Bad News Turned Good: Reversal Under Censorship
Online Appendix

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All statements below fix some strategy profile $(r^L(\mathbf{p}), r^H(\mathbf{p}))_{\mathbf{p}\in[0,1]^2}$, which in turn produces functions $D(\mathbf{p})$ and $f^s(\mathbf{p})$. Some statements further require this strategy profile to constitute an equilibrium.

Lemmas 1 and 2 are used heavily throughout the Appendix. They are monolithic in essence, but it proved more convenient to stagger their proofs for different bands, since they use different supplementary results.

Lemma 3. *1. $D(\mathbf{p}) \in [-1, -q)$ if and only if $f^s(\mathbf{p}) > p^s$.*

2. $D(\mathbf{p}) = -q$ if and only if either $f^s(\mathbf{p}) = p^s$ or $r^H(\mathbf{p}) = r^L(\mathbf{p}) = 0$.

3. $D(\mathbf{p}) \in (-q, 1-q]$ if and only if $f^s(\mathbf{p}) < p^s$.

Proof. To show the first claim, observe that $D(\mathbf{p}) < -q$ is equivalent to

$$\begin{aligned}(1-q)\cdot(1-r^H(\mathbf{p})) - (1-r^L(\mathbf{p})) &< -q\\ \Leftrightarrow (1-q)\cdot r^H(\mathbf{p}) - r^L(\mathbf{p}) &> 0\\ \Leftrightarrow \left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right)\cdot\left(\frac{p^s}{1-p^s}\right)^{-1} \equiv (1-q)\cdot\frac{r^H(\mathbf{p})}{r^L(\mathbf{p})} &> 1.\end{aligned}$$

Two other claims can be obtained by reversing the inequalities or equating both sides. Finally, if $r^H(\mathbf{p}) = r^L(\mathbf{p}) = 0$, then (4) directly gives that $D(\mathbf{p}) = (1-q) - 1 = -q$. □

Lemma 4. *For any $k \geqslant 0$ the following hold:*

1. *For all $\mathbf{p} = (p^n, p^s) \in \mathcal{B}_k^{\uparrow}$, if there exists $\tilde{\mathbf{p}} = (p^n, \tilde{p}^s)$ with $\tilde{p}^s \in [\bar{p}, p^s]$ and $D(\tilde{\mathbf{p}}) \geqslant 0$, then $\tau(\mathbf{p}) = +\infty$. Otherwise $\tau(\mathbf{p})$ can be represented as*

$$\tau(\mathbf{p}) = -\int_{\bar{p}}^{p^s} \frac{1}{\lambda z(1-z)\cdot\pi(p^n, z)D(p^n, z)}dz. \tag{A1}$$

2. *For any $\mathbf{p} = (p^n, p^s) \in \mathcal{B}_k^{\uparrow}$, if $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$ and $\tau(\mathbf{p}) < +\infty$, then $\tau(p^n, \cdot)$ is differentiable in its second argument at p^s.*[1]

3. *If $D(\mathbf{p}) \leqslant -\varepsilon < 0$ for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$, then $\tau(\mathbf{p})$ is finite for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$.*

[1] At $\mathbf{p}$ with $p^s = \bar{p}$ by the derivative of $\tau(p^n, \cdot)$ we understand its right derivative.

4. *Suppose $g(\mathbf{p}) : \mathcal{B}_k^{\uparrow} \to [\bar{p}, 1]$ is defined indirectly as $\tau\left(f^n(p^n), g(\mathbf{p})\right) = \psi\left(\tau(\mathbf{p})\right)$ for some differentiable and strictly increasing function ψ, and $\tau(\mathbf{p})$ is finite for any $\mathbf{p} \in \mathcal{B}_k^{\uparrow} \cup \mathcal{B}_{k-1}^{\uparrow}$ with $p^s < 1$, strictly increasing and differentiable in p^s on $[\bar{p}, 1)$.*[2] *Then $g(\mathbf{p})$ is a strictly increasing and differentiable function of p^s. In particular, we have the following representation:*

$$\ln\left(\frac{g(\mathbf{p})}{1-g(\mathbf{p})}\right) = J(\mathbf{p}) + \ln\left(\frac{p^s}{1-p^s}\right)$$

where $J(\mathbf{p})$ is a differentiable function of p^s.

Proof. 1. If there exists $\tilde{\mathbf{p}} = (p^n, \tilde{p}^s)$ with $\tilde{p}^s \in [\bar{p}, p^s]$ and $D(\tilde{\mathbf{p}}) \geqslant 0$, then $\mathbf{p}_t$ never reaches $\mathcal{B}_k^{\downarrow}$, so $\tau(\mathbf{p}) = +\infty$ by definition. Now let p_t^s denote the solution to (4) with the initial condition $p_0^s = p^s$. If $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, p_t^s is a strictly decreasing function of t. Therefore, there exists an inverse function $t(p_t^s)$ measuring the time it takes for belief to drift from the initial value p^s to p_t^s. Its derivative is given by

$$\frac{dt(p_t^s)}{dp_t^s} = \left(\lambda p_t^s(1-p_t^s) \cdot \pi(p^n, p_t^s) D(p^n, p_t^s)\right)^{-1},$$

and $t(p^s) = 0$. Therefore, $t(p_t^s) = \int\limits_{p^s}^{p_t^s} \frac{1}{\lambda z(1-z)\cdot\pi(p^n,z)D(p^n,z)} dz$. As $D(\bar{p}) < 0$, the threshold is crossed in zero time. Then substituting $p_t^s = \bar{p}$ we get the result.[3]

2. If $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, then representation (A1) is valid. Taking the derivative with respect to p^s we get

$$\frac{d\tau(\mathbf{p})}{dp^s} = -\left(\lambda p^s(1-p^s) \cdot \pi(p^n, p^s) D(p^n, p^s)\right)^{-1}. \tag{A2}$$

As long as $0 < \bar{p} \leqslant p^s < 1$ and $D(p^n, p^s) < 0$, the derivative is finite and positive.

3. In case $D(\mathbf{p}) \leqslant -\varepsilon < 0$ the improper integral in (A1) converges for any $\mathbf{p}$ and therefore $\tau(\mathbf{p}) < +\infty$.

4. Differentiability of $g(\mathbf{p})$ follows directly from the differentiability and monotonicity of a composition and an inverse function. Differentiability of $J(\mathbf{p})$ is then straightforward as $\ln\left(\frac{g(\mathbf{p})}{1-g(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right)$ is a sum of differentiable functions and is therefore differentiable.

□

Band $\mathcal{B}_0$

Lemma 5. *1. $D(\mathbf{p}) \geqslant -q + \varepsilon$ for some $\varepsilon \in (0, q]$ implies $\ln\left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right) \leqslant \ln(1-\varepsilon)$.*

2. $D(\mathbf{p}) \leqslant -q - \varepsilon$ for some $\varepsilon \in (0, 1-q]$ implies $\ln\left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right) \geqslant \ln(1+\varepsilon)$.

[2] If $\mathbf{p} \in \mathcal{B}_0$, then we let $\mathcal{B}_{k-1} = \mathcal{B}_k = \mathcal{B}_0$.

[3] This proof does not imply that the integral converges. Hence even if $D(p^n, \tilde{p}^s) < 0$ for all $\tilde{p}^s \in [\bar{p}, p^s]$, it may still be that $\tau(\mathbf{p}) = +\infty$.

Proof. We prove only the first claim, the second one is analogous. $D(\mathbf{p}) \geqslant -q + \varepsilon$ implies

$$-(1-q) \cdot r^H(\mathbf{p}) + r^L(\mathbf{p}) \geqslant \varepsilon$$

and further

$$\begin{aligned} \ln\left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right) &= \ln\left(\frac{(1-q)\cdot r^H(\mathbf{p})}{r^L(\mathbf{p})}\right) \\ &\leqslant \ln\left(1 - \frac{\varepsilon}{r^L(\mathbf{p})}\right) \leqslant \ln(1-\varepsilon). \end{aligned}$$

□

Proof of Lemma 1 for $\mathcal{B}_0^{\uparrow}$. Suppose there exists $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ with $\tau(\tilde{\mathbf{p}}) = +\infty$. Then consider states $\mathbf{p}_{inf,1} := (\tilde{p}^n, p^s_{inf,1})$ and $\mathbf{p}_{inf,2} := (f^n(\tilde{p}^n), p^s_{inf,2})$, where $p^s_{inf,1} = \inf\{p^s \mid \tau(\tilde{p}^n, p^s) = +\infty\}$ and $p^s_{inf,2} = \inf\{p^s \mid \tau(f^n(\tilde{p}^n), p^s) = +\infty\}$.[4] We start by showing that

$$\ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) - \ln\left(\frac{p^s_{inf,2}}{1-p^s_{inf,2}}\right) \geqslant -\ln\left(1 - \frac{q}{2}\right). \tag{A3}$$

By Lemma 4 there can be three (mutually non-exclusive) sub-cases to consider.

Case 1 $D(\mathbf{p}_{inf,1}) \geqslant 0$. Then $\tau(\mathbf{p}_{inf,1}) = +\infty$, and $r^L(\mathbf{p}_{inf,1}) > 0$.[5] Therefore, a low-type seller must weakly prefer to disclose a bad review, and thus $\tau(f(\mathbf{p}_{inf,1})) = +\infty$. Then $p^s_{inf,2} \leqslant f^s(\mathbf{p}_{inf,1})$ by definition of $p^s_{inf,2}$, and $\ln\left(\frac{f^s(\mathbf{p}_{inf,1})}{1-f^s(\mathbf{p}_{inf,1})}\right) - \ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) \leqslant \ln(1-q)$ by Lemma 5, which together imply (A3).

Case 2 $D(\mathbf{p}_{inf,1}) < 0$ and there exists a sequence $\{\tilde{p}^s_k\}$ such that $\tilde{p}^s_k \downarrow p^s_{inf,1}$ and $D(\tilde{\mathbf{p}}_k) > -\frac{1}{k}$, where $\tilde{\mathbf{p}}_k := (\tilde{p}^n, \tilde{p}^s_k)$. Then for any $\varepsilon > 0$ and sufficiently high K we have $D(\tilde{\mathbf{p}}_K) > -\frac{q}{4}$, $r^L(\tilde{\mathbf{p}}_K) > 0$, and $\ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) - \ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) < \varepsilon$. As $\tau(\tilde{\mathbf{p}}_K) = +\infty$ and $r^L(\tilde{\mathbf{p}}_K) > 0$, we must have $\tau(f(\tilde{\mathbf{p}}_K)) = +\infty$, and therefore $\mathbf{p}_{inf,2} \leqslant f(\tilde{\mathbf{p}}_K)$. Finally, by Lemma 5 we then have $\ln\left(\frac{f^s(\tilde{\mathbf{p}}_K)}{1-f^s(\tilde{\mathbf{p}}_K)}\right) - \ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) \leqslant \ln\left(1 - \frac{3q}{4}\right)$. It is then true that

$$\begin{aligned} \ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right) - \ln\left(\frac{p^s_{inf,2}}{1-p^s_{inf,2}}\right) &> \ln\left(\frac{\tilde{p}^s_K}{1-\tilde{p}^s_K}\right) - \varepsilon - \ln\left(\frac{f^s(\tilde{\mathbf{p}}_K)}{1-f^s(\tilde{\mathbf{p}}_K)}\right) \\ &\geqslant -\ln\left(1 - \frac{3q}{4}\right) - \varepsilon. \end{aligned}$$

The last term is greater than $-\ln\left(1 - \frac{q}{2}\right)$ for sufficiently small ε.

[4] As the set is non-empty and bounded from below by $\bar{p}$, the infimum exists.

