu_H) + \frac{V}{\underline{V}_2} J_2^*(\mu) & \text{for } V < \underline{V}_2, \\ J_2^*(\mu) & \text{for } V \in [\underline{V}_2, \bar{V}_2], \\ \frac{1+\delta-V}{1+\delta-\bar{V}_2} J^*(\mu) - \frac{V-\bar{V}_2}{1+\delta-\bar{V}_2}\alpha_B C & \text{for } V > \bar{V}_2. \end{cases} \tag{80}$$

**Period 1** In the first period, the principal given $(\mu_1, V_1)$ maximizes

$$J_1(\mu_1, V_1) = \max_{\iota\in\{0,1\}, e\in\{0,1\}, V_\omega\in[0,1+\delta]} \mu_1 u^P(\iota, e) + (1-\mu_1)u^P(\iota, 0) + \delta \sum_\omega (\alpha_\omega + \mu_1\beta_\omega e) J_2(\mu_\omega, V_\omega) \quad (81)$$

subject to

$$\mu_\omega = \frac{\mu_1(\alpha_\omega + \beta_\omega)}{\alpha_\omega + \mu_1\beta_\omega}, \quad (82)$$

$$\gamma \le \delta \sum\nolimits_\omega \beta_\omega V_\omega \text{ if } e = 1, \quad (83)$$

$$V_1 = 1 - \gamma \cdot 1_{\{e=1\}} + \delta \sum_\omega (\alpha_\omega + \beta_\omega e) V_\omega. \quad (84)$$

Except for the claim about $V_1$ in Proposition 6, we assume that (PK) is not binding. Hence, we omit $V_1$ until we prove Proposition 6. If $e = 0$, then $\iota = 1$ is optimal since there is no learning or incentive reason not to intervene. Since the continuation payoff is concave in $V_\omega$, it is optimal to have $V_\omega = \frac{1}{\delta}(V_1 - 1)$ and obtain

$$-\alpha_b l - \alpha_{gB} C + \delta J_2^*\left(\mu_1, \frac{1}{\delta}(V_1 - 1)\right). \quad (85)$$

Especially, if (PK) is not binding, then the principal's value is

$$-\alpha_b l - \alpha_{gB} C + \delta J_2^*(\max\{\mu_1, \mu_H\}). \quad (86)$$

If $e = 1$ is implemented, then we can write the problem as

$$J_1(\mu_1) = \max_{\iota, V_\omega} \mu u^P(\iota, 1) + (1-\mu)u^P(\iota, 0) + \delta \sum_\omega (\alpha_\omega + \mu\beta_\omega) J_2(\mu_\omega, V_\omega) \quad (87)$$

subject to

$$\mu_\omega = \frac{\alpha_\omega + \beta_\omega}{\alpha_\omega + \mu_1\beta_\omega} \text{ and } \gamma \le \delta \sum\nolimits_\omega \beta_\omega V_\omega. \quad (88)$$

Since the derivation of the optimal continuation payoffs requires tedious algebra, we first summarize the result: For $\iota = 0$, there are following four cases, depending on the value of $\mu_1$ and $\mu_H$:

1. $\mu_H < \mu_{bB} < \mu_{gB} < \mu_{gG} < \mu_{bG}$: $V_{bG} = V_{gG} = \bar{V}_2$, and find the maximum $V_{gB} \in [0, \underline{V}_2]$ and $V_{bB} \in [0, \underline{V}_2]$ to satisfy the incentive compatibility constraint (IC), $\gamma \le \delta \sum_\omega \beta_\omega V_\omega$. We first decrease $V_{bB}$ before we decrease $V_{gB}$: $V_{gB} < \underline{V}_2$ only if $V_{bB} = 0$. The principal's payoff is

$$J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(-C) + \delta\mu_1\beta_B(-C) - \sum_{\omega\in\{bB, gB\}} \frac{\delta\beta_\omega(\underline{V}_2 - V_\omega)}{\underline{V}_2}\left(\frac{\alpha_\omega}{\beta_\omega}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \quad (89)$$

2. $\mu_{bB} < \mu_H < \mu_{gB} < \mu_{gG} < \mu_{bG}$: $V_{bG} = V_{gG} = \bar{V}_2$, $V_{bB} = 0$, and find the maximum $V_{gB} \in [0, \underline{V}_2]$ to satisfy the incentive compatibility constraint (IC), $\gamma \leq \delta \sum_\omega \beta_\omega V_\omega$. The principal's payoff is

$$\begin{aligned} J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(\alpha_G + \mu_1\beta_G)(-C) + \delta\beta_B\mu_1(\alpha_G + \beta_G)(-C) \\ + \delta\alpha_B(\alpha_{gB} + \mu_1\beta_{gB})(-C) + \delta\beta_B\mu_1(\alpha_{gB} + \beta_{gB})(-C) \\ + \delta(\alpha_{bB} + \mu_1\beta_{bB})(\alpha_B + \mu_H\beta_B)(-C) \\ - \frac{\delta\beta_{gB}(\underline{V}_2 - V_\omega)}{\underline{V}_2}\left(\frac{\alpha_{gB}}{\beta_{gB}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \end{aligned} \tag{90}$$

