Competitive Information Disclosure to an Auctioneer: Online Appendix

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This Online Appendix contains additional results on equilibrium signal structures and a more formal definition of ϵ- and δ-extensions.

OA1 More Results on Equilibrium Signal Structures

In Subsection OA1.1, we establish stronger versions of Lemma 6. We then make use of this in Subsections OA1.2 and OA1.3, where we consider the case of two possible valuations and the case of two bidders and three possible valuations, respectively. In particular, we show that under the assumptions in Section VII, there are no other equilibria than those identified in Propositions 3 and 4.

OA1.1 Strengthening Lemma 6

We strengthen Lemma 6 to Lemma OA2 and ultimately Lemma OA3 below.[1] We will need a generalization of δ-extensions that allows to raise ironed virtual valuations to a

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[1]While the main goal is to complement the equilibrium analysis of Section VII, we establish these results for the general model as the additional assumptions in Section VII do not facilitate the proofs.

level that may differ from zero and, furthermore, to target particular posteriors. Let $b_i \in B_i$ be any signal structure of bidder i. Let $p_i \in \mathcal{P}_i$ and $x \in \mathbb{R}$ be such that

$$m_i > 1, \quad H_i(v_i^k, p_i) < x \leq H_i(v_i^{k+1}, p_i), \quad \text{and} \quad x < v_i^k \quad \text{for some } k < m_i. \tag{OA.1}$$

Instead of posterior p_i, a Δ-*extension* b_i^Δ *of* b_i *to* x *at* p_i draws posterior $p_i^{\Delta(p_i)}$ with probability $1 - [1 - P_i(v_i^k)]\Delta(p_i)$ and posterior p_i'' with probability $[1 - P_i(v_i^k)]\Delta(p_i)$, where $\Delta(p_i) \in (0, 1)$, $V_i(p_i'') = \{v_i^{k+1}, \ldots, v_i^{m_i}\}$,

$$p_i''(v_i) = \frac{p_i(v_i)}{1 - P_i(v_i^k)} \quad \forall v_i \in V_i(p_i''),$$

$V_i(p_i^{\Delta(p_i)}) = V_i(p_i)$, and

$$p_i^{\Delta(p_i)}(v_i) = \begin{cases} \frac{p_i(v_i)}{1-[1-P_i(v_i^k)]\Delta(p_i)} & \text{if } v_i \leq v_i^k, \\ \frac{[1-\Delta(p_i)]p_i(v_i)}{1-[1-P_i(v_i^k)]\Delta(p_i)} & \text{if } v_i > v_i^k. \end{cases}$$

Note that the expected posterior under b_i^Δ conditional on b_i drawing posterior p_i is p_i. Hence, b_i^Δ satisfies (1) and thus $b_i^\Delta \in B_i$ just as b_i. We state the following result, whose proof is analogous to the proof of Lemma 3 and therefore omitted.

Lemma OA1. *a) For every $i \in N$, every $x \in \mathbb{R}$, and every $p_i \in \mathcal{P}_i$ that satisfies* (OA.1)*, there is a $\Delta(p_i) \in (0, 1)$ such that*

$$H_i(v_i, p_i^{\Delta(p_i)}) = \begin{cases} x & \text{if } v_i = v_i^k, \\ H_i(v_i, p_i) & \text{if } v_i \in \{v_i^{k+1}, \ldots, v_i^{m_i}\}. \end{cases} \tag{OA.2}$$

Moreover, $H_i(v_i, p_i'') = H_i(v_i, p_i)$ for all $v_i \in V_i(p_i'')$.

*b) Let f be any optimal strategy of the auctioneer, $i \in N$, and $\mathbf{b} \in B$. For $x \in \mathbb{R}$ and a Borel set $F \subseteq \mathcal{P}_i$, let $\hat{\mathcal{P}}_i = \{\, p_i \in F \mid$ (OA.1) *holds for* $x \,\}$. Let b_i^Δ be such that for every $p_i \in \hat{\mathcal{P}}_i$,* (OA.2) *holds. Then,*

$$\begin{aligned} U_i^f(b_i^\Delta, \mathbf{b}_{-i}) \geq & \int_{\mathcal{P}_i \setminus \hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p})} [v_i - H_i(v_i, p_i)] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v}) \mathrm{d}b_{-i}(\mathbf{p}_{-i}) \mathrm{d}b_i(p_i) \\ & + \int_{\hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i > v_i^k} [v_i - H_i(v_i, p_i)] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v}) \mathrm{d}b_{-i}(\mathbf{p}_{-i}) \mathrm{d}b_i(p_i) \\ & + \int_{\hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i = v_i^k} (v_i - x) q_i^f(\mathbf{v}, (p_i^{\Delta(p_i)}, \mathbf{p}_{-i})) p(\mathbf{v}) \mathrm{d}b_{-i}(\mathbf{p}_{-i}) \mathrm{d}b_i(p_i). \end{aligned}$$