[5] The latter is true because if $r^L(\mathbf{p}_{inf,1}) = 0$, then $D(\mathbf{p}_{inf,1}) \leqslant -q$.

Case 3 $D(\mathbf{p}_{inf,1}) < 0$ and there exists a sequence $\{\tilde{p}^s_k\}$ such that $\tilde{p}^s_k \uparrow p^s_{inf,1}$ and $k < \tau(\tilde{\mathbf{p}}_k) < +\infty$, where $\tilde{\mathbf{p}}_k := (\tilde{p}^n, \tilde{p}^s_k)$. As $\tau(\mathbf{p}_{inf,1}) = +\infty$ in this sub-case and $\tau(\tilde{\mathbf{p}}_k) < +\infty$, for any k there exists $\hat{\mathbf{p}}_k = (\tilde{p}^n, \hat{p}^s_k)$ with $\hat{p}^s_k \in [\tilde{p}^s_k, p^s_{inf,1}]$ such that $\tau(\hat{\mathbf{p}}_k) > k$ and $D(\hat{\mathbf{p}}_k) > -\frac{1}{k}$. Note that $\hat{\mathbf{p}}_k \to \mathbf{p}_{inf,1}$ as $k \to +\infty$. Now suppose (A3) does not hold. Fix some arbitrary $\delta > 0$. For any $\delta > 0$ we have $\tau(f^n(\tilde{p}^n), p^s_{inf,2} - \delta)) < +\infty$, so we can find $k > \frac{4}{q}$ such that $\tau(\hat{\mathbf{p}}_k) > \tau(f^n(\tilde{p}^n), p^s_{inf,2} - \delta))$. By Lemma 5 we know that $\ln\left(\frac{f^s(\hat{\mathbf{p}}_k)}{1-f^s(\hat{\mathbf{p}}_k)}\right) - \ln\left(\frac{\hat{p}^s_k}{1-\hat{p}^s_k}\right) \leqslant \ln\left(1 - \frac{3q}{4}\right)$. As $r^L(\hat{p}^s_k) > 0$, we must have $\tau(f(\hat{\mathbf{p}}_k)) \geqslant \tau(\hat{\mathbf{p}}_k)$, and therefore by the monotonicity of $\tau(f^n(\tilde{p}^n), p^s)$ in its second argument we must have $f^s(\hat{\mathbf{p}}_k) > p^s_{inf,2} - \delta$. However,

$$\ln\left(\frac{f^s(\hat{\mathbf{p}}_k)}{1 - f^s(\hat{\mathbf{p}}_k)}\right) - \ln\left(\frac{p^s_{inf,2}}{1 - p^s_{inf,2}}\right) < \ln\left(1 - \frac{3q}{4}\right) - \ln\left(1 - \frac{q}{2}\right) < 0,$$

which implies that $f^s(\hat{\mathbf{p}}_k) < p^s_{inf,2}$, and by taking sufficiently small δ we achieve a contradiction. $\square$

Having shown (A3), consider the sequence $\{\mathbf{p}_{inf,k}\}$ where $\mathbf{p}_{inf,k} := \left((f^n)^{k-1}(\tilde{p}^n), p^s_{inf,k}\right)$ and $p^s_{inf,k} = \inf\{p^s \mid \tau\left((f^n)^{k-1}(\tilde{p}^n), p^s\right) = +\infty\}$. Equation (A3) then implies that $p^s_{inf,k} < \bar{p}$ for all $k > M := \left\lceil \frac{\ln\left(\frac{p^s_{inf,1}}{1-p^s_{inf,1}}\right)}{\ln(1-\frac{q}{2})} \right\rceil$, i.e., we have $\mathbf{p}_{inf,k} \in \mathcal{B}^{\downarrow}_0$, and there exists $\varepsilon_k > 0$ such that $\mathbf{p} \in \mathcal{B}^{\downarrow}_0$ for all $\mathbf{p} = (p^n_{inf,k}, p^s)$ with $p^s \in [p^s_{inf,k}, p^s_{inf,k} + \varepsilon_k)$. However, by definition we have $\tau(\mathbf{p}) = 0$ for all $\mathbf{p} \in \mathcal{B}^{\downarrow}_0$, which brings us to a contradiction with the definition of $\mathbf{p}_{inf,M}$.

Proof of Lemma 2 for $\mathcal{B}^{\uparrow}_0$. Proofs for this and other regions proceed by contradiction: we show that the low-type seller can neither have strict preference towards revealing a review ($r^L(\mathbf{p}) = 1$), nor towards deleting a review ($r^L(\mathbf{p}) = 0$).

Suppose that at some $\mathbf{p} \in \mathcal{B}^{\uparrow}_0 \cap \mathcal{R}$ a low-type seller strictly prefers to reveal a bad review, i.e., $r^L(\mathbf{p}) = 1$. Then $D(\mathbf{p}) \geqslant 0$ and $\tau(\mathbf{p}) = +\infty$, which contradicts Lemma 1 for $\mathcal{B}^{\uparrow}_0$. If $r^L(\mathbf{p}) = 0$ and $r^H(\mathbf{p}) > 0$ instead, then revealing a bad review brings the maximal continuation profit to a low-type seller, while deleting it yields strictly less if no new bad review arrives in time $\tau(\mathbf{p})$, which is finite by Lemma 1 for all $\mathbf{p} \in \mathcal{B}^{\uparrow}_0$, so the probability of this happening is strictly positive. That contradicts $r^L(\mathbf{p}) = 0$. As $r^L(\mathbf{p}) < 1$ for all $\mathbf{p} \in \mathcal{B}^{\uparrow}_0$, we have that a low-type seller weakly prefers to conceal a bad review at every state in $\mathcal{B}^{\uparrow}_0$. Therefore, the value of a low-type seller is equal to the value he gets by deleting all further bad reviews: $V^L(\mathbf{p}) = \int_0^{\tau(\mathbf{p})} e^{-rt} \cdot \mu dt$. As $V^L(\mathbf{p}) = V^L(f(\mathbf{p}))$, we must then have $\tau(\mathbf{p}) = \tau(f(\mathbf{p}))$. $\square$

For further proofs we introduce a new object: the *average drift* at state $\mathbf{p} = (p^n, p^s)$ is defined as

$$\bar{D}(\mathbf{p}) := \frac{1}{\lambda\pi(\mathbf{p})\tau(\mathbf{p})}\left(\ln\left(\frac{p^s}{1 - p^s}\right) - \ln\left(\frac{\bar{p}}{1 - \bar{p}}\right)\right).$$

By Lemma 4 $\tau(\mathbf{p})$ is differentiable in p^s, and by Lemma 1 for $\mathcal{B}^{\uparrow}_0$, in any equilibrium $\tau(\mathbf{p}) < +\infty$ for all $\mathbf{p} \in \mathcal{B}^{\uparrow}_0$. Therefore, in any equilibrium $\bar{D}(\mathbf{p})$ is well defined in $\mathcal{B}^{\uparrow}_0$ and is differentiable with respect to p^s for any $p^s < 1$. Lemma 2 for $\mathcal{B}^{\uparrow}_0$ also states that $\tau(\mathbf{p}) = \tau(f(\mathbf{p}))$, and therefore function $J(\mathbf{p})$ is well-defined for all $\mathbf{p} \in \mathcal{B}^{\uparrow}_0$ by Lemma 4.

Lemma 6. *1. Suppose there exists a state $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ such that $\bar{D}(\tilde{\mathbf{p}}) \leqslant -q-\varepsilon$ for some $\varepsilon \in (0, 1-q]$. Then there exists $\hat{p}^s \in [\bar{p}, \tilde{p}^s]$ such that $\bar{D}(\tilde{p}^n, \hat{p}^s) \leqslant -q-\varepsilon$ and $J(\tilde{p}^n, \hat{p}^s) \geqslant \ln(1+\varepsilon)$.*

2. Suppose there exists a state $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_0^{\uparrow}$ such that $\bar{D}(\tilde{\mathbf{p}}) \geqslant -q+\varepsilon$ for some $\varepsilon \in (0, q)$. Then there exists $\hat{p}^s \in [\bar{p}, \tilde{p}^s]$ such that $\bar{D}(\tilde{p}^n, \hat{p}^s) \geqslant -q+\varepsilon$ and $J(\tilde{p}^n, \hat{p}^s) \leqslant \ln(1-\varepsilon)$.

Proof. We only show the first statement; the second is proved analogously. Consider a set $S := \{p^s \in [\bar{p}, \tilde{p}^s] \mid J(\tilde{p}^n, p^s) \geqslant \ln(1+\varepsilon)\}$. First, it is nonempty, as otherwise by Lemma 5 we have $D(\mathbf{p}) > -q-\varepsilon$ for all $\mathbf{p}$ with $p^s \in [\bar{p}, \tilde{p}^s]$, which violates $\bar{D}(\tilde{\mathbf{p}}) \leqslant -q-\varepsilon$.[6] Second, S is closed (as $J(\mathbf{p})$ is continuous in p^s) so its upper contour sets are closed in p^s. Finally, S is trivially bounded from above by $\tilde{p}^s$. Therefore, there exists $\hat{p}^s := \sup S \in S$. Moreover, for all $p^s > \hat{p}^s$ we have $J(\tilde{p}^n, p^s) < \ln(1+\varepsilon)$ and, therefore, $D(\tilde{p}^n, p^s) > -q-\varepsilon$, which implies $\bar{D}(\tilde{p}^n, \hat{p}^s) \leqslant -q-\varepsilon$. The second property of $\hat{p}^s$ follows directly from the definition of S. □

Proof of Proposition 1. First note that any strategy profile that generates $f^s(\mathbf{p}) = p^s$ for all $\mathbf{p} \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ constitutes an equilibrium. Indeed, by Lemma 3 $f^s(\mathbf{p}) = p^s$ implies $D(\mathbf{p}) = -q$ for all $\mathbf{p}$, and therefore $\tau(\mathbf{p}) = \tau(f(\mathbf{p}))$ for all $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$, making both types of sellers indifferent between disclosing and concealing a bad review.

Proof of the converse is split into two steps. In step 1 below we show that if there exists $\mathbf{p} \in \mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ such that $J(\mathbf{p}) \neq 0$, then there exists a state $\tilde{\mathbf{p}}$ such that either $\bar{D}(\tilde{\mathbf{p}}) \leqslant -q-\varepsilon$ and $J(\tilde{\mathbf{p}}) \geqslant \ln(1+\varepsilon)$, or $\bar{D}(\tilde{\mathbf{p}}) \geqslant -q+\varepsilon$ and $J(\tilde{\mathbf{p}}) \leqslant \ln(1-\varepsilon)$.[7] Then in step 2 we achieve a contradiction in both of these cases.

Step 1 Suppose there exists $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$ such that $J(\mathbf{p}) \neq 0$. If $\bar{D}(\mathbf{p}) \neq -q$, then the claim is valid by Lemma 6. Now suppose that $\bar{D}(\mathbf{p}) = -q$. Then as $J(\mathbf{p}) \neq 0$, it must be that $\bar{D}(f(\mathbf{p})) \neq -q$ and we can apply Lemma 6 to $f(\mathbf{p})$.