3. $\mu_{bB} < \mu_{gB} < \mu_H < \mu_{gG} < \mu_{bG}$: $V_{bG} = V_{gG} = \bar{V}_2$, and $V_{gB} = V_{bB} = 0$. The principal's payoff is

$$\begin{aligned} J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(\alpha_G + \mu_1\beta_G)(-C) + \delta\beta_B\mu_1(\alpha_G + \beta_G)(-C) \\ + \delta(\alpha_B + \mu_1\beta_B)(\alpha_B + \mu_H\beta_B)(-C). \end{aligned} \tag{91}$$

4. $\mu_{bB} < \mu_{gB} < \mu_{gG} < \mu_H < \mu_{bG}$: $V_{bG} = \bar{V}_2$ and $V_{gB} = V_{bB} = 0$, and find the minimum $V_{gG} \in \left[0, \bar{V}_2\right]$ to satisfy IC. The principal's payoff is

$$\begin{aligned} J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(\alpha_{bG} + \mu_1\beta_{bG})(-C) + \delta\beta_B\mu_1(\alpha_{bG} + \beta_{bG})(-C) \\ + \delta(\alpha_{gG} + \mu_1\beta_{gG})(\alpha_B + \mu_H\beta_B)(-C) + \delta(\alpha_B + \mu_1\beta_B)(\alpha_B + \mu_H\beta_B)(-C) \\ + \frac{\delta\beta_{gG}V_{gG}}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \end{aligned} \tag{92}$$

For $\iota = 1$, there are following four cases, depending on the value of $\mu_1$ and $\mu_H$:

1. $\mu_H < \mu_b, \mu_{gB} < \mu_{gG}$: $V_{gG} = \bar{V}_2$, and find the maximum $V_{gB} \in [0, \underline{V}_2]$ and $V_b \in [0, \underline{V}_2]$ to satisfy IC. We first decrease $V_\omega$ with lower $\frac{\beta_\omega}{\alpha_\omega}$: $V_{\omega^{**}} < \underline{V}_2$ for $\omega^{**} = \arg\max_{\omega \in \{b, gB\}} \frac{\beta_\omega}{\alpha_\omega}$ only if $V_{\omega^*} = 0$ for $\omega^* = \arg\min_{\omega \in \{b, gB\}} \frac{\beta_\omega}{\alpha_\omega}$.

   The principal's payoff is

$$\begin{aligned} J_1(\mu_1) = (\alpha_b + \mu_1\beta_b)(-l) + (\alpha_{gB} + \mu_1\beta_{gB})(-C) + \delta\alpha_B(-C) + \delta\mu_1\beta_B(-C) \\ - \sum_{\omega \in \{b, gB\}} \frac{\delta\beta_\omega(\underline{V}_2 - V_\omega)}{\underline{V}_2}\left(\frac{\alpha_\omega}{\beta_\omega}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \end{aligned} \tag{93}$$

2. $\min\{\mu_b, \mu_{gB}\} < \mu_H < \max\{\mu_b, \mu_{gB}\} < \mu_{gG}$: $V_{bG} = V_{gG} = \bar{V}_2$, $V_{\omega^*} = 0$ for $\omega^* = \arg\min_{\omega \in \{b, gB\}} \frac{\beta_\omega}{\alpha_\omega}$, and find the maximum $V_{\omega^{**}} \in [0, \underline{V}_2]$ to satisfy IC for $\omega^{**} = \arg\max_{\omega \in \{b, gB\}} \frac{\beta_\omega}{\alpha_\omega}$.

The principal's payoff is

$$\begin{aligned}J_1(\mu_1) = (\alpha_b + \mu_1\beta_b)(-l) + (\alpha_{gB} + \mu_1\beta_{gB})(-C) \\ + \delta\alpha_B(\alpha_{gG} + \alpha_{\omega^{**}} + \mu_1(\beta_{gG} + \beta_{\omega^{**}}))(-C) \\ + \delta\beta_B\mu_1(\alpha_{gG} + \alpha_{\omega^{**}} + (\beta_{gG} + \beta_{\omega^{**}}))(-C) \\ + \delta(\alpha_{\omega^*} + \mu_1\beta_{\omega^*})(\alpha_B + \mu_H\beta_B)(-C) \\ - \frac{\delta\beta_{\omega^{**}}(\underline{V}_2 - V_\omega)}{\underline{V}_2}\left(\frac{\alpha_{\omega^{**}}}{\beta_{\omega^{**}}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \quad (94)\end{aligned}$$

3. $\max\{\mu_b, \mu_{gB}\} < \mu_H < \mu_{gG}$: $V_{gG} = \bar{V}_2$ and $V_{gB} = V_b = 0$. The principal's payoff is

$$\begin{aligned}J_1(\mu_1) = (\alpha_b + \mu_1\beta_b)(-l) + (\alpha_{gB} + \mu_1\beta_{gB})(-C) \\ + \delta\alpha_B(\alpha_{gG} + \mu_1\beta_{gG})(-C) + \delta\beta_B\mu_1(\alpha_{gG} + \beta_{gG})(-C) \\ + \delta(\alpha_{gB} + \alpha_b + \mu_1(\beta_{gB} + \beta_b))(\alpha_B + \mu_H\beta_B)(-C). \quad (95)\end{aligned}$$