We use Lemma OA1 to prove the next result, which strengthens Lemma 6.

Lemma OA2. *Suppose* $\mathbf{b}$ *is a Nash equilibrium of a disclosure game. Then, there is a bidder* $i \in N$ *such that*

$$b_i(\{\, p_i \in \mathcal{P}_i \mid H_i(v_i^1, p_i) \geq \min_j \bar{v}_j^1 \,\}) = 1. \tag{OA.3}$$

Proof. Let f be any optimal strategy for the auctioneer. By contradiction, suppose $\mathbf{b}$ is a Nash equilibrium of the disclosure game defined by f and (OA.3) does not hold. For every bidder $i \in N$, define

$$\underline{x}_i = \inf\{\, H_i(v_i^1, p_i) \mid p_i \in \operatorname{supp}(b_i) \,\}, \tag{OA.4}$$

where possibly $\underline{x}_i = -\infty$. Let $\underline{x} = \max_i \underline{x}_i$. Then for every $\rho > \underline{x}$,

$$b_i(\{\, p_i \in \mathcal{P}_i \mid H_i(v_i^1, p_i) < \rho \,\}) > 0 \quad \forall i \in N. \tag{OA.5}$$

Note that (OA.3) is equivalent to $\underline{x} \geq \min_i \bar{v}_i^1$. Since (OA.3) does not hold by our hypothesis, $\underline{x} < \min_i \bar{v}_i^1$. By Lemma 6, $\underline{x} \geq 0$. Moreover, one can show that there exists a bidder $j \in N$ such that

$$b_j(\{\, p_j \in \mathcal{P}_j \mid H_j(v_j^1, p_j) > \underline{x} \,\}) = 1. \tag{OA.6}$$

Indeed, Lemma 6 states (OA.6) for $\underline{x} = 0$. To show that (OA.6) holds for $0 \leq \underline{x} < \min_i \bar{v}_i^1$, one can proceed as in the proof of Lemma 6 but replace δ-extensions by Δ-extensions to $\underline{x}$ at all p_j such that $H_j(v_j^1, p_j) < \underline{x}$.

Consider any bidder i. Since there are only finitely many possible valuations, there exists $\hat{\rho} \in (\underline{x}, \bar{v}_i^1)$ such that for every $\rho \in (\underline{x}, \hat{\rho})$, b_i assigns positive probability to

$$\mathcal{P}_{i,\rho} = \{\, p_i \in \mathcal{P}_i \mid H_i(v_i^k, p_i) < \rho \text{ and } \hat{\rho} \leq H_i(v_i^{k+1}, p_i) \text{ for some } k < m_i \,\}.$$

Consider a Δ-extension b_i^Δ of b_i to $\hat{\rho}$ at all $p_i \in \mathcal{P}_{i,\rho}$. By Lemma OA1b), we can choose b_i^Δ such that $U_i^f(b_i^\Delta, \mathbf{b}_{-i}) - U_i^f(\mathbf{b})$ is weakly greater than

$$\int_{\mathcal{P}_{i,\rho}} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i = v_i^k} (v_i - \hat{\rho}) q_i^f(\mathbf{v}, (p_i^{\Delta(p_i)}, \mathbf{p}_{-i})) p(\mathbf{v}) \mathrm{d}b_{-i}(\mathbf{p}_{-i}) \mathrm{d}b_i(p_i)$$
$$- \int_{\mathcal{P}_{i,\rho}} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v} \in V(\mathbf{p}): v_i \leq v_i^k} [v_i - H_i(v_i, p_i)] q_i^f(\mathbf{v}, \mathbf{p}) p(\mathbf{v}) \mathrm{d}b_{-i}(\mathbf{p}_{-i}) \mathrm{d}b_i(p_i).$$