Step 2 Suppose there exists $\mathbf{p}_1$ such that $\bar{D}(\mathbf{p}_1) \leqslant -q-\varepsilon$ and $J(\mathbf{p}_1) \geqslant \ln(1+\varepsilon)$. Denote $K := 1 + \ln(1+\varepsilon) \cdot \tau(\mathbf{p}_1)^{-1}$. As $\tau(f(\mathbf{p}_1)) = \tau(\mathbf{p}_1)$ and $D(\mathbf{p}) \geqslant -1$, it must be that $\bar{D}(f(\mathbf{p}_1)) \leqslant K \cdot (-q-\varepsilon)$. Then by Lemma 6 there exists $\mathbf{p}_2 = (p_2^s, p_2^n)$ with $p_2^s \in [\bar{p}, f^s(\mathbf{p}_1)]$ and $p_2^n := f^n(p_1^n)$ such that $\bar{D}(\mathbf{p}_2) \leqslant K(-q-\varepsilon)$ and $J(\mathbf{p}_2) \geqslant \ln(1+K(q+\varepsilon)-q) > \ln(1+\varepsilon)$. Iterating this procedure $M := [-\log_K(q+\varepsilon)]+1$ times we arrive at a state $\mathbf{p}_M$ such that $\bar{D}(\mathbf{p}_M) \leqslant K^M(-q-\varepsilon) < -1$, which is impossible.

Alternatively, suppose there exists $\mathbf{p}_1$ such that $\bar{D}(\mathbf{p}_1) \geqslant -q+\varepsilon$ and $J(\mathbf{p}_1) \leqslant \ln(1-\varepsilon)$. Then as $\tau(f(\mathbf{p}_1)) = \tau(\mathbf{p}_1)$ and $J(\mathbf{p}_1) < 0$, it must be that $\bar{D}(f(\mathbf{p}_1)) > -q+\varepsilon$. Then by Lemma 6 there exists $\mathbf{p}_2 = (p_2^s, p_2^n)$ with $p_2^s \in [\bar{p}, f^s(\mathbf{p}_1)]$ and $p_2^n := f^n(p_1^n)$ such that $\bar{D}(\mathbf{p}_2) \geqslant -q+\varepsilon$ and $J(\mathbf{p}_2) \leqslant \ln(1-\varepsilon)$. At the same time, $\ln\left(\frac{p_2^s}{1-p_2^s}\right) - \ln\left(\frac{p_1^s}{1-p_1^s}\right) < \ln(1-\varepsilon)$. Iterating this procedure $M := \left[\left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p_1^s}{1-p_1^s}\right)\right) \cdot \frac{1}{\ln(1-\varepsilon)}\right] + 1$ times we achieve a state $\mathbf{p}_M = (p_M^s, p_M^n)$ such that $p_M^s < \bar{p}$ and $\tau(\mathbf{p}_M) = \tau(\mathbf{p}_1)$, – a contradiction. □

Proof of Corollary 2. Proposition 1 and Lemma 3 imply that $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_0$ in any equilibrium. Therefore, (A1) states that $\tau(\mathbf{p})$ for any given $\mathbf{p}$ must be the same in any equilibrium.

[6] If $D(\mathbf{p}) \neq -q$, then $\mathbf{p} \in \mathcal{R}$ and (2) imply $J(\mathbf{p}) = \ln\left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right)$.

[7] Note that $J(\mathbf{p}) = 0$ implies $f^s(\mathbf{p}) = p^s$.

Representation (5) then implies that the same is true for $V^L(\mathbf{p})$. The high type's value $V^H(\mathbf{p})$ is also the same in any equilibrium, since it can be written for $\mathbf{p} \in \mathcal{B}_0$ as

$$V^H(\mathbf{p}) = \int_0^{\tau(\mathbf{p})} e^{-rt}\left(\mu + (1-\mu)\cdot(1-e^{-\lambda q\mu t})\right)dt + \int_{\tau(\mathbf{p})}^{+\infty} e^{-rt}\left(1-e^{-\lambda q\mu\tau(\mathbf{p})}\right)dt$$

$$= \frac{\mu\,(r+\lambda q)}{r(r+\lambda q\mu)}\cdot\left(1-e^{-(r+\lambda q\mu)\tau(\mathbf{p})}\right).$$

Finally, consumers' behavior and, hence, payoffs are always the same at a given $\mathbf{p}$ in any equilibrium. Therefore, for a given $\mathbf{p} \in \mathcal{B}_0$ all players' payoffs are the same in any continuation equilibrium. □

Band $\mathcal{B}_1$

Lemma 7. *If $\mu < 1/2$, then $\mathcal{B}_1 \cap \mathcal{R} = \emptyset$.*

Proof. If $\mathbf{p} \in \mathcal{B}_1$ then the low-type seller can guarantee himself

$$V^L(\mathbf{p}) \geqslant \frac{1-\mu}{r} + (1-e^{-r\tau(\mathbf{p})})\cdot\frac{\mu}{r} > \frac{1-\mu}{r}$$

by deleting all future reviews and retaining naive consumers forever and sophisticated consumers for time $\tau(\mathbf{p})$. Disclosing any bad review makes naive consumers quit the market until a good review arrives (which is never for a low-type seller), so

$$V^L(f(\mathbf{p})) = (1-e^{-r\tau(f(\mathbf{p}))})\cdot\frac{\mu}{r} \leqslant \frac{\mu}{r}.$$

As one can see, if $\mu < 1/2$, then $V^L(f(\mathbf{p})) < V^L(\mathbf{p})$, hence the low-type seller is never willing to disclose a bad review. □

Proof of Lemma 2 for $\mathcal{B}_1$. Whenever $\mu < 1/2$, by Lemma 7 we have $\mathcal{B}_1 \cap \mathcal{R} = \emptyset$ so the statement is trivially true. Thus from now on assume $\mu \geqslant 1/2$. We divide the proof into two parts corresponding to two subregions of $\mathcal{B}_1$.

Case 1: $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ There it must be that $(1-q)\cdot r^H(\mathbf{p}) > r^L(\mathbf{p})$, as by sacrificing the pool of naive consumers any seller must gain the pool of sophisticated consumers for at least some period of time, so $f^s(\mathbf{p}) > \bar{p} > p^s$. In particular, this implies that $r^L(\mathbf{p}) = 1$ is not possible in any equilibrium.

As for the second case, suppose instead that $r^L(\mathbf{p}) = 0$ and $r^H(\mathbf{p}) > 0$. Then any single bad review reveals a high-type seller and trades off the pool of naive consumers for the whole pool of sophisticated consumers forever. Either group under the respective scenario stays on the marker forever, and the other group joins after a good review. Thus $r^L(\mathbf{p}) = 0$ is optimal for the low-type seller only if $\mu = 1/2$. In that case the low-type seller is indifferent between disclosing a bad review and concealing it. If, however, $\mu > 1/2$, then the combination of $r^L(\mathbf{p}) = 0$ and $r^H(\mathbf{p}) > 0$ is impossible, and thus $r^L(\mathbf{p}) > 0$.

Case 2: $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$ If $\mu = 1/2$, then a strategy profile constitutes an equilibrium in $\mathcal{B}_1^{\uparrow}$ if and only if $r^H(\mathbf{p}) = r^L(\mathbf{p}) = 0$ for states with $p^s > \bar{p}$, and $r^L(p^n, \bar{p}) = 0$. At $\bar{p}$ the low-type seller can then retain one and only one of two types of consumers on the market while another is driven out forever, and he is therefore indifferent between revealing and deleting (but in equilibrium deletes all bad reviews).

Thus for the remainder of the proof we assume that $\mu > 1/2$ and consider $\mathbf{p} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$. If $r^L(\mathbf{p}) = 1$, then $D(\mathbf{p}) \geqslant 0$, so staying silent at $\mathbf{p}$ gives the maximum possible continuation payoff to any seller. On the other hand, by disclosing at $\mathbf{p}$ any seller loses naive consumers for at least some time and therefore gets strictly less, – a contradiction with the optimality of $r^L(\mathbf{p}) = 1$.

We are left to show that the low-type seller cannot strictly prefer to delete a bad review. Suppose by way of contradiction that there exists $\tilde{\mathbf{p}} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ such that $r^L(\tilde{\mathbf{p}}) = 0$ and $r^H(\tilde{\mathbf{p}}) > 0$. Then $f^s(\tilde{\mathbf{p}}) = 1$ and $\tau(f(\tilde{\mathbf{p}})) = +\infty$. Moreover, non-disclosure is on path for the low type in this and all future states, thus deleting this and all future bad reviews must be weakly better for the low-type seller than disclosing a bad review at $\tilde{\mathbf{p}}$ and all further bad reviews afterwards:

$$\int_0^{+\infty} e^{-rt}(1-\mu)dt + \int_0^{\tau(\tilde{\mathbf{p}})} e^{-rt}\mu dt \geqslant \int_0^{\tau(f(\tilde{\mathbf{p}}))} e^{-rt}\mu dt$$

$$\Leftrightarrow \frac{1-\mu}{\mu} + e^{-r\tau(f(\tilde{\mathbf{p}}))} \geqslant e^{-r\tau(\tilde{\mathbf{p}})}. \tag{A4}$$

On the other hand, the high-type seller's value from disclosing a bad review at $\tilde{\mathbf{p}}$ is

$$V^H(f(\tilde{\mathbf{p}})) = \int_0^{\tau(f(\tilde{\mathbf{p}}))} e^{-rt}\left(\mu + (1-\mu)\cdot(1-e^{-\lambda q\mu t})\right)dt + \tag{A5}$$

$$+ \int_{\tau(f(\tilde{\mathbf{p}}))}^{+\infty} e^{-rt}\left(1 - e^{-\lambda q\mu\tau(f(\tilde{\mathbf{p}}))}\right)dt$$

$$= \frac{\mu(r+\lambda q)}{r(r+\lambda q\mu)} \cdot \left(1 - e^{-(r+\lambda q\mu)\tau(f(\tilde{\mathbf{p}}))}\right). \tag{A6}$$

The value that the high-type seller gets from deleting a bad review at $\tilde{\mathbf{p}}$ is at least as large as the value from deleting all bad reviews from $\tilde{\mathbf{p}}$ onwards:

$$V^H(\tilde{\mathbf{p}}) \geqslant \int_0^{\tau(\tilde{\mathbf{p}})} e^{-rt}dt + \int_{\tau(\tilde{\mathbf{p}})}^{+\infty} e^{-rt}\left(1 - \mu e^{-\lambda q(\mu\tau(\tilde{\mathbf{p}}) + (1-\mu)t)}\right)dt$$

$$= \frac{1}{r}\cdot\left(1 - \frac{r\mu}{r+\lambda q(1-\mu)}e^{-(r+\lambda q)\tau(\tilde{\mathbf{p}})}\right)$$

$$\geqslant \frac{1}{r} - \frac{\mu}{r+\lambda q(1-\mu)}\cdot\left(\frac{1-\mu}{\mu}\right)^{1+\frac{\lambda q}{r}},$$

where the last inequality follows from (A4) after recalling that $\tau(f(\tilde{\mathbf{p}})) = +\infty$. As $r^H(\tilde{\mathbf{p}}) > 0$, it must be that $V^H(f(\tilde{\mathbf{p}})) \geqslant V^H(\tilde{\mathbf{p}})$, which implies:

$$\frac{\mu(r+\lambda q)}{r(r+\lambda q\mu)} - \frac{1}{r} + \frac{\mu}{r+\lambda q(1-\mu)}\cdot\left(\frac{1-\mu}{\mu}\right)^{1+\frac{\lambda q}{r}} \geqslant 0$$

$$\Leftrightarrow \frac{1+\frac{\lambda q}{r}\mu}{1+\frac{\lambda q}{r}(1-\mu)}\cdot\left(\frac{1-\mu}{\mu}\right)^{\frac{\lambda q}{r}} \geqslant 1 \tag{A7}$$

Note that (A7) holds with equality for $\mu = 1/2$ and that $\left(1 + \frac{\lambda q}{r} x\right)(1-x)^{\frac{\lambda q}{r}}$ is a decreasing function of x for all $x \in (0,1)$. This means that (A7) is violated whenever $\mu > 1/2$, so there does not exist any $\tilde{\mathbf{p}} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ with $r^L(\tilde{\mathbf{p}}) = 0$.