4. $\mu_{gG} < \mu_H$: $V_{gB} = V_b = 0$, and find the minimum $V_{gG} \in [0, \bar{V}_2]$ to satisfy IC. The principal's payoff is

$$\begin{aligned}J_1(\mu_1) = (\alpha_b + \mu_1\beta_b)(-l) + (\alpha_{gB} + \mu_1\beta_{gB})(-C) \\ + \delta(\alpha_{gG} + \mu_1\beta_{gG})(\alpha_B + \mu_H\beta_B)(-C) \\ + \delta(\alpha_{gB} + \alpha_b + \mu_1(\beta_{gB} + \beta_b))(\alpha_B + \mu_H\beta_B)(-C) \\ + \frac{\delta\beta_{gG}V_{gG}}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \quad (96)\end{aligned}$$

## 2.2 Period-1 Problem without Promise Keeping

Suppose $\iota = 0$ is optimal. Given (72), there are following four cases:

1. $\mu_H < \mu_{bB} < \mu_{gB} < \mu_{gG} < \mu_{bG}$: In this case, the principal would like to set $V_{gG} = V_{bG} = \bar{V}_2$, $V_{gB} = V_{bB} = \underline{V}_2$. This satisfies IC if and only if $\delta\beta_G\bar{V}_2 + \delta\beta_B\underline{V}_2 \geq \gamma$.

   If IC is not satisfied, then the principal either increases $V_{gG}$ or $V_{bG}$, or she decreases $V_{gB}$ or $V_{bB}$. To relax the IC constraint by one unit by changing $V_\omega$, we must change $V_\omega$ by $\frac{1}{\delta\beta_\omega}$. The marginal effect of increasing $V_{gG}$ by $\frac{1}{\delta\beta_{gG}}$ units on $\delta\sum_\omega(\alpha_\omega + \mu_1\beta_\omega)J_2(\mu_\omega, V_\omega)$ is

$$\frac{1}{\beta_{gG}}(\alpha_{gG} + \mu_1\beta_{gG})\frac{\mu_{gG}\beta_B C}{1+\delta-\bar{V}_2} = \mu_1\left(\frac{\alpha_{gG}}{\beta_{gG}} + 1\right)\frac{\beta_B C}{1+\delta-\bar{V}_2} < 0. \quad (97)$$

   Similarly, the marginal payoff of increasing $V_{bG}$ by $\frac{1}{\delta\beta_{bG}}$ is $\mu_1\frac{\beta_B C}{1+\delta-\bar{V}_2} < 0$.

The cost to change $V_{gB}$ by $\frac{1}{\delta\beta_{gB}}$ is

$$\frac{1}{\beta_{gB}}(\alpha_{gB}+\mu_1\beta_{gB})\frac{J_2^*(\mu_{gB})-J_2^*(\mu_H)}{\underline{V}_2}$$
$$=\frac{1}{\underline{V}_2}\left(\frac{\alpha_{gB}}{\beta_{gB}}(\mu_H-\mu_1)-\mu_1(1-\mu_H)\right)\beta_B C. \quad (98)$$

Similarly, the cost to change $V_{bB}$ by $\frac{1}{\delta\beta_{bB}}$ is $\frac{1}{\underline{V}_2}\left(\frac{\alpha_{bB}}{\beta_{bB}}(\mu_H-\mu_1)-\mu_1(1-\mu_H)\right)\beta_B C$.

Since $1+\delta-\bar{V}_2<\delta<1<\underline{V}_2$, and $\frac{\alpha_\omega}{\beta_\omega}(\mu_H-\mu_1)-\mu_1(1-\mu_H)\in[0,\mu_1]$ given $\mu_\omega\geq\mu_H$, it is optimal to decrease $V_{gB}$ or $V_{bB}$ instead of increasing $V_{gG}$ or $V_{bG}$. Moreover, given that $\frac{\beta_{bB}}{\alpha_{bB}}<\frac{\beta_{gB}}{\alpha_{gB}}$, we have $0<\frac{\alpha_{bB}}{\beta_{bB}}(\mu_H-\mu_1)-\mu_1(1-\mu_H)<\frac{\alpha_{gB}}{\beta_{gB}}(\mu_H-\mu_1)-\mu_1(1-\mu_H)$, so it is most efficient to decrease $V_{bB}$ then decrease $V_{gB}$.

If decreasing $V_{bB}$ is enough, that is, if $\delta\beta_G\bar{V}_2+\delta\beta_{gB}\underline{V}_2\geq\gamma$, then the principal decreases $V_{bB}$ such that IC holds with equality: $\delta\beta_G\bar{V}_2+\delta\beta_{gB}\underline{V}_2+\delta\beta_{bB}V_{bB}=\gamma$. Otherwise, she sets $V_{bB}=0$ and decreases $V_{gB}$ such that IC is satisfied with equality: $\delta\beta_G\bar{V}_2+\delta\beta_{gB}V_{gB}=\gamma$. Since $V_{gB}=0$ satisfies IC by assumption, there exists $V_{gB}\in[0,\underline{V}_2]$ to satisfy IC for sure.