There is a constant K such that for every $\rho \in (x, \hat{\rho})$ and $p_i \in \mathcal{P}_{i,\rho}$,

$$\int_{\mathcal{P}_{-i}} \sum_{\mathbf{v}_{-i} \in V_{-i}(\mathbf{p}_{-i})} q_i^f((v_i^k, \mathbf{v}_{-i}), (p_i^{\Delta(p_i)}, \mathbf{p}_{-i}))p_{-i}(\mathbf{v}_{-i})\mathrm{d}b_{-i}(\mathbf{p}_{-i}) = K > 0,$$

where the inequality follows from (OA.5) and $\hat{\rho} > \underline{x} \geq 0$. On the other hand, for any sequence (ρ^s) in $(\underline{x}, \hat{\rho})$ such that $\lim_{s\to\infty} \rho^s = \underline{x}$, and with $p_i^s \in \mathcal{P}_{i,\rho^s}$ and $v_i^s \in V_i(p_i^s) \setminus \{v_i^{k+1}, \dots, v_i^{m_i}\}$, we have

$$\lim_{s\to\infty} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v}_{-i} \in V_{-i}(\mathbf{p}_{-i})} q_i^f((v_i^s, \mathbf{v}_{-i}), (p_i^s, \mathbf{p}_{-i}))p_{-i}(\mathbf{v}_{-i})\mathrm{d}b_{-i}(\mathbf{p}_{-i}) = 0$$

by (OA.6). It follows that for small ρ, $U_i^f(b_i^\Delta, \mathbf{b}_{-i}) - U_i^f(\mathbf{b}) > 0$. Hence, b_i is not a best response against $\mathbf{b}_{-i}$, and consequently $\mathbf{b}$ is not a Nash equilibrium; a contradiction. $\square$

We use Lemma OA2 to prove the next result. It strengthens Lemma OA2 in that if the lowest possible valuation is the same across all bidders, there are at least *two* bidders whose ironed virtual valuation is weakly higher than that valuation.

Lemma OA3. *Let $\bar{v}_i^1 = \bar{v}^1$ for all $i \in N$. Suppose $\mathbf{b}$ is a Nash equilibrium of a disclosure game. Then, there are at least two bidders $i \in N$ such that*

$$b_i(\{\, p_i \in \mathcal{P}_i \mid H_i(v_i^1, p_i) \geq \bar{v}^1 \,\}) = 1. \tag{OA.7}$$

Proof. Let f be any optimal strategy for the auctioneer. Suppose $\mathbf{b}$ is a Nash equilibrium of the disclosure game defined by f. By Lemma OA2, (OA.7) holds for at least one bidder $i \in N$. By contradiction, suppose it holds for no bidder $j \neq i$. That is, using the notation $\underline{x}_j$ defined in (OA.4), $\underline{x}_j < \bar{v}^1$ for all $j \neq i$. Consider bidder i. By $\bar{v}_i^1 = \bar{v}^1$ and Lemma 1a), b_i draws with probability $\bar{p}_i(\bar{v}_i^1)$ posterior p_i with support $V_i(p_i) = \{\bar{v}_i^1\}$. With the remaining probability, b_i draws a posterior $p_i \in \hat{\mathcal{P}}_i = \{\, p_i \in \mathcal{P}_i \mid \bar{v}_i^1 \notin V_i(p_i) \,\}$ with $H_i(v_i^1, p_i) \geq \bar{v}^1$. Choose $\gamma > 0$ such that

$$\bar{v}_i^1 - \frac{1}{\gamma}(\bar{v}_i^{\bar{m}_i} - \bar{v}_i^1) \in \left(\max\left\{\max_{j\neq i} \underline{x}_j, 0\right\}, \bar{v}_i^1\right).$$

For any $p_i \in \hat{\mathcal{P}}_i$, define $p_i' \in \mathcal{P}_i$ by $V_i(p_i') = \{\bar{v}_i^1\} \cup V_i(p_i) = \{\bar{v}_i^1, v_i^1, \dots, v_i^{m_i}\}$ and

$$p_i'(v_i) = \begin{cases} \frac{\gamma}{1+\gamma} & \text{if } v_i = \bar{v}_i^1, \\ \frac{p_i(v_i)}{1+\gamma} & \text{if } v_i \in \{v_i^1, \dots, v_i^{m_i}\}. \end{cases} \tag{OA.8}$$