Finally, as $r^L(\mathbf{p}) < 1$ for all $\mathbf{p} \in \mathcal{B}_1$ and the low-type seller is indifferent between disclosing and concealing a bad review at all $\mathbf{p} \in \mathcal{B}_0^{\uparrow} \cup \mathcal{B}_1$, (A4) holds with equality for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$. □

Proof of Lemma 1 for $\mathcal{B}_{1+}^{\uparrow}$. We prove the claim only for $\mathcal{B}_1^{\uparrow}$. Induction to all further bands is straightforward. Assume the contrary. Then there exists $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_1^{\uparrow}$ with $\tau(\tilde{\mathbf{p}}) = +\infty$. Consider a state $\tilde{\mathbf{p}}_{inf} := (\tilde{p}^n, \tilde{p}^s_{inf}) \in \mathcal{B}_1^{\uparrow}$ where $\tilde{p}^s_{inf} = \inf\{p^s \mid \tau(\tilde{p}^n, p^s) = +\infty\}$. According to Lemma 4, there can be three sub-cases. Either $D(\tilde{\mathbf{p}}_{inf}) \geqslant 0$, or $D(\tilde{\mathbf{p}}_{inf}) < 0$ and there exists a sequence $\tilde{p}^s_k$ converging to $\tilde{p}^s_{inf}$ either from below or from above such that $D(\tilde{p}^n, \tilde{p}^s_k) \to 0$.

If $D(\tilde{\mathbf{p}}_{inf}) \geqslant 0$ or $\tilde{p}^s_k$ converges to $\tilde{p}^s_{inf}$ from above, there exists $\hat{\mathbf{p}}$ such that $\tau(\hat{\mathbf{p}}) = +\infty$ (i.e., no disclosure at $\hat{\mathbf{p}}$ grants the maximal continuation payoff) and $D(\hat{\mathbf{p}}) > -q$, with the latter implying that $\hat{\mathbf{p}} \in \mathcal{R}$. By deleting all bad reviews the seller can retain both groups of consumers in the market forever starting from $\hat{\mathbf{p}}$. However, we know that $V^\theta(f(\hat{\mathbf{p}}))$ is strictly smaller than the maximal possible payoff for seller of type θ, since this is true for any $\mathbf{p} \in \mathcal{B}_0$ with $p^s < 1$. Revealing a bad review at $\hat{\mathbf{p}}$ is thus strictly suboptimal for either type of the seller, which contradicts $\hat{\mathbf{p}} \in \mathcal{R}$.

If $\tilde{p}^s_k$ converges to $\tilde{p}^s_{inf}$ from below, then for any $\varepsilon > 0$ and any $C > 0$ there exists $\hat{\mathbf{p}}$ such that $D(\hat{\mathbf{p}}) > -\varepsilon$ and $\tau(\hat{\mathbf{p}}) > C$. The latter property is satisfiable, as improper integral in (A1) diverges, and therefore for any $C > 0$ there exists some k such that $\tau(\tilde{p}^n, \tilde{p}^s_{inf} - \frac{1}{k}) > C$. As for the former, we know that $\tau(\tilde{p}^n, \tilde{p}^s_{inf}) - \tau(\tilde{p}^n, \tilde{p}^s_{inf} - \frac{1}{k}) = +\infty$, and therefore there exists $\hat{p}^s \in [\tilde{p}^s_{inf} - \frac{1}{k}, \tilde{p}^s_{inf}]$ such that $D(\tilde{p}^n, \hat{p}^s) > -\varepsilon$. As the seller's value $V^\theta(\mathbf{p})$ in any state $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$ with $p^s < 1$ is strictly less than the maximal one and as C can be made arbitrarily large, we can find C large enough that the value of disclosure is strictly less than the value of staying silent. Since $\hat{\mathbf{p}} \in \mathcal{R}$ as long as $\varepsilon < q$, we achieve a contradiction. □

Proof of Proposition 2. It is shown in Lemma 7 that if $\mu < 1/2$, then $r^L(\mathbf{p}) = r^H(\mathbf{p}) = 0$ for all $\mathbf{p} \in \mathcal{B}_1 \cap \mathcal{R}$ is the only possible equilibrium strategy profile. To show the second condition, recall from Lemma 2 that a low-type seller must be indifferent between revealing a bad review at $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$ and not, and that his indifference condition can be written as

$$\frac{1-\mu}{\mu} + e^{-r\tau(f(\mathbf{p}))} = e^{-r\tau(\mathbf{p})}. \tag{A8}$$

As $\tau(f(\mathbf{p})) \leqslant +\infty$, it should be that $\tau(\mathbf{p}) \leqslant \frac{1}{r} \ln \frac{\mu}{1-\mu}$. Therefore, as $D(\mathbf{p}) \geqslant -1$, we have $\ln \frac{p^*}{1-p^*} - \ln \frac{\bar{p}}{1-\bar{p}} \leqslant \frac{\lambda}{r} \ln \frac{\mu}{1-\mu}$ which gives the result. □

Proof of Proposition 3. The claim was already established for $\mathcal{B}_1^{\downarrow}$ in the proof of Lemma 2 for $\mathcal{B}_1$. We are left to show it for $\mathcal{B}_1^{\uparrow}$. As Lemma 4 shows, we can construct a mapping g such that

$$\frac{1-\mu}{\mu} + e^{-r\tau(g(\mathbf{p}))} = e^{-r\tau(\mathbf{p})}, \tag{A9}$$

so $g(\mathbf{p}) = f^s(\mathbf{p})$ for all $\mathbf{p} \in \mathcal{R}$. Further, $g(\mathbf{p})$ can be represented as

$$\ln\left(\frac{g(\mathbf{p})}{1-g(\mathbf{p})}\right) = J(\mathbf{p}) + \ln\left(\frac{p^s}{1-p^s}\right) \tag{A10}$$

for some function $J(\mathbf{p})$ which is differentiable in p^s. Taking the derivative of both sides of (A9) with respect to p^s, we get

$$e^{-r\tau(\mathbf{p})} \cdot \frac{d\tau(\mathbf{p})}{dp^s} = e^{-r\tau(g(\mathbf{p}))} \cdot \frac{d\tau(g(\mathbf{p}))}{dg(\mathbf{p})} \frac{dg(\mathbf{p})}{dp^s}. \tag{A11}$$

As is shown in Lemma 4, $\frac{d\tau(\mathbf{p})}{dp^s} = (\lambda p^s (1-p^s)\, \pi(\mathbf{p}) D(\mathbf{p}))^{-1}$. Differentiating (A10) we get

$$\frac{dg(\mathbf{p})}{dp^s} = \frac{e^{-J(\mathbf{p})} + p^s(1-p^s)\frac{dJ(\mathbf{p})}{dp^s} e^{-J(\mathbf{p})}}{\left[p^s + (1-p^s)e^{-J(\mathbf{p})}\right]^2}.$$

Plugging the three derivatives, we get that (A11) corresponds to

$$e^{-r\tau(\mathbf{p})} \cdot \mu q = e^{-r\tau(g(\mathbf{p}))} \cdot (-D(\mathbf{p})) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(\mathbf{p})}{dp^s}\right].$$

Plugging (A9) into the expression above we get

$$(-D(\mathbf{p})) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(\mathbf{p})}{dp^s}\right] = q \cdot \left[\mu + (1-\mu) \cdot e^{r\tau(g(\mathbf{p}))}\right]. \tag{A12}$$

Consider state $(p^n, \bar{p}) \in \mathcal{B}_1$. We know $\tau(p^n, \bar{p}) = 0$, therefore (A9) implies $\tau(g(p^n, \bar{p})) > 0$, which in turn means $J(p^n, \bar{p}) > 0$. For any $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$ we have $\tau(g(\mathbf{p})) > \tau(g(p^n, \bar{p})) > 0$, therefore there exists $\varepsilon > 0$ such that the RHS of (A12) is larger than $q + \varepsilon$. If additionally $\frac{dJ(\mathbf{p})}{dp^s} < 0$, (A12) implies $D(\mathbf{p}) < -q - \varepsilon$, and consequently $J(\mathbf{p}) \geqslant \ln(1+\varepsilon)$ by Lemma 5. It then follows from continuity of $J(\mathbf{p})$ in p^s that $J(\mathbf{p}) > 0$ for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$. For there to exist $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$ such that $J(\mathbf{p}) \leqslant 0$ there should exist $\tilde{\mathbf{p}}$ such that $J(\tilde{\mathbf{p}}) \in (0, \ln(1+\varepsilon))$ and $\frac{dJ(\tilde{\mathbf{p}})}{dp^s} < 0$, which is ruled out by the argument above. ☐

Lemma 8. *If $\mu \geqslant 1/2$, then for any set $\tilde{\mathcal{R}} \subseteq \mathcal{B}_1^{\downarrow}$ there exists an equilibrium with $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \tilde{\mathcal{R}}$.*

Proof. As Lemma 2 states, for $\mu \geqslant 1/2$ the low-type seller is indifferent between disclosing a bad review and concealing it at all $\mathbf{p} \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$. This indifference is given by (A8), and using $\tau(\mathbf{p}) = 0$ for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$ as well as the fact that $\tau(f(\mathbf{p})) = \frac{1}{\lambda q \mu}\left[\ln\left(\frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})}\right) - \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right]$ it can be rewritten as[8]

$$\left(\frac{\bar{p}}{1-\bar{p}} \cdot \frac{1-f^s(\mathbf{p})}{f^s(\mathbf{p})}\right)^{\frac{r}{\lambda q \mu}} = 2 - \frac{1}{\mu}$$

$$\Leftrightarrow \frac{f^s(\mathbf{p})}{1-f^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{(1-q) \cdot r^H(\mathbf{p})}{r^L(\mathbf{p})} = \frac{\bar{p}}{1-\bar{p}} \left(2 - \frac{1}{\mu}\right)^{-\frac{\lambda q \mu}{r}}.$$

Next we consider incentives of a high-type seller. Since $r^H(\mathbf{p}) > 0$, he should weakly prefer to reveal a bad review. We further show that this is always true whenever $\mu \geqslant 1/2$ (and the preference is strict if $\mu > 1/2$), and therefore $r^H(\mathbf{p}) = 1$ constitutes an equilibrium in $\mathcal{B}_1^{\downarrow}$. The value from revealing a bad review can be computed by plugging (A8) and $\tau(\mathbf{p}) = 0$ into (A6) to obtain

$$V^H(f(\mathbf{p})) = \left(\frac{1}{r} - \frac{1-\mu}{r+\lambda q \mu}\right) \cdot \left(1 - \left(2 - \frac{1}{\mu}\right)^{1+\frac{\lambda q \mu}{r}}\right). \tag{A13}$$

[8]As $D(\mathbf{p}) = -q$ in $\mathcal{B}_0^{\uparrow}$, the expression for $\tau(f(\mathbf{p}))$ follows from (A1) and the fact that $\pi(\mathbf{p}) = \mu$ for $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$.