In total, we have $V_{bG}=V_{gG}=\bar{V}_2$, and largest $V_{gB}\in[0,\underline{V}_2]$ and $V_{bB}\in[0,\underline{V}_2]$ to satisfy IC $\gamma\leq\delta\sum_\omega\beta_\omega V_\omega$. Moreover, $V_{gB}<\underline{V}_2$ only if $V_{bB}=0$. The principal's payoff is

$$\begin{aligned}J_1(\mu_1)&=(\alpha_B+\mu_1\beta_B)(-C)+\delta\sum_{\omega\in\{bG,gG\}}(\alpha_\omega+\mu_1\beta_\omega)J_2^*(\mu_\omega)\\&+\delta\sum_{\omega\in\{bB,gB\}}(\alpha_\omega+\mu_1\beta_\omega)\left(\frac{\underline{V}_2-V_\omega}{\underline{V}_2}J_2^*(\mu_H)+\frac{V_\omega}{\underline{V}_2}J_2^*(\mu_\omega)\right)\\&=(\alpha_B+\mu_1\beta_B)(-C)+\delta\alpha_B(-C)+\delta\mu_1\beta_B(-C)\\&-\sum_{\omega\in\{bB,gB\}}\frac{\delta\beta_\omega(\underline{V}_2-V_\omega)}{\underline{V}_2}\left(\frac{\alpha_\omega}{\beta_\omega}(\mu_H-\mu_1)-\mu_1(1-\mu_H)\right)\beta_B C. \quad (99)\end{aligned}$$

2. $\mu_{bB}<\mu_H<\mu_{gB}<\mu_{gG}<\mu_{bG}$: In this case, the principal would like to set $V_{gG}=V_{bG}=\bar{V}_2$, $V_{gB}=\underline{V}_2$, and $V_{bB}=0$. This satisfies IC if and only if $\delta\beta_G\bar{V}_2+\delta\beta_{gB}\underline{V}_2\geq\gamma$.

   If this condition is not satisfied, then the principal either increases $V_{gG}$ or $V_{bG}$, or decreases $V_{gB}$. As in Case 1, it is optimal to decrease $V_{gB}$ instead of increasing $V_{gG}$ or $V_{bG}$. Hence, the optimal continuation payoff will be: $V_{gG}=V_{bG}=\bar{V}_2$, $V_{bB}=0$, and $V_{gB}\geq 0$ solves IC with equality: $\delta\beta_G\bar{V}_2+\delta\beta_{gB}V_{gB}=\gamma$.

   Hence, we have $V_{bG}=V_{gG}=\bar{V}_2$, $V_{bB}=0$, and largest $V_{gB}\in[0,\underline{V}_2]$ to satisfy IC

$\gamma \leq \delta \sum_\omega \beta_\omega V_\omega$. The principal's payoff is

$$\begin{aligned}J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(\alpha_G + \mu_1\beta_G)(-C) + \delta\beta_B\mu_1(\alpha_G + \beta_G)(-C)\\ + \delta\alpha_B(\alpha_{gB} + \mu_1\beta_{gB})(-C) + \delta\beta_B\mu_1(\alpha_{gB} + \beta_{gB})(-C)\\ + \delta(\alpha_{bB} + \mu_1\beta_{bB})(\alpha_B + \mu_H\beta_B)(-C)\\ - \frac{\delta\beta_{gB}(\underline{V}_2 - V_\omega)}{\underline{V}_2}\left(\frac{\alpha_{gB}}{\beta_{gB}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \quad (100)\end{aligned}$$

3. $\mu_{bB} < \mu_{gB} < \mu_H < \mu_{gG} < \mu_{bG}$: In this case, the principal would like to set $V_{gG} = V_{bG} = \bar{V}_2$, $V_{gB} = V_{bB} = 0$. This satisfies IC by assumption. The principal's payoff is

$$\begin{aligned}J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) + \delta\alpha_B(\alpha_G + \mu_1\beta_G)(-C) + \delta\beta_B\mu_1(\alpha_G + \beta_G)(-C)\\ + \delta(\alpha_B + \mu_1\beta_B)(\alpha_B + \mu_H\beta_B)(-C). \quad (101)\end{aligned}$$

4. $\mu_{bB} < \mu_{gB} < \mu_{gG} < \mu_H < \mu_{bG}$: In this case, the principal would like to set $V_{bG} = \bar{V}_2$, $V_{gG} = V_{gB} = V_{bB} = 0$. This satisfies IC if and only if $\delta\beta_{bG}\bar{V}_2 \geq \gamma$.

   Otherwise, the principal has to either increase $V_{bG}$ or increase $V_{gG}$. On the one hand, the marginal payoff of increasing $V_{gG}$ by $\frac{1}{\delta\beta_{gG}}$ units on $\delta\sum_\omega(\alpha_\omega + \mu_1\beta_\omega)J_2(\mu_\omega, V_\omega)$ is, if $\mu_{gG} > \underline{\mu}$, then

$$\begin{aligned}\frac{1}{\beta_{gG}}(\alpha_{gG} + \mu_1\beta_{gG})\frac{J_2^*(\mu_{gG}) - J_2^*(\mu_H)}{\bar{V}_2}\\ = \frac{1}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C, \quad (102)\end{aligned}$$

   and if $\mu_{gG} < \underline{\mu}$, then

$$\mu_1\left(\frac{\alpha_{gG}}{\beta_{gG}} + 1\right)\frac{\beta_B C}{1+\delta}. \quad (103)$$