Since

$$J_i(\bar{v}_i^1, p_i') = \bar{v}_i^1 - \frac{1}{\gamma}(v_i^1 - \bar{v}_i^1) < \bar{v}_i^1 \leq H_i(v_i^1, p_i),$$

we have

$$H_i(v_i, p_i') = \begin{cases} J_i(v_i, p_i') & \text{if } v_i = \bar{v}_i^1, \\ H_i(v_i, p_i) & \text{if } v_i \in \{v_i^1, \dots, v_i^{m_i}\}. \end{cases}$$

Choose $\alpha, \beta \in (0, 1)$ such that

$$[1 - \bar{p}_i(\bar{v}_i^1)](1 - \beta)\gamma = \bar{p}_i(\bar{v}_i^1)(1 - \alpha). \tag{OA.9}$$

Consider the following distribution b_i' on $\mathcal{P}_i$:

- With probability $\bar{p}_i(\bar{v}_i^1)\alpha$, p_i with $V_i(p_i) = \{\bar{v}_i^1\}$ is drawn.
- With probability $\beta b_i(\hat{\mathcal{P}}_i)$, a $p_i \in \hat{\mathcal{P}}_i$ is drawn from distribution $b_i / b_i(\hat{\mathcal{P}}_i)$.
- With probability $(1-\beta)(1+\gamma)b_i(\hat{\mathcal{P}}_i)$, a $p_i \in \hat{\mathcal{P}}_i$ is drawn from distribution $b_i / b_i(\hat{\mathcal{P}}_i)$ and is replaced by p_i' as defined in (OA.8).

Distribution b_i' is indeed a distribution on $\mathcal{P}_i$ since using (OA.9) and $b_i(\hat{\mathcal{P}}_i) = 1 - \bar{p}_i(\bar{v}_i^1)$,

$$\int_{\mathcal{P}_i} \mathrm{d}b_i'(p_i) = \bar{p}_i(\bar{v}_i^1)\alpha + [\beta + (1 - \beta)(1 + \gamma)]b_i(\hat{\mathcal{P}}_i) = 1.$$

Moreover, (OA.8), (OA.9), and $b_i(\hat{\mathcal{P}}_i) = 1 - \bar{p}_i(\bar{v}_i^1)$ imply

$$\int_{\mathcal{P}_i} p_i(\bar{v}_i^1)\mathrm{d}b_i'(p_i) = \bar{p}_i(\bar{v}_i^1)\alpha + (1 - \beta)(1 + \gamma) \int_{\hat{\mathcal{P}}_i} \frac{\gamma}{1 + \gamma}\mathrm{d}b_i(p_i) = \bar{p}_i(\bar{v}_i^1),$$

and for any valuation $v_i \in \bar{V}_i$ other than $\bar{v}_i^1$,

$$\int_{\mathcal{P}_i} p_i(v_i)\mathrm{d}b_i'(p_i) = \beta \int_{\hat{\mathcal{P}}_i} p_i(v_i)\mathrm{d}b_i(p_i) + (1 - \beta)(1 + \gamma) \int_{\hat{\mathcal{P}}_i} \frac{p_i(v_i)}{1 + \gamma}\mathrm{d}b_i(p_i) = \int_{\hat{\mathcal{P}}_i} p_i(v_i)\mathrm{d}b_i(p_i).$$

Hence, b_i' satisfies (1) and thus $b_i' \in B_i$ just as b_i. Now,

$$\begin{aligned} U_i^f(b_i', \mathbf{b}_{-i}) - U_i^f(\mathbf{b}) = \int_{\hat{\mathcal{P}}_i} \int_{\mathcal{P}_{-i}} \sum_{\mathbf{v}_{-i} \in V_{-i}(\mathbf{p}_{-i})} &[\bar{v}_i^1 - H_i(\bar{v}_i^1, p_i')] q_i^f((\bar{v}_i^1, \mathbf{v}_{-i}), (p_i', \mathbf{p}_{-i})) \\ &\cdot \frac{\gamma}{1 + \gamma} p_{-i}(\mathbf{v}_{-i}) \mathrm{d}b_{-i}(\mathbf{p}_{-i})(1 - \beta)(1 + \gamma)\mathrm{d}b_i(p_i) > 0, \end{aligned}$$

where the inequality follows from $\bar{v}_i^1 > H_i(\bar{v}_i^1, p_i') > \max\left\{\max_{j \neq i} \underline{x}_j, 0\right\}$. Thus, b_i is not a best response against $\mathbf{b}_{-i}$, and so $\mathbf{b}$ is not a Nash equilibrium; a contradiction. □

OA1.2 Two Possible Valuations

It now easily follows that for the case of two possible valuations, there are no other equilibria than those identified in Proposition 3.