The value of staying silent at $\mathbf{p}$ is no greater than supremum over all T of expected payoffs from staying silent until T and then receiving and disclosing a bad review exactly at T (with $T = +\infty$ allowed as an option to stay silent forever). The remainder of this proof shows that this amount is smaller than $V^H(f(\mathbf{p}))$, which finalizes the argument. The supremum is equal to

$$\bar{V} = \sup_T \left\{ \int_0^T e^{-rt} \left[1 - \mu \cdot e^{-\lambda q(1-\mu)t}\right] dt + \right.$$
$$\left. + e^{-rT} \left(e^{-\lambda q(1-\mu)T} \cdot V^H\left(f(\mathbf{p}_T)\right) + \left(1 - e^{-\lambda q(1-\mu)T}\right) \cdot \frac{1}{r} \right) \right\}.$$

By simplifying the expression above we obtain

$$\bar{V} = \sup_T \left(\frac{1}{r} - \frac{\mu}{r + \lambda q(1-\mu)} \right) \cdot \left(1 - e^{-(r+\lambda q(1-\mu))T}\right) + e^{-(r+\lambda q(1-\mu))T} \cdot V^H\left(f(\mathbf{p}_T)\right),$$

which is a convex combination of $\left(\frac{1}{r} - \frac{\mu}{r+\lambda q(1-\mu)}\right)$ and $V^H(f(\mathbf{p}_T))$. The latter is given by (A13) and is thus independent of T. Therefore, to finalize the argument we need to show that

$$V^H(f(\mathbf{p})) \geqslant \left(\frac{1}{r} - \frac{\mu}{r + \lambda q(1-\mu)} \right),$$

which would mean that staying silent at $\mathbf{p}$ is weakly worse than revealing a bad review at $\mathbf{p}$, and the equality is attained only when $\mu = 1/2$. Indeed, the condition above is equivalent to

$$\frac{1}{1 + \frac{\lambda q(1-\mu)}{r}} \cdot \left(2 - \frac{1}{\mu}\right) \geqslant \left(2 - \frac{1}{\mu}\right)^{1+\frac{\lambda q \mu}{r}}. \tag{A14}$$

For $\mu = 1/2$ the inequality is trivially satisfied with equality. And for $\mu \in (1/2, 1)$ we have

$$\left(1 - \frac{1-\mu}{\mu}\right)^{\frac{\lambda q \mu}{r}} < e^{-\frac{\lambda q(1-\mu)}{r}} < \frac{1}{1 + \frac{\lambda q(1-\mu)}{r}},$$

which concludes the argument. □

Band $\mathcal{B}_{2+}$

Proof of Lemma 2 for $\mathcal{B}_{2+}$. Suppose not: there exists some $\mathbf{p} \in \mathcal{R}$ at which the low-type seller has strict preference. Depending on the direction of this preference, two cases are possible:

Case 1: $r^L(\mathbf{p}) = 0$, $r^H(\mathbf{p}) > 0$ Then $f^s(\mathbf{p}) = 1$ and $f^n(\mathbf{p}) \geqslant \bar{p}$, so by revealing this bad review and deleting all future ones the seller can guarantee himself the maximal possible continuation payoff. Therefore, deleting bad review at $\mathbf{p}$ cannot be strictly better than leaving it – a contradiction.

Case 2: $r^L(\mathbf{p}) = 1$, $r^H(\mathbf{p}) \leqslant 1$ It implies $D(\mathbf{p}) \geqslant 0$. If $p^s \geqslant \bar{p}$, then this contradicts Lemma 1 for $\mathbf{p} \in \mathcal{B}^{\uparrow}_{1+}$. If, however, $p^s < \bar{p}$, then by Lemma 3 $D(\mathbf{p}) \geqslant 0$ implies that $f^s(\mathbf{p}) < p^s$ for bad reviews revealed at $\mathbf{p}$, and therefore $f^s(\mathbf{p}) < \bar{p}$. The low-type seller's value from revealing a bad review in $\mathcal{B}^{\downarrow}_1$ is equal to the value of deleting all future bad reviews starting from $f(\mathbf{p})$. Deleting a bad review in

$\mathcal{B}_2^{\downarrow}$ can guarantee at least the same value by case of deleting all bad reviews. This means that despite we've assumed $r^L(\mathbf{p}) = 1$, the low-type seller is indeed indifferent between disclosure and concealment at $\mathbf{p}$.

We have shown that the low-type seller's value at any $\mathbf{p} \in \mathcal{B}_{2+}$ is equal to that from deleting all bad reviews starting from $\mathbf{p}$, and the value of disclosure at $\mathbf{p}$ is equal to the value he gets deleting all bad reviews starting from $f^s(\mathbf{p})$. Thus the indifference condition of the low-type seller results in

$$\int_0^{\tau(\mathbf{p})} e^{-rt} dt + (1-\mu) e^{-r\tau(\mathbf{p})} \int_0^{+\infty} e^{-rt} dt = \int_0^{\tau(f(\mathbf{p}))} e^{-rt} dt + (1-\mu) e^{-r\tau(f(\mathbf{p}))} \int_0^{+\infty} e^{-rt} dt,$$

which can be further reduced to

$$\tau(f(\mathbf{p})) = \tau(\mathbf{p}). \tag{A15}$$

This concludes the proof. □

Proof of Proposition 4. We show the claim for $\mathcal{B}_2^{\uparrow}$. Induction to $\mathcal{B}_k^{\uparrow}$ with $k > 2$ is straightforward.

As Lemma 4 shows, we can construct mapping g such that $\tau(g(\mathbf{p})) = \tau(\mathbf{p})$, and for some function $J(\mathbf{p})$ which is continuous in p^s we have:

$$\ln\left(\frac{g(\mathbf{p})}{1-g(\mathbf{p})}\right) = J(\mathbf{p}) + \ln\left(\frac{p^s}{1-p^s}\right).$$

Now suppose that there exists $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_2^{\uparrow}$ such that $f^s(\tilde{\mathbf{p}}) < \tilde{p}^s$. Then $g(\tilde{\mathbf{p}}) = f^s(\tilde{\mathbf{p}})$, and therefore $J(\tilde{\mathbf{p}}) < 0$. As $J(\mathbf{p})$ is a continuous function of p^s and $J(\tilde{p}^n, \bar{p}) = 0$, there exists $\hat{p}^s < \tilde{p}^s$ such that $J(\tilde{p}^n, \hat{p}^s) = 0$ and $J(\tilde{p}^n, p^s) < 0$ for all $p^s \in (\hat{p}^s, \tilde{p}^s]$. Thus $g(\tilde{p}^n, p^s) \leqslant p^s$ for all $p^s \in [\hat{p}^s, \tilde{p}^s]$. Therefore, by Lemmas 3 and 4 we must have $D(\tilde{p}^n, p^s) \geqslant -q$ for all $p^s \in [\hat{p}^s, \tilde{p}^s]$. However, $D(\mathbf{p}) \leqslant -q$ for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$ which violates $\tau(g(\tilde{\mathbf{p}})) - \tau(g(\hat{\mathbf{p}})) = \tau(\tilde{\mathbf{p}}) - \tau(\hat{\mathbf{p}})$ given representation (A1), where $\hat{\mathbf{p}} = (\tilde{p}^n, \hat{p}^s)$. □

Proofs of the Main Results

Proof of Theorem 1. The statement of the Theorem follows directly from Propositions 1, 3 and 4. □

Proof of Corollary 1. From Theorem 1 and expression (2) we have

$$(1-q) \cdot r^H(\mathbf{p}) \geqslant r^L(\mathbf{p}).$$

Therefore, if $r^H(\mathbf{p}) = 0$, then we must have $r^L(\mathbf{p}) = 0$ and $\mathbf{p} \notin \mathcal{R}$. If $r^H(\mathbf{p}) > 0$, then

$$r^H(\mathbf{p}) > (1-q) \cdot r^H(\mathbf{p}) \geqslant r^L(\mathbf{p})$$

which proves the claim. □

Proof of Theorem 2. To prove the first part note that first, by Corollary 2 all continuation equilibria are payoff-equivalent in $\mathcal{B}_0^{\uparrow}$. Next, if $\mu < 1/2$, then Lemma 7 implies that $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_1$, and therefore all continuation equilibria are payoff-equivalent in $\mathcal{B}_1$ as well. As $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \emptyset$ by Lemma 7 and p^s can never cross $\bar{p}$ from below, seller's value $V^{\theta}(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$ is equal in any equilibrium to the value of keeping naive consumers in the market forever. Finally, in any equilibrium $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\uparrow}$: by Theorem 1 and Lemma 3, $D(\mathbf{p}) \leqslant -q$, and if there exists an equilibrium and $\mathbf{p} \in \mathcal{B}_{2+}^{\uparrow}$ with $D(\mathbf{p}) < -q$, then $J(\mathbf{p}) > 0$ by Lemma 3, which violates $\tau(f(\mathbf{p})) = \tau(\mathbf{p})$ as $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_1$. This implies that $\tau(\mathbf{p})$ is constant across equilibria, which together with the above gives payoff-equivalence in $\mathcal{B}_{2+}^{\uparrow}$.

If $\mu = 1/2$, then $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow} \cap \mathcal{R}$ with $p^s > \bar{p}$. Since on any path of play the game only passes through one state in $\mathcal{B}_1^{\uparrow}$ with $p^s = \bar{p}$ (which is the only state in $\mathcal{B}_1^{\uparrow}$ where $D(\mathbf{p}) < -q$ is possible), and drift there is still negative, $\tau(\mathbf{p})$ in any equilibrium must be the same as in case $\mu < 1/2$ (where $\mathcal{B}_1^{\uparrow} \cap \mathcal{R} = \emptyset$) for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$. The same logic as above can then establish that $D(\mathbf{p}) = -q$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\uparrow}$. Finally, in case $\mu = 1/2$ it may be that $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} \neq \emptyset$, but both types of seller are in any such $\mathbf{p}$ indifferent between revealing and deleting a bad review, and therefore receive the same payoff as if $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \emptyset$.

The remainder of the proof is devoted to constructing an equilibrium that satisfies the requirements of the second part of Theorem 2. We propose a strategy profile and show that it satisfies all equilibrium conditions.

Construct the strategy profile as follows. Let $\mathcal{B}_0^{\uparrow} \cap \mathcal{R} = \emptyset$, and for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ build the strategy profile $\left(r^H, r^L\right)$ in such a way that $\mathcal{B}_1^{\downarrow} \cap \mathcal{R} = \mathcal{B}_1^{\downarrow}$, – the latter is possible by Lemma 8.

For $\mu > 1/2$ the inequality in (A14) is strict for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$, so by continuity of preferences of the high-type seller there exists $\varepsilon_1 > 0$ such that he strictly prefers to reveal at all $\mathbf{p} \in \{\mathcal{B}_1^{\uparrow} \mid p^s \in [\bar{p}, \bar{p}+\varepsilon_1)\}$, i.e., these states can belong to $\mathcal{R}$ in equilibrium. In all such states it must be that $r^H(\mathbf{p}) = 1$, and $r^L(\mathbf{p})$ is then defined implicitly by (A8). The latter can be reduced to the following differential equation for $J(\mathbf{p})$:

$$\left(1 - (1-q)e^{-J(\mathbf{p})}\right) \cdot \left[1 + p^s(1-p^s) \cdot \frac{dJ(\mathbf{p})}{dp^s}\right] = \\ = q \cdot \left[\mu + (1-\mu) \cdot \left(\frac{1-\bar{p}}{\bar{p}} \cdot \frac{p^s}{1-p^s}\right)^{\frac{r}{\lambda q \mu}} \cdot e^{\frac{r}{\lambda q \mu} J(\mathbf{p})}\right] \quad \text{(A16)}$$

with an initial condition $J(p^n, \bar{p}) = -\frac{\lambda q \mu}{r} \ln\left(2 - \frac{1}{\mu}\right)$.[9] Then $r^L(\mathbf{p})$ can be obtained from $J(\mathbf{p}) = \ln(1-q) - \ln r^L(\mathbf{p})$. By the existence theorem (see Pontryagin (1962), chapter 4, §21) a solution to (A16) exists in some neighborhood of $(p^n, \bar{p})$, i.e., there exists $\varepsilon_2 > 0$ such that $J(\mathbf{p})$, and consequently $r^L(\mathbf{p})$, is well-defined for all $\mathbf{p} = (p^n, p^s)$ with $p^s \in [\bar{p}, \bar{p} + \varepsilon_2)$. Take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ and set $r^L(\mathbf{p})$ for all $\mathbf{p} \in \{\mathcal{B}_1^{\uparrow} \mid p^s < \bar{p} + \varepsilon\}$ as prescribed by the procedure above. At the remaining states $\mathbf{p} \in \{\mathcal{B}_1^{\uparrow} \mid p^s \geqslant \bar{p} + \varepsilon\}$ set $r^H(\mathbf{p}) = r^L(\mathbf{p}) = 0$.