   By definition of $\underline{\mu}$, the cost of increasing $V_{gG}$ is no more than

$$\frac{1}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C. \quad (104)$$

   On the other hand, the marginal payoff to increase $V_{bG}$ by $\frac{1}{\delta\beta_{bG}}$ is

$$\mu_1\frac{\beta_B C}{1+\delta-\bar{V}_2}. \quad (105)$$

   Since $1+\delta-\bar{V}_2 \leq \delta$, $\bar{V}_2 \geq 1$, and $\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H) \in [0, \mu_1]$ given $\mu_{gG} \leq \mu_H$, it is optimal to increase $V_{gG}$ such that IC holds with equality. Hence, the optimal continuation payoff will be: $V_{bG} = \bar{V}_2$, $V_{gB} = V_{bB} = 0$, and smallest $V_{gG} \geq 0$ to satisfy $\delta\beta_{bG}\bar{V}_2 + \delta\beta_{gG}V_{gG} \geq \gamma$. The principal's payoff is

$$\begin{aligned}
J_1(\mu_1) = (\alpha_B + \mu_1\beta_B)(-C) \\
+ \delta\alpha_B(\alpha_{bG} + \mu_1\beta_{bG})(-C) + \delta\beta_B\mu_1(\alpha_{bG} + \beta_{bG})(-C) \\
+ \delta(\alpha_{gG} + \mu_1\beta_{gG})(\alpha_B + \mu_H\beta_B)(-C) \\
+ \delta(\alpha_B + \mu_1\beta_B)(\alpha_B + \mu_H\beta_B)(-C) \\
+ \frac{\delta\beta_{gG}V_{gG}}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1 - \mu_H)\right)\beta_B C. \quad (106)
\end{aligned}$$

Suppose next that $\iota = 1$ is optimal. Let $\omega^* \in \{b, gB\}$ and $\omega^{**} \in \{b, gB\}$ such that $\mu_{\omega^*} < \mu_{\omega^{**}}$. There are following four possible cases:

1. $\mu_H < \mu_{\omega^*} < \mu_{\omega^{**}} < \mu_{gG}$: In this case, the principal would like to set $V_{gG} = \bar{V}_2$, $V_{gB} = V_b = \underline{V}_2$. This satisfies IC if and only if

$$\delta\beta_{gG}\bar{V}_2 + \delta(\beta_{gB} + \beta_b)\underline{V}_2 \geq \gamma. \quad (107)$$

If this condition is not satisfied, then the principal either increases $V_{gG}$, or decreases $V_b$ or $V_{gB}$. On the one hand, the marginal payoff of increasing $V_{gG}$ by $\frac{1}{\delta\beta_{gG}}$ is

$$\mu_1\left(\frac{\alpha_{gG}}{\beta_{gG}} + 1\right)\frac{\beta_B C}{1 + \delta - \bar{V}_2}. \quad (108)$$

On the other hand, for $\omega \in \{b, gB\}$, the cost to change $V_\omega$ by $\frac{1}{\delta\beta_\omega}$ is

$$\frac{1}{\beta_\omega}(\alpha_\omega + \mu_1\beta_\omega)\frac{J_2^*(\mu_\omega) - J_2^*(\mu_H)}{\underline{V}_2} = \frac{1}{\underline{V}_2}\left(\frac{\alpha_\omega}{\beta_\omega}(\mu_H - \mu_1) - \mu_1(1 - \mu_H)\right)\beta_B C. \quad (109)$$

Since $1 + \delta - \bar{V}_2 < \delta < \underline{V}_2$ and $\frac{\alpha_\omega}{\beta_\omega}(\mu_H - \mu_1) - \mu_1(1 - \mu_H) \in [0, \mu_1]$, it is optimal to decrease $V_{gB}$ or $V_b$ instead of increasing $V_{gG}$. Moreover, the marginal payoff of changing $V_{\omega^*}$ by $\frac{1}{\delta\beta_{\omega^*}}$ is higher. Hence, the optimal continuation payoff will be: $V_{gG} = \bar{V}_2$, $V_{\omega^*} \in [0, \underline{V}_2]$, and $V_{\omega^{**}} = \underline{V}_2$ if there exists $V_{\omega^*} \in [0, \underline{V}_2]$ to solve IC with equality: $\delta\beta_{gG}\bar{V}_2 + \delta\beta_{\omega^{**}}\underline{V}_2 + \delta\beta_{\omega^*}V_{\omega^*} = \gamma$. Otherwise, $V_{gG} = \bar{V}_2$, $V_{\omega^*} = 0$, and $V_{\omega^{**}} \in [0, \underline{V}_2]$, where $V_{\omega^{**}} \in [0, \underline{V}_2]$ solves IC with equality: $\delta\beta_{gG}\bar{V}_2 + \delta\beta_{\omega^{**}}V_{\omega^{**}} = \gamma$.

The calculation of the principal's payoff is the same as in the case with $\iota = 0$, so it is omitted.

2. $\mu_{\omega^*} < \mu_H < \mu_{\omega^{**}} < \mu_{gG}$: In this case, the principal would like to set $V_{gG} = \bar{V}_2$, $V_{\omega^{**}} = \underline{V}_2$, and $V_{\omega^*} = 0$. If this is not enough, then by the same calculation as Case 1, it is more efficient to decrease $V_{\omega^{**}}$. Hence we take $V_{\omega^{**}} \in [0, \underline{V}_2]$ to solve IC when it holds with equality:

$$\delta\beta_{gG}\bar{V}_2 + \delta\beta_{\omega^{**}}V_{\omega^{**}} = \gamma. \quad (110)$$

3. $\mu_{\omega^*} < \mu_{\omega^{**}} < \mu_H < \mu_{gG}$: In this case, the principal would like to set $V_{gG} = \bar{V}_2$ and $V_{\omega^{**}} = V_{\omega^*} = 0$. By definition, this continuation payoff satisfies IC.