Proposition OA1. *Suppose $\bar{V}_i = \{v^L, v^H\}$ for all $i \in N$. If $\mathbf{b}^*$ is a Nash equilibrium of a disclosure game, then for at least two bidders $i \in N$, b_i^* draws with probability $\bar{p}_i(v^L)$ the posterior p_i' such that $V_i(p_i') = \{v^L\}$ and with probability $\bar{p}_i(v^H)$ the posterior p_i'' such that $V_i(p_i'') = \{v^H\}$.*

Proof. Follows directly from Lemma OA3 and Lemma 1a). $\square$

OA1.3 Two Bidders and Three Possible Valuations

In Proposition 4, we identified an equilibrium for the case of two symmetric bidders with three possible valuations. We will now show that there are no other equilibria.

We will use Lemma OA4 below, which does not require that the priors are identical as in Proposition 4. So let $N = \{1, 2\}$ and suppose $\bar{V}_i = \{v^1, v^2, v^3\}$ for both $i \in N$. Let f be any optimal strategy for the auctioneer, and suppose (b_1^*, b_2^*) is a Nash equilibrium of the disclosure game defined by f. By Lemma OA3 and Lemma 1a), each b_i^* draws the posterior p_i such that $V_i(p_i) = \{v^1\}$ with probability $\bar{p}_i(v^1)$. By (1), it follows that (almost) every other posterior p_i drawn by b_i^* has support $V_i(p_i) \subseteq \{v^2, v^3\}$ and can thus be identified with the variable $y_i = p_i(v^2) = 1 - p_i(v^3)$.

Accordingly, we identify the signal structure b_i^* with a distribution function F_i^* over $[0, 1]$: conditional on $V_i(p_i) \neq \{v^1\}$, the posterior p_i is identified with y_i drawn from F_i^*. Given posterior p_i with $p_i(v^2) = y_i > 0$ and given bidder i's realized valuation is $v_i = v^2$, let

$$\bar{Q}_i(y_i) = \int_{\mathcal{P}_j} \sum_{v_j \in V_j(p_j)} q_i^f((v^2, v_j), \mathbf{p}) p_j(v_j) \mathrm{d}b_j^*(p_j)$$

be bidder i's expected allocation probability given b_j^* of bidder $j \neq i$. Define $\bar{Q}_i(0) = 0$. Identifying b_i^* with F_i^*, bidder i's payoff is

$$U_i^f(b_1^*, b_2^*) = [1 - \bar{p}_i(v^1)] \int_0^1 (1 - y_i)(v^3 - v^2) \bar{Q}_i(y_i) \mathrm{d}F_i^*(y_i),$$

where we used that $[v^2 - H_i(v^2, p_i)]p_i(v^2) = (1 - y_i)(v^3 - v^2)$ for p_i with $p_i(v^2) = y_i$. Since (b_1^*, b_2^*) is a Nash equilibrium,

$$\begin{aligned} \forall i \in N: \ F_i^* \in \arg\max_{F_i} & \int_0^1 (1 - y_i)\bar{Q}_i(y_i)\mathrm{d}F_i(y_i) \\ \text{s.t. } [1 - \bar{p}_i(v^1)] & \int_0^1 y_i \mathrm{d}F_i(y_i) = \bar{p}_i(v^2), \end{aligned} \tag{OA.10}$$

where we omit the factor $[1-\bar{p}_i(v^1)](v^3-v^2)$ from the objective and where the constraint ensures (1).

Let $\underline{y} = (v^3 - v^2)/(v^3 - v^1)$. Note that $\underline{y}$ is the value of y_i such that the virtual valuation of v^2 equals v^1, that is, $H_i(v^2, p_i) = v^1$ for $p_i(v^2) = \underline{y}$.[2] The following lemma states several properties of F_i^*.