The strategy profile in $\mathcal{B}_{2+}$ is constructed as follows. For any $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$ let $r^H(\mathbf{p}) = 1$ and $r^L(\mathbf{p}) = (1-q) \cdot \left(\frac{p^s}{1-p^s} \cdot \frac{1-\bar{p}}{\bar{p}}\right)^{\frac{1}{2}}$, which together lead to $J(\mathbf{p}) = \frac{1}{2} \cdot \left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p^s}{1-p^s}\right)\right) > 0$, meaning $\bar{p} > f^s(\mathbf{p}) > p^s$. In $\mathcal{B}_{2+}^{\uparrow}$ let $r^H(\mathbf{p}) = r^L(\mathbf{p}) = 0$ for $\mathbf{p} \in \{\mathcal{B}_{2+}^{\uparrow} \mid p^s = \bar{p}\}$. Let $r^H(\mathbf{p}) = 1$ for all $\mathbf{p} \in \{\mathcal{B}_{2+}^{\uparrow} \mid p^s > \bar{p}\}$. We compute $r^L(\mathbf{p})$ inductively over bands, where the induction statement is "$r^L(\mathbf{p})$ is constructed for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$ and it is such that $D(\mathbf{p}) \leqslant -q$". This is true by construction for

[9] This initial condition is such that $J(\mathbf{p})$ is continuous at $(p^n, \bar{p})$.

$k=1$, which starts the induction. Suppose it holds for $k-1$. For $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$ we construct $r^L(\mathbf{p})$ so that (A15) holds. In particular, consider a change of variable $z = \ln\left(\frac{p^s}{1-p^s}\right)$ and let $J(p^n, z)$ represent, with abuse of notation, the respective transformation of $J(\mathbf{p})$, i.e., $J(p^n, z) = \ln(1-q) - \ln r^L\left(p^n, \frac{e^z}{1+e^z}\right)$. Then taking the derivatives of both sides of (A15) with respect to z, we obtain the following differential equation for $J(p^n, z)$:

$$\left(1-(1-q)e^{-J(p^n,z)}\right) \cdot \left[1 + \frac{dJ(p^n, z)}{dz}\right] = -D\left(f^n(\mathbf{p}), z + J(p^n, z)\right) \tag{A17}$$

with the initial condition $J\left(p^n, \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right) = 0$.[10]

We next show that a solution to (A17) exists and is nonnegative for all $z \geqslant \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$. Suppose that there exists $\mathbf{p} = (p^n, p^s) \in \mathcal{B}_k^{\uparrow}$ such that $J\left(p^n, \ln\left(\frac{p^s}{1-p^s}\right)\right) = -\varepsilon < 0$. As a solution to an ODE, $J(p^n, z)$ is a continuous function of z. Therefore, there exists $\tilde{p}^s \in (\bar{p}, p^s)$ such that $J\left(p^n, \ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)\right) = \max\{-\frac{1}{2}\varepsilon, \frac{1}{2}\ln(1-q)\}$. Then

$$\left.\frac{dJ(p^n, z)}{dz}\right|_{z=\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)} = \left.\frac{-D(f^n(\mathbf{p}), z + J(p^n, z))}{1-(1-q)e^{-J(p^n,z)}}\right|_{z=\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)} - 1 > \frac{q}{q} - 1 = 0.$$

Therefore, as we increase z from $\ln\left(\frac{\tilde{p}^s}{1-\tilde{p}^s}\right)$, $J(p^n, z)$ could never fall below $-\frac{\varepsilon}{2}$, while we have assumed $J\left(p^n, \ln\left(\frac{p^s}{1-p^s}\right)\right) = -\varepsilon$ – a contradiction. As $\varepsilon > 0$ was taken arbitrarily, it shows that $J(p^n, z) \geqslant 0$ for all $z \geqslant \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$. We next can take arbitrary solution to (A17) in the neighborhood of its initial condition, the existence of which is ensured by the existence theorem (see Pontryagin (1962), chapter 4, §21). It can be extended for all $z \geqslant \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)$ if and only if $J(p^n, z) < +\infty$ for all such z (see Pontryagin (1962), chapter 4, §24) which is true as

$$\left|\frac{dJ(p^n, z)}{dz}\right| < \frac{1}{q} - 1 = \frac{1-q}{q} < +\infty.$$

Consequently, by Lemma 3 we obtain that $D(\mathbf{p}) \leqslant -q$ for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$, which concludes this part of the proof.

We next show that the constructed strategy profile constitutes an equilibrium. We first show that the low-type seller is indifferent whether to reveal a bad review or to conceal it at all $\mathbf{p} \in \mathcal{B}_{2+} \cap \mathcal{R}$. If $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$, then by construction $0 < r^L(\mathbf{p}) < 1$. From Lemma 8 we also know that $0 < r^L(\mathbf{p}) < 1$ for $\mathbf{p} \in \mathcal{B}_1^{\downarrow} \cap \mathcal{R}$. Then the value of a low-type seller in any $\mathbf{p} \in \mathcal{B}_{1+}^{\downarrow} \cap \mathcal{R}$ is equal to the value he receives in case he deletes all future bad reviews: $V^L(\mathbf{p}) = \frac{1-\mu}{r}$. Therefore, a low-type seller is indeed indifferent between disclosing a bad review and deleting it for any $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow} \cap \mathcal{R}$. For $\mathbf{p} \in \mathcal{B}_{1+}^{\uparrow} \cap \mathcal{R}$ the indifference directly follows from the way $r^L(\mathbf{p})$ is constructed and the fact that $r^L(\mathbf{p}) < 1$.[11]

By construction, the high-type seller strictly prefers to reveal bad reviews at all $\mathbf{p} \in \mathcal{B}_1 \cap \mathcal{R}$. We proceed by showing that the high-type seller weakly prefers to reveal a bad review at all $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow} \cap \mathcal{R}$.

[10] The RHS of (A17) is not smooth in $J(p^n, z)$, but is piecewise smooth. Therefore, as a solution to (A17) we take a composition of two solutions which are pasted together using continuity.

[11] The latter is true as $J(\mathbf{p}) < +\infty$ for all $\mathbf{p} \in \mathcal{B}_{1+}^{\uparrow} \cap \mathcal{R}$.

Concealing a review at $\mathbf{p} \in \mathcal{B}_2^{\downarrow} \cap \mathcal{R}$ cannot yield him a payoff higher than if he could choose time T at which a bad review will arrive and will be revealed:

$$V^H(\mathbf{p}) \leqslant \max_{T>0} \left\{ \int_0^T e^{-(r+\lambda q(1-\mu))t} \cdot (1-\mu) \cdot \left(1 + \frac{\lambda q}{r}\right) dt + \right.$$
$$\left. + e^{-(r+\lambda q(1-\mu))T} \cdot V^H\left(f\left(\mathbf{p}_T\right)\right) \right\}$$
$$= \max_{T>0} \left\{ \left(1 - e^{-(r+\lambda q(1-\mu))T}\right) \cdot \left(\frac{1}{r} - \frac{\mu}{r + \lambda q\,(1-\mu)}\right) + \right.$$
$$\left. + e^{-(r+\lambda q(1-\mu))T} \cdot V^H\left(f\left(\mathbf{p}_T\right)\right) \right\}$$
$$\leqslant \max_{T>0} V^H\left(f\left(\mathbf{p}_T\right)\right) = V^H\left(f\left(\mathbf{p}\right)\right)$$

where process $\mathbf{p}_t$ is given by (4) with initial condition $\mathbf{p}_0 = \mathbf{p}$. The last inequality holds because

$$V^H(\mathbf{p}) \geqslant \int_0^{+\infty} e^{-rt} \left[e^{-\lambda q(1-\mu)t} \cdot (1-\mu) + \left(1 - e^{-\lambda q(1-\mu)t}\right)\right] dt =$$
$$= \left(\frac{1}{r} - \frac{\mu}{r + \lambda q(1-\mu)}\right)$$

for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ since the high-type seller can delete all future bad reviews. The last equality holds because $V^H(f(\mathbf{p}_T))$ is independent of T. Indeed, distributions of arrival times of the next buying consumer are the same for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. Therefore, $V^H(\mathbf{p})$ is the same for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. The resulting inequality $V^H(\mathbf{p}) \leqslant V^H(f(\mathbf{p}))$ implies that the high-type seller weakly prefers to reveal a bad review at all $\mathbf{p} \in \mathcal{B}_2^{\downarrow}$. The argument above can be extended by induction to all further bands in order to obtain that $V^H(\mathbf{p}) \leqslant V^H(f(\mathbf{p}))$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$.

We are left to show that the high type at least weakly prefers to reveal a bad review in $\mathcal{B}_{2+}^{\uparrow}$. We show the argument for $\mathcal{B}_2^{\uparrow}$, and the argument for $\mathcal{B}_k^{\uparrow}$ with higher k then follows by induction. Fix some state $\mathbf{p} = (p^n, p^s) \in \mathcal{B}_2^{\uparrow} \cap \mathcal{R}$. The high-type seller's value in case he decides to conceal a bad review at $\mathbf{p}$ is bounded from above by his payoff when he can receive and reveal a bad review at any time T of his choice:

$$V^H(\mathbf{p}) \leqslant \max \left\{ \max_{T \leqslant \tau(\mathbf{p})} \int_0^T e^{-rt} \left(1 + \frac{\lambda q}{r}\right) dt + e^{-rT} \cdot V^H(f(p_T)), \right.$$
$$\left. \int_0^{\tau(\mathbf{p})} e^{-rt} \left(1 + \frac{\lambda q}{r}\right) dt + e^{-r\tau(\mathbf{p})} \cdot V^H(p^n, \bar{p}) \right\}, \quad \text{(A18)}$$

where we use that $p^s_{\tau(\mathbf{p})} = \bar{p}$. On the one hand, since deleting all bad reviews is always feasible for the high-type seller, we have

$$V^H(f(\mathbf{p})) \geqslant \int_0^{\tau(f(\mathbf{p}))} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-r\tau(f(\mathbf{p}))}\cdot V^H(f^n(\mathbf{p}),\bar{p})$$
$$\geqslant \int_0^{\tau(\mathbf{p})} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-r\tau(\mathbf{p})}\cdot V^H(p^n,\bar{p})$$

where the second inequality follows because by construction $\tau(\mathbf{p}) = \tau(f(\mathbf{p}))$, and $V^H(f^n(\mathbf{p}),\bar{p}) \geqslant V^H(p^n,\bar{p})$ as shown above.[12] On the other hand, for any $T \leqslant \tau(\mathbf{p})$ we can write

$$V^H(f(\mathbf{p})) \geqslant \int_0^{T} e^{-rt}\left(1+\frac{\lambda q}{r}\right)dt + e^{-rT}\cdot V^H(f(\mathbf{p}_T))$$

because the high-type seller can reveal no reviews during $[0,T]$, and because if $\mathbf{p}_T \in \mathcal{R}$, then the process given by (4) with starting point $f(\mathbf{p})$ reaches value $f(\mathbf{p}_T)$ at exactly time T (this is since $\tau(\mathbf{p}) = \tau(f(\mathbf{p}))$ for all $\mathbf{p} \in \mathcal{B}_2^{\uparrow} \cap \mathcal{R}$), while if $\mathbf{p}_T \notin \mathcal{R}$, then $V^H(f(\mathbf{p}_T)) = 0$. Everything said above implies that $V^H(f(\mathbf{p})) \geqslant V^H(\mathbf{p})$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\uparrow}$, which concludes the proof that strategy profile is an equilibrium.