4. $\mu_{\omega^*} < \mu_{\omega^{**}} < \mu_{gG} < \mu_H$: As in Case 4 for $\iota = 0$, the principal sets $V_{gG} \in [0, 1+\delta]$ and $V_{\omega^{**}} = V_{\omega^*} = 0$, where $V_{gG}$ solves IC with equality: $\delta\beta_{gG}V_{gG} = \gamma$.

## 2.3 Proof of Proposition 4

**Discount Factor $\delta$:** Let $J_1(\iota, \delta)$ be the principal's payoff given $e = 1$, $\iota \in \{0, 1\}$, and $\delta \in [0, 1]$, keeping all the other parameters fixed. We show that the marginal benefit of no intervention is increasing in $\delta \in [0, 1]$:

$$\frac{d}{d\delta}\left(J_1(0,\delta) - J_1(1,\delta)\right) \geq 0. \tag{111}$$

Note that the following variables are independent of $\delta$: $J_2^*(\cdot)$, $\underline{V}_2$, and $1 + \delta - \bar{V}_2$. Given $e = 1$, let $\omega(\iota)$ be the outcome $\omega$ such that $V_\omega$ changes as we change $\delta$, and let $w(\iota, \delta) := \delta \sum_{\tilde{\omega} \neq \omega(\iota)} \beta_{\tilde{\omega}} V_{\tilde{\omega}}(\iota)$ be the expected sum of the other continuation payoffs. Note that $\omega(\iota)$ is non-empty if and only if IC is binding.

Suppose that it is not the case that $\omega(\iota) = gG$ and $\mu_{gG} < \underline{\mu}$. Note that $\delta\beta_{\omega(\iota)}V_{\omega(\iota)}(\iota) + w(\iota, \delta) = \gamma$ if IC is binding, and so $\frac{d}{d\delta}\left(\delta\beta_{\omega(\iota)}V_{\omega(\iota)}(\iota)\right) = -\sum_{\tilde{\omega} \neq \omega(\iota)} \beta_{\tilde{\omega}} V_{\tilde{\omega}}(\iota)$. As seen in Appendix $B.2$, the marginal effect of changing $V_{\omega(\iota)}(\iota)$ by $1/(\delta\beta_{\omega(\iota)})$ unit (or changing $\delta\beta_{\omega(\iota)}V_{\omega(\iota)}$ by one unit) is independent of $\delta$. Hence, there exists $mc_{\omega(\iota)}(\iota) \geq 0$ such that the marginal effect of increasing $\delta$ is given by

$$\frac{d}{d\delta}J_1(\iota, \delta) = mc_{\omega(\iota)}(\iota) + J_2(\iota, \delta), \tag{112}$$

where $J_2(\iota, \delta)$ is the expected continuation payoff from period 2. In particular, $mc_{\omega(\iota)}(\iota)$ corresponds to the absolute value of the marginal payoff of increasing $\delta\beta_{\omega(i)}V_{\omega(\iota)}$ by one unit, multiplied by $\sum_{\tilde{\omega} \neq \omega(\iota)} \beta_{\tilde{\omega}} V_{\tilde{\omega}}(\iota)$ (the change in $\delta\beta_{\omega(\iota)}V_{\omega(\iota)}(\iota)$ when we change $\delta$).

Since $J_2$ is piecewise linear and no intervention guarantees the higher continuation payoff for the principal, the benefit in terms of continuation payoff from no intervention is larger than the cost that the principal has to pay in state $\omega(\iota)$:

$$mc_{\omega(1)}(1) - mc_{\omega(0)}(0) \leq J_2(0, \delta) - J_2(1, \delta). \tag{113}$$

Hence, we have the desired inequality (111).

Next, suppose $\mu_{gG} < \underline{\mu} < \mu_H$. If $\omega(1) = \emptyset$ (IC is not binding for intervention), then we have $\omega(0) = \emptyset$ since no intervention allows the principal to monitor the effort more precisely. Hence, (111) holds.

If $\omega(0) = \omega(1) = gG$, then for $\iota = 0$, we have $w(\iota, \delta) = 0$ and $\delta\beta_{gG}V_{gG}$ is fixed at $\gamma - \delta\beta_{bG}\bar{V}_2$. The expected continuation payoff from $V_{gG}$, $\delta\left(\alpha_{gG} + \mu_1\beta_{gG}\right) J_2\left(\mu_{gG}, V_{gG}\right)$, is

$$\begin{aligned}
&\delta\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\left(\frac{1+\delta - V_{gG}}{1+\delta} J^*(\mu_H) - \frac{V_{gG}}{1+\delta}\alpha_B C\right) \\
&\quad = \delta\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\left(\alpha_B + \mu_H\beta_B\right)(-C) - \frac{1}{1+\delta}\frac{\gamma}{\beta_{gG}}\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\mu_H\beta_B(-C) \\
&\qquad + \frac{\delta}{1+\delta}\frac{\beta_{bG}\bar{V}_2}{\beta_{gG}}\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\mu_H\beta_B(-C). \quad (114)
\end{aligned}$$

For $\iota = 1$, in contrast, $\delta\beta_{gG}V_{gG}$ is fixed at $\gamma$. The expected continuation payoff from $V_{gG}$, $\delta(\alpha_{gG} + \mu_1\beta_{gG})J_2(\mu_{gG}, V_{gG})$, is

$$\begin{aligned}
&\delta\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\left(\frac{1+\delta - V_{gG}}{1+\delta} J^*(\mu_H) - \frac{V_{gG}}{1+\delta}\alpha_B C\right) \\
&\quad = \delta\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\left(\alpha_B + \mu_H\beta_B\right)(-C) - \frac{1}{1+\delta}\frac{\gamma}{\beta_{gG}}\left(\alpha_{gG} + \mu_1\beta_{gG}\right)\mu_H\beta_B(-C), \quad (115)
\end{aligned}$$

since $\delta\left(\alpha_\omega + \mu_1\beta_\omega\right) J_2\left(\mu_\omega, V_\omega\right) = \delta J_2(\mu_H)$ for each $\omega \neq gG$. Direct calculation implies (111). The proof for $\omega(0) = \emptyset$ and $\omega(1) = gG$ is analogous.

**Cost of Effort $\gamma$:** Let $J_1(\iota, \gamma)$ be the principal's payoff given $e = 1$, $\iota \in \{0, 1\}$, and $\gamma$, keeping all the other parameters fixed. There exists a set of parameters such that

$$\frac{d}{d\gamma}\left(J_1(0,\gamma) - J_1(1,\gamma)\right) \quad (116)$$

is negative for some $\gamma$ while it is positive for others.

For example, suppose that, with $\iota = 0$, $\mu_H < \mu_{bB} < \mu_{gB} < \mu_{gG} < \mu_{bG}$, and $V_{gB} = \underline{V}_2$ and $V_{bB} \in (0, \underline{V}_2)$ to satisfy IC:

$$\delta\left(\beta_{bG} + \beta_{gG}\right)\bar{V}_2 + \delta\beta_{gB}\underline{V}_2 + \delta\beta_{bB}V_{bB} = \gamma. \quad (117)$$

At the same time, assume that, with $\iota = 1$, $\mu_H < \mu_b < \mu_{gB} < \mu_{gG}$, and $V_{gB} = \underline{V}_2$ and $V_b \in (0, \underline{V}_2)$ to satisfy IC:

$$\delta\beta_{gG}\bar{V}_2 + \delta\beta_{gB}\underline{V}_2 + \delta\beta_b V_b = \gamma. \quad (118)$$

Since $\frac{d}{d\gamma}\bar{V}_2 = -\frac{1-\alpha_G}{\beta_G}$, $\beta_{bG} > 0$, $\beta_b < 0$, and $\beta_{bB} < 0$, given (117) and (118), we have $\frac{d}{d\gamma}\beta_{bB}V_{bB} > \frac{d}{d\gamma}\beta_b V_b$. Since $\mu_H < \mu_{bB}, \mu_b$, increasing $\beta_{bB}V_{bB}$ and $\beta_b V_b$ (decreasing $V_{bB}$ and $V_b$) is costly. In particular, depending on the relative values of $\frac{d}{d\gamma}\beta_{bB}V_{bB}$, $\frac{d}{d\gamma}\beta_b V_b$, $\frac{d}{dV_{bB}}\left(\alpha_{bB} + \beta_{bB}\right)\delta J_2\left(\mu_{bB}, V_{bB}\right)$, and $\frac{d}{dV_b}\left(\alpha_b + \beta_b\right)\delta J_2\left(\mu_b, V_b\right)$, $\frac{d}{d\gamma}\left(J_1(0,\gamma) - J_1(1,\gamma)\right)$ can be negative or positive. A specific numerical example is available upon request.

## 2.4 Proof of Proposition 5

For $\iota = 0$, since the instantaneous payoff $-(\alpha_B + \beta_B) C$ stays the same, we focus on the continuation payoff. Except for Case 4 of Appendix $B.2$, the continuation payoff is constant given $\beta_B$ and $\beta_G$ being fixed.

In Case 4, $V_{bG} = \bar{V}_2$, $V_{gB} = V_{bB} = 0$, and we set $V_{gG} \in \left[0, \bar{V}_2\right]$ as the smallest value to satisfy IC: $\delta\beta_{bG}\bar{V}_2 + \delta\beta_{gG}V_{gG} = \gamma$. Hence, the marginal effect of increasing $\beta_{bG}$ (and decreasing $\beta_{gG}$) on $\delta\beta_{gG}V_{gG}$ is $-\delta\bar{V}_2$. The marginal effect of increasing $\beta_{bG}$ (and decreasing $\beta_{gG}$) on the principal's payoff is, given the result of Appendix $B.2$,

$$\begin{aligned}
&\delta\alpha_B\mu_1(-C) + \delta\beta_B\mu_1(-C) - \delta\mu_1(\alpha_B + \mu_H\beta_B)(-C) \\
&\qquad + \frac{-\delta\bar{V}_2}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)\beta_B C \\
&\qquad + \frac{\delta\beta_{gG}V_{gG}}{\bar{V}_2}\frac{\alpha_{gG}}{(\beta_{gG})^2}(\mu_H - \mu_1)\beta_B C \\
&\qquad = \delta\frac{\alpha_{gG}}{\beta_{gG}}(\mu_H - \mu_1)(-\beta_B) C\left(1 - \frac{V_{gG}}{\bar{V}_2}\right). \quad (119)
\end{aligned}$$

Since $\mu_{gG} < \mu_H$ implies $\mu_H - \mu_1 > 0$ and we have $V_{gG} < \bar{V}_2$, the marginal payoff is positive.