Finally, to conclude the proof of Theorem 2 we need to show that the equilibrium has the desired properties. We start with the fact that $f^s(\mathbf{p}) > p^s$ for all $\mathbf{p} \in \mathcal{R}$. By construction the strategy profile already implies $f^s(\mathbf{p}) > p^s$ for all $\mathbf{p} \in \left(\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_{2+}^{\downarrow}\right) \cap \mathcal{R}$. We next establish the claim for $\mathbf{p} \in \mathcal{B}_2^{\uparrow}$. From (A17) we know that $J(\mathbf{p}) \geqslant 0$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\uparrow}$. Suppose by way of contradiction that there exists some $\tilde{\mathbf{p}} = (\tilde{p}^n, \tilde{p}^s) \in \mathcal{B}_2^{\uparrow}$ such that $f^s(\tilde{\mathbf{p}}) = \tilde{p}^s$, i.e., $J(\tilde{\mathbf{p}}) = 0$. Assume first that $f(\tilde{\mathbf{p}}) \in \mathcal{R}$ which by Lemma 3 implies $D(f(\tilde{\mathbf{p}})) < -q$. Then there exists $\varepsilon > 0$ such that $D(f^n(\tilde{\mathbf{p}}), p^s) \leqslant -q-\varepsilon$ for all $p^s \in [f^s(\tilde{\mathbf{p}}) - \frac{\varepsilon}{4}, f^s(\tilde{\mathbf{p}})]$ as, by construction, $D(\mathbf{p})$ is continuous in p^s.[13] At the same time, we have that $J(\mathbf{p}) < J(\tilde{\mathbf{p}}) + \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$ for all $\mathbf{p} = (\tilde{p}^n, p^s)$ with $p^s \in [\tilde{p}^s - \frac{\varepsilon}{4}, \tilde{p}^s]$. By converse of Lemma 5 this implies that $D(\mathbf{p}) > -q - \frac{\varepsilon}{2}$ for those $\mathbf{p}$. Therefore $\tau(\tilde{\mathbf{p}}) - \tau(\tilde{p}^n, \tilde{p}^s - \frac{\varepsilon}{4}) > \tau(f(\tilde{\mathbf{p}})) - \tau(f^n(\tilde{\mathbf{p}}), f^s(\tilde{\mathbf{p}}) - \frac{\varepsilon}{4})$. Consequently, $J(\tilde{p}^n, \tilde{p}^s - \frac{\varepsilon}{4}) < 0$, – a contradiction. Now assume $f(\tilde{\mathbf{p}}) \notin \mathcal{R}$, that is $D(f(\tilde{\mathbf{p}})) = -q$. Then (A17) can be solved explicitly. Its general solution satisfies

$$(1-q)(z + J(p^n, z)) + q\ln\left(1 - e^{J(p^n,z)}\right) = C \tag{A19}$$

where C is a constant pinned down by the boundary condition for z_0 where $z_0 = \inf\{z \mid f(p^n, z) \notin \mathcal{R}\}$ and $J(p^n, z_0) > 0$ is given as a solution to (A17) for $z \in \left[\ln\left(\frac{\bar{p}}{1-\bar{p}}\right), z_0\right]$ with initial condition $J\left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right) = 0$. Therefore, C is well-defined and finite. As we have assumed $J(\tilde{\mathbf{p}}) = 0$ for some $\tilde{\mathbf{p}}$, substituting it into (A19) we achieve $C = -\infty$, – a contradiction.

All said above shows that $J(\mathbf{p}) > 0$ for all $\mathbf{p} \in \mathcal{B}_2^{\uparrow}$. By Lemma 3 it implies $D(\mathbf{p}) < -q$ for all $\mathbf{p} \in \mathcal{B}_2^{\uparrow}$, and the argument then extends to further bands straightforwardly.

[12] Values at the cutoff are equal to respective values under the cutoff since the latter are constant, and total payoff is insensitive to alterations of flow payoff in a single state (i.e., the fact that sophisticated consumers are buying in $p^s = \bar{p}$ does not affect payoffs).

[13] Otherwise there exists a sequence $\{p_k^s\}$ such that $p_k^s \to f^s(\tilde{\mathbf{p}})$ and $D(f^n(\tilde{\mathbf{p}}), p_k^s) \to -q$ as $k \to +\infty$ which contradicts the continuity of $D(\mathbf{p})$ in $\mathcal{B}_1^{\uparrow}$.

To see that this equilibrium is not payoff-equivalent to an equilibrium with $\mathcal{R} = \emptyset$, note that, for instance, the equilibrium constructed above has $D(\mathbf{p}) < -q$ for $\mathbf{p} \in \{\mathcal{B}_1^{\uparrow} \mid p^s \in [\bar{p}, \bar{p}+\varepsilon)\}$, as opposed to $D(\mathbf{p}) = -q$ in the fully censored equilibrium, meaning that $\tau(\mathbf{p})$ is smaller in the former for all $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$. Noticing that $\tau(\mathbf{p})$ directly enters the low-type seller's value in $\mathcal{B}_1^{\uparrow}$ concludes the argument. □

Proofs for Section V

Proof of Proposition 5. By Lemma 2 the low-type seller is indifferent between revealing a bad review and deleting it at all $\mathbf{p} \in \mathcal{B}_{1+} \cap \mathcal{R}$. Therefore, $V^L(\mathbf{p}) = \frac{1-\mu}{r}$ for all $\mathbf{p} \in \mathcal{B}_{1+}^{\downarrow}$ irrespective of equilibrium. For $\mathbf{p} \in \mathcal{B}_{1+}^{\uparrow}$ we have

$$V^L(\mathbf{p}) = \frac{1-\mu}{r} + (1 - e^{-r\tau(\mathbf{p})}) \cdot \frac{\mu}{r} = \frac{1 - \mu e^{-r\tau(\mathbf{p})}}{r}.$$

Therefore, to show the claim we need to establish that larger $\mathcal{R}$ implies pointwise weakly smaller $\tau(\mathbf{p})$. The claim holds for $\mathcal{B}_0^{\uparrow}$ (larger $\mathcal{B}_0^{\uparrow} \cap \mathcal{R}$ has no effect on $\tau(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$). Proceed by induction and show that if the claim holds for $\mathcal{B}_{k-1}^{\uparrow}$, then it also holds for $\mathcal{B}_k^{\uparrow}$. For any $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$ we show that if $\tau'(\mathbf{p}) = \tau''(\mathbf{p})$, then $\frac{d\tau'(\mathbf{p})}{dp} \leqslant \frac{d\tau''(\mathbf{p})}{dp}$ where objects indexed by single and double primes denote respective objects in the two equilibria under consideration with $\mathcal{R}'$ and $\mathcal{R}'' \subset \mathcal{R}'$ respectively. Three cases are possible for every $\mathbf{p}$ with $\tau'(\mathbf{p}) = \tau''(\mathbf{p})$:

1. If $\mathbf{p} \notin \mathcal{R}'$, then $D'(\mathbf{p}) = D''(\mathbf{p}) = -q$.
2. If $\mathbf{p} \in \mathcal{R}' \backslash \mathcal{R}''$, then $D'(\mathbf{p}) \leqslant -q = D''(\mathbf{p})$, where the first inequality follows from Theorem 1 and Lemma 3.
3. If $\mathbf{p} \in \mathcal{R}''$, then $\tau'(f(\mathbf{p})) \leqslant \tau''(f(\mathbf{p}))$ implies that $J'(\mathbf{p}) \geqslant J''(\mathbf{p})$, which in turn means that $D'(\mathbf{p}) \leqslant D''(\mathbf{p})$ because both equilibria are semi-separating.

Therefore, $D'(\mathbf{p}) \leqslant D''(\mathbf{p})$ for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$. Since $\tau(p^n, \bar{p}) = 0$ for all p^n, (A1) implies that $\tau'(\tilde{\mathbf{p}}) \leqslant \tau''(\tilde{\mathbf{p}})$ for all $\mathbf{p} \in \mathcal{B}_k^{\uparrow}$. □

Proof of Proposition 6. As the seller of a high quality product never receives any bad review, after any bad review beliefs jump to $f^s(\mathbf{p}) = f^n(\mathbf{p}) = 0$ and no future consumers ever buy the product again. Revealing a bad review thus grants the worst continuation payoff, and is therefore strictly dominated by deleting it for any seller who can guarantee non-zero continuation payoff which is true if either $p^n \geqslant \bar{p}$ or $p^s > \bar{p}$. □

Proof of Proposition 7. First let us introduce some extra notation for the general setting. Let $\mathcal{B}_{-1} = \{(p^n, p^s) \in \mathcal{B}_0 \mid (f_-^n)^{-1}(p^n) \geqslant \bar{p}\}$ and $\mathcal{B}_{-k} = \{(p^n, p^s) \mid \big((f_-^n)^{-1}(\mathbf{p}), p^s\big) \in \mathcal{B}_{-k+1}\}$ for $k > 1$.[14] By analogy with $\mathcal{B}_k$ for $k > 0$, $\mathcal{B}_{-k}$ measure distance between p^n and $\bar{p}$: if $\mathbf{p} \in \mathcal{B}_{-k}$ for $k > 0$, then k less bad reviews would be required to bring naive consumers back to the market.

Let us also refresh the expressions for belief updating for the general case. Rational consumers' beliefs are updated in the general setting as:

$$\frac{f_+^s(\mathbf{p})}{1 - f_+^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{q_+^H r_+^H(\mathbf{p})}{q_+^L r_+^L(\mathbf{p})}; \qquad \frac{f_-^s(\mathbf{p})}{1 - f_-^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{q_-^H r_-^H(\mathbf{p})}{q_-^L r_-^L(\mathbf{p})} \tag{A20}$$

[14]Here function f^n is meant in the sense of $[0,1] \to [0,1]$ (i.e., $f^n(p^n)$) since, as we remember, $f^n(\mathbf{p})$ does not depend on p^s.

after good and bad reviews respectively, and as

$$\dot{p}^s = \lambda p^s (1-p^s) \cdot \left[q_+^H \left(1 - r_+^H(\mathbf{p})\right) + q_-^H \left(1 - r_-^H(\mathbf{p})\right) - \right.$$
$$\left. - q_+^L \left(1 - r_+^L(\mathbf{p})\right) - q_-^L \left(1 - r_-^L(\mathbf{p})\right)\right] \quad \text{(A21)}$$

in the absence of reviews. Naive consumers' reaction to good and bad reviews respectively is given by:

$$\frac{f_+^s(\mathbf{p})}{1-f_+^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{q_+^H}{q_+^L}; \qquad \frac{f_-^s(\mathbf{p})}{1-f_-^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{q_-^H}{q_-^L}.$$

We construct the equilibrium as follows. For good reviews let $\mathcal{R}_+ = \mathcal{B}_{-1}^{\uparrow}$ and $r_+^\theta(\mathbf{p}) = 1$ for either θ and all $\mathbf{p} \in \mathcal{R}_+$. For bad reviews let $\mathcal{R}_- = \cup_{k \geqslant 1} \mathcal{B}_k^{\downarrow}$ and $r_-^H(\mathbf{p}) = 1$ for all $\mathbf{p} \in \mathcal{R}_-$. Let $r_-^L(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$ be constructed as in Theorem 2. Finally, $r_-^L(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ is constructed below.