For $\iota = 1$, the instantaneous payoff $-(\alpha_b + \beta_b) l - (\alpha_{bB} + \beta_{bB})C$ decreases since $\beta_b$ increases while other variables stay the same. In addition, the continuation payoff decreases as well. Since the verification is analogous in all the four cases listed in Appendix $B.2$, here, we focus on Case 1: $\mu_H < \mu_b, \mu_{gB} < \mu_{gG}$. Recall that $V_{gG} = \bar{V}_2$, and we take the largest $V_{gB} \in [0, \underline{V}_2]$ and $V_b \in [0, \underline{V}_2]$ to satisfy IC: $\delta\beta_{gG}\bar{V}_2 + \delta\beta_{gB}V_{gB} + \delta\beta_b V_b \geq \gamma$. Hence, $\delta\beta_{gB}V_{gB}$ and $\delta\beta_b V_b$ increase as $\beta_{gG}$ decreases: $\frac{d\delta\beta_\omega V_\omega}{d\beta_{bG}} - \frac{d\delta\beta_\omega V_\omega}{d\beta_{gG}} \geq 0$. Given the result of Appendix $B.2$, the total effect on the continuation payoff is

$$\begin{aligned}
&\sum_{\omega\in\{b,gB\}} \underbrace{\left(\frac{d\delta\beta_\omega V_\omega}{d\beta_{bG}} - \frac{d\delta\beta_\omega V_\omega}{d\beta_{gG}}\right)}_{(+)} \frac{1}{\underline{V}_2} \underbrace{\left(\frac{\alpha_\omega}{\beta_\omega}(\mu_H - \mu_1) - \mu_1(1-\mu_H)\right)}_{(+)} \beta_B C \\
&\qquad + \frac{\delta(\underline{V}_2 - V_b)}{\underline{V}_2} \underbrace{\frac{\alpha_b}{\beta_b}(\mu_H - \mu_1)}_{(+)\text{ given } \mu_1 > \mu_H \text{ and } \beta_b < 0} \beta_B C < 0. \quad (120)
\end{aligned}$$

## 2.5 Proof of Proposition 6

**Promised Value** $V_1$: Suppose $\mu_1 = \mu_H$, and $\gamma > \delta\beta_{bG}$, that is, keeping the agent only after $\omega = bG$ for one period is not sufficient to incentivize the effort. Given this assumption, since $\alpha_{gG} > 0$, to implement $e = 1$, the $H$-type agent obtains the payoff more than one (that is, the principal has to pay the rent to the agent). Hence, with $V_1 = 1$, the principal has to implement $e = 0$. So, optimal $\iota$ is 1. Similarly, with $V_1 = 1 + \delta + \delta^2$, the principal has to retire the agent, so $e = 0$ and intervention is optimal.

On the other hand, suppose $V_1 \in [1, 1+\delta+\delta^2]$ is such that (PK) is not binding. Then, the results in Appendix $B.2$ implies that $e = 1$ and $\iota = 0$ are optimal for sufficiently small $\beta_{bB}$.

**Initial Belief** $\mu_1$: We consider non-binding (PK), and let $J_1(\mu_1, \iota)$ be the principal's payoff given $e = 1$, $\iota \in \{0, 1\}$, and $\mu_1 \in [0, 1]$, keeping all the other parameters fixed. Suppose the $\mu_1$ is sufficiently small such that $\mu_{gG} < \mu_H$. In addition, the reward $\bar{V}_2$ after $\omega = bG$ is sufficient to satisfy IC with $V_{gG} = V_{gB} = V_{bB} = 0$. Then, $\frac{d}{d\mu} J_1(\mu, 0)$ is equal to

$$\beta_B(-C) + \delta\alpha_B\beta_{bG})(-C) + \delta\beta_B(\alpha_{bG} + \beta_{bG})(-C) \\ + \delta\beta_{gG}(\alpha_B + \mu_H\beta_B)(-C) + \delta\beta_B(\alpha_B + \mu_H\beta_B)(-C), \quad (121)$$

while $\frac{d}{d\mu} J_1(\mu, 1)$ is equal to

$$\beta_b(-l) + \beta_{gB}(-C) + \delta\beta_{gG}(\alpha_B + \mu_H\beta_B)(-C) \\ + \delta(\beta_{gB} + \beta_b)(\alpha_B + \mu_H\beta_B)(-C) + \frac{\gamma}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}}(-1) - (1-\mu_H)\right)\beta_B C. \quad (122)$$

Hence,

$$\frac{d}{d\mu} J_1(\mu, 0) - \frac{d}{d\mu} J_1(\mu, 1) \\ = \beta_{bG} l - \beta_{bB}(C - l) + \delta|\beta_B| C\left(\beta_{bG}(1-\mu_H) - \frac{\gamma}{\bar{V}_2}\left(\frac{\alpha_{gG}}{\beta_{gG}} + (1-\mu_H)\right)\right). \quad (123)$$

At the limit where $\delta \to 1$ and $\left|\delta\beta_{bG}\bar{V}_2 - \gamma\right| \to 0$, this value is

$$\beta_{bG}\left(l - \frac{\alpha_{gG}}{\beta_{gG}}|\beta_B| C\right) - \beta_{bB}(C - l). \quad (124)$$

This can be positive or negative, depending on the parameters. Hence, for sufficiently large $\delta$ and small $\left|\delta\beta_{bG}\bar{V}_2 - \gamma\right|$, the sign of $\frac{d}{d\mu} J_1(\mu, 0) - \frac{d}{d\mu} J_1(\mu, 1)$ is not determined.

# References

**Stokey, Nancy L.** 1989. *Recursive methods in economic dynamics.* Harvard University Press.