We construct $r_-^L(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ in such a way as to make the low-type seller indifferent between revealing a bad review and not. In such construction, $V^L(\mathbf{p}) = \frac{1-\mu}{r}$ for any $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ (and actually all $\mathbf{p} \in \mathcal{B}_{1+}^{\downarrow}$ given the remainder of the construction), so deleting all future bad reviews is optimal. On the other hand, for any $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$ we have

$$\begin{aligned} V^L(\mathbf{p}) &= \int_0^{\tau(\mathbf{p})} e^{-rt} \left[e^{-\lambda\mu q_+^L t} \cdot \mu + \left(1 - e^{-\lambda\mu q_+^L t}\right) \cdot 1 \right] dt + \\ &\quad + e^{-r\tau(\mathbf{p})} \left(1 - e^{-\lambda\mu q_+^L \tau(\mathbf{p})}\right) \cdot \frac{1}{r} \\ &= \left(1 - e^{-(r+\lambda\mu q_+^L)\tau(\mathbf{p})}\right) \cdot \left(\frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}\right). \end{aligned}$$

To clarify, this expression describes payoff from selling to sophisticated consumers until $\tau(\mathbf{p})$ and to all consumers after a good review arrives if this happens before $\tau(\mathbf{p})$. The latter is valid because condition $q_+^H \cdot q_-^H \geqslant q_+^L \cdot q_-^L$ ensures that revealing one additional good review in any $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$ brings naive consumers back to the market.

Given the strategies defined above, $D(\mathbf{p}) = -(q_+^H - q_+^L) < 0$ for all $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$, hence

$$\tau(\mathbf{p}) = \frac{1}{\lambda\mu(q_+^H - q_+^L)} \left(\ln\left(\frac{p^s}{1-p^s}\right) - \ln\left(\frac{\bar{p}}{1-\bar{p}}\right)\right) < \infty \quad \text{(A22)}$$

for all $\mathbf{p} = (p^n, p^s) \in \mathcal{B}_{-1}^{\uparrow}$. Furthermore, $\tau(\mathbf{p})$ is continuous and strictly increasing in p^s, so $V^L(\mathbf{p})$ is continuous and strictly increasing in p^s as well. Finally, $\tau(\mathbf{p}) \to \infty$ as $p^s \to 1$ and $\tau(\mathbf{p}) \to 0$ as $p^s \to \bar{p}$, therefore $V^L(\mathbf{p})$ spans the whole interval $\left[0, \frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}\right]$ across $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$.

Fix some $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. Let $\hat{\mathbf{p}} \in \mathcal{B}_{-1}^{\uparrow}$ be such that $V^L(\hat{\mathbf{p}}) = \frac{1-\mu}{r}$. It exists for reasons described above: $\frac{1-\mu}{r} < \frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}$ whenever $\mu > 1/2$. Finally, let $r_-^L(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ be such that $f_-(\mathbf{p}) = (f_-^n(p^n), \hat{p}^s)$ (closed-form expression for $r_-^L(\mathbf{p})$ can be obtained from (A20)).

The construction above trivially implies $f_-^s(\mathbf{p}) > \bar{p} > p^s$ for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. It also generates $f_+(\mathbf{p}) > p^s$ for all $\mathbf{p} \in \mathcal{R}_+$. Construction in Theorem 2 also implies that $f_-^s(\mathbf{p}) > p^s$ for all $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$. This verifies the first property in the Proposition. The second property is trivial – $\mathcal{R}_-$ is nonempty for the strategy profile constructed above. Therefore, to conclude the proof we need to verify two things: that the constructed strategy profile constitutes an equilibrium and that this equilibrium is payoff-distinct from fully censored equilibrium in any meaning of the latter.

We start by verifying that the strategy profile above constitutes an equilibrium. First, either type of the seller at least weakly prefers to reveal good reviews at all $\mathbf{p} \in \mathcal{R}_+$. This is because $f_+(\mathbf{p}) \in \mathcal{B}_1^{\uparrow}$ so $D(f(\mathbf{p})) = 0$ and $\tau(f(\mathbf{p})) = \infty$.[15] Simply speaking, revealing a good review moves seller to an absorbing state in which he can retain both naive and sophisticated consumers in the market forever. This attains the maximal payoff, so is always at least weakly optimal.

Low-type seller is by construction indifferent between deleting and revealing bad reviews at all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. This indifference extends to $\mathcal{B}_{2+}^{\downarrow}$. If in any $\mathbf{p} \in \mathcal{B}_2^{\downarrow}$ the low-type seller chooses to delete a bad review, he can achieve a payoff of $\frac{1-\mu}{r}$ by deleting all future bad reviews as well. At the same time, revealing a bad review at $\mathbf{p}$ (or any future state) grants him $V^L(f(\mathbf{p})) = \frac{1-\mu}{r}$ which is exactly the same payoff. The argument can be iterated further to show that the low type is indifferent at all $\mathcal{B}_{2+}^{\downarrow}$.

The only equilibrium property left to verify is the high type's preference. Suppose that the high-type seller is currently in some state $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. If he deletes all future bad reviews, then his payoff equals $\frac{1-\mu}{r}$. If, however, he has a bad review in hand and reveals it, then he arrives at some $f(\mathbf{p})$ with $f^s(\mathbf{p}) = \hat{p}^s$ and receives

$$
\begin{aligned}
V^H(f(\mathbf{p})) = & \int_0^{\tau(f(\mathbf{p}))} e^{-rt} \left[e^{-\lambda\mu q_+^H t} \cdot \mu + \left(1 - e^{-\lambda\mu q_+^H t}\right) \cdot 1 \right] dt \\
& + e^{-r\tau(f(\mathbf{p}))} \left(1 - e^{-\lambda\mu q_+^H \tau(f(\mathbf{p}))}\right) \frac{1}{r} \\
= & \left(1 - e^{-(r+\lambda\mu q_+^H)\tau(f(\mathbf{p}))}\right) \cdot \left(\frac{1}{r} - \frac{1-\mu}{r + \lambda\mu q_+^H}\right).
\end{aligned}
$$

Given that $q_+^H > q_+^L$ and the low type's indifference requires $\left(1 - e^{-(r+\lambda\mu q_+^L)\tau(f(\mathbf{p}))}\right) \cdot \left(\frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}\right) = \frac{1-\mu}{r}$, trivially $V^H(f(\mathbf{p})) > \frac{1-\mu}{r}$. Doing the usual argument with the high-type seller solving a relaxed problem in which he has a choice of when to reveal the bad review (used in proofs of Lemma 8 and Theorem 2), we can arrive at the conclusion that he strictly prefers to reveal a bad review at $\mathbf{p}$. Using the same argument as in the proof of Theorem 2 we can then show that this strict preference propagates to $\mathcal{B}_{2+}^{\downarrow}$. This concludes the proof that the constructed strategy profile is an equilibrium.

Finally, we want to show that the equilibrium above is payoff-distinct from fully censored equilibrium in either sense of the latter (i.e., where $\mathcal{R}_- = \emptyset$ and $\mathcal{R}_+$ is either same as above, or also empty). In either case it is enough to consider $V^H(\mathbf{p})$ at any $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$. In either fully censored equilibrium we have $V^H(\mathbf{p}) = \frac{1-\mu}{r}$ because the high-type seller is unable to reveal any reviews. In contrast, in the equilibrium constructed above $V^H(\mathbf{p}) > \frac{1-\mu}{r}$ because this inequality is true for all $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$ and the high-type seller jumps to $\mathcal{B}_0^{\uparrow}$ from $\mathcal{B}_1^{\downarrow}$ (by receiving and revealing a bad review) with a positive probability in finite time. □

Proof of Proposition 8. We construct the equilibrium in a way analogous to Proposition 7 but accounting for fake reviews. For good reviews let $\mathcal{R}_+ = \mathcal{B}_{-1}^{\uparrow}$ and $r_+^\theta(\mathbf{p}) = \phi_+^\theta(\mathbf{p}) = 1$ for either θ and all $\mathbf{p} \in \mathcal{R}_+$. For bad reviews let $\mathcal{R}_- = \cup_{k \geqslant 1} \mathcal{B}_k^{\downarrow}$ and $r_-^H(\mathbf{p}) = \phi_-^H(\mathbf{p}) = 1$ for all $\mathbf{p} \in \mathcal{R}_-$. For any $\mathbf{p} \in \mathcal{B}_{2+}^{\downarrow}$ let $r_-^L(\mathbf{p})$ and $\phi_-^L(\mathbf{p})$ be an arbitrary solution of the equation $\ln\left(\frac{f_-^s(\mathbf{p})}{1-f_-^s(\mathbf{p})}\right) - \ln\left(\frac{p^s}{1-p^s}\right) = \frac{1}{2} \cdot \left(\ln\left(\frac{\bar{p}}{1-\bar{p}}\right) - \ln\left(\frac{p^s}{1-p^s}\right)\right)$.[16]

[15] In case $q_+^H \cdot q_-^H < q_+^L \cdot q_-^L$, which we do not consider in this proposition, one would need to either ensure that prior p_0 is such that $f_+^n(f_-(\mathbf{p})) \geqslant \bar{p}$ for all $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ on equilibrium path, or to verify that the argument to follow holds even if more than one good review is required to achieve $\mathcal{B}_1^{\uparrow}$ from any $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$.

[16] This is analogous to the construction in Theorem 2. It ensures that $f_-^s(\mathbf{p}) > p^s$ and $f_-(\mathbf{p}) \in \mathcal{B}^{\downarrow}$.

Finally, $r_-^L(\mathbf{p})$ and $\phi_-^L(\mathbf{p})$ for $\mathbf{p} \in \mathcal{B}_1^{\downarrow}$ are constructed in such a way as to make the low type indifferent between revealing bad reviews and not. Similarly to Proposition 7 in such construction we have $V^L(\mathbf{p}) = \frac{1-\mu}{r}$ for any $\mathbf{p} \in \mathcal{B}_1^{\uparrow}$, while for any $\mathbf{p} \in \mathcal{B}_0^{\uparrow}$: $V^L(\mathbf{p}) = \left(1 - e^{-(r+\lambda\mu q_+^L)\tau(\mathbf{p})}\right) \cdot \left(\frac{1}{r} - \frac{1-\mu}{r+\lambda\mu q_+^L}\right)$.

Given the strategies defined above, $D(\mathbf{p}) = -(q_+^H - q_+^L) < 0$ for all $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$ as in Proposition 7 (since effects of fake positive reviews on $D(\mathbf{p})$ imposed by high and low type cancel each other out). Further, $\frac{f_+^s(\mathbf{p})}{1-f_+^s(\mathbf{p})} = \frac{p^s}{1-p^s} \cdot \frac{\lambda q_+^H + \lambda_\phi}{\lambda q_+^L + \lambda_\phi}$ for all $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$, meaning that $f_+^s(\mathbf{p}) > p^s$ so $f_+(\mathbf{p}) \in \mathcal{B}_1^{\uparrow}$ for all $\mathbf{p} \in \mathcal{B}_{-1}^{\uparrow}$.

From here the fact that this strategy profile is an equilibrium and all required equilibrium properties can be verified in exactly the same way as in Proposition 7. □

References

Pontryagin, L.S. 1962. *Ordinary Differential Equations.* Addison-Wesley.