Lemma OA4. *Let $N = \{1, 2\}$ and $\bar{V}_i = \{v^1, v^2, v^3\}$ for both $i \in N$. Let (b_1^*, b_2^*) be a Nash equilibrium of a disclosure game. Let $S(F_i^*)$ be the intersection of the support of F_i^* with $(0, 1]$ and suppose $S(F_i^*) \not\subseteq \{\underline{y}, 1\}$ for both $i \in N$. Then, there exists $\overline{y} \in (\underline{y}, 1)$ such that $S_i(F_i^*) = [\underline{y}, \overline{y}]$ and F_i^* has no atom in $(\underline{y}, \overline{y}]$ for both $i \in N$.*

Proof. We proceed in four steps, proving the following properties: (i) F_i^* has no atom in $(\underline{y}, 1)$; (ii) $\min S(F_i^*) = \underline{y}$; (iii) $\max S(F_i^*) = \overline{y} \in (\underline{y}, 1)$; (iv) $S(F_i^*)$ is convex.[3] We repeatedly use that if there exist $e' \in [0, 1)$ and $e'' \in (e', 1]$ such that

$$\begin{aligned} \forall \lambda \in (0, 1): \quad & \lambda(1 - e')\bar{Q}_j(e') + (1 - \lambda)(1 - e'')\bar{Q}_j(e'') \\ & > [1 - \lambda e' - (1 - \lambda)e'']\bar{Q}_j(\lambda e' + (1 - \lambda)e''), \end{aligned} \tag{OA.11}$$

then F_i^* assigns probability zero to (e', e'') by optimality (OA.10).

(i) By contradiction, suppose F_i^* has an atom at $e \in (\underline{y}, 1)$. Then,

$$\lim_{y_j \downarrow e} (1 - y_j)\bar{Q}_j(y_j) > \lim_{y_j \uparrow e} (1 - y_j)\bar{Q}_j(y_j).$$

It follows that there exist $e' < e$ and $e''' > e$ such that, for every λ for which $\lambda e' + (1 - \lambda)e''' \in (e', e)$,

$$\lambda(1 - e')\bar{Q}_j(e') + (1 - \lambda)(1 - e''')\bar{Q}_j(e''') > [1 - \lambda e' - (1 - \lambda)e''']\bar{Q}_j(\lambda e' + (1 - \lambda)e''').$$

[2] For posteriors with binary support, virtual valuations J_i are always increasing so that ironed virtual valuations H_i coincide with virtual valuations.

[3] By definition, the support of F_i^* is closed, so that $\min S(F_i^*)$ and $\max S(F_i^*)$ exist.

By optimality (OA.10), F_j^* assigns probability zero to (e', e). Let $e'''' = \min\{\, y_j \in S(F_j^*) \mid y_j \geq e \,\}$. For every $y_i \in (e', e'''')$, it holds that $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)\lim_{\hat{e}\downarrow e'} \bar{Q}_i(\hat{e})$, whereas for every $y_i > e''''$, $(1-y_i)\bar{Q}_i(y_i) > (1-y_i)\lim_{\hat{e}\downarrow e'} \bar{Q}_i(\hat{e})$. We may assume that F_j^* has no atom at e, for otherwise at least one bidder can obtain a strictly higher payoff through an ϵ-extension by Lemma 2b). Since F_j^* has no atom at e, $(1-y_i)\bar{Q}_i(y_i)$ is continuous at e. It follows that there exists $e'' > e$ such that (OA.11) holds for e' and e''. This is a contradiction to e being in the support of F_i^*.

(ii) By Lemma OA3 and Lemma 1a), $\min S(F_i^*) \geq \underline{y}$. It remains to show $\min S(F_i^*) = \underline{y}$ for both $i \in N$. The proof proceeds by contradiction. Suppose first $\min S(F_j^*) > \underline{y}$ and $\min S(F_i^*) \in (\underline{y}, \min S(F_j^*)]$, where $j \neq i$. Then, for every $y_i \in (\underline{y}, \min S(F_j^*))$ we have $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)\bar{p}_j(v^1)$, whereas for every $y_i > \min S(F_j^*)$, $(1-y_i)\bar{Q}_i(y_i) > (1-y_i)\bar{p}_j(v^1)$. By (i), we may assume that F_j^* has no atom at $\min S(F_j^*)$, so that $(1-y_i)\bar{Q}_i(y_i)$ is continuous at $\min S(F_j^*)$. It follows that there exist $e' \in (\underline{y}, \min S(F_i^*))$ and $e'' > \min S(F_j^*) \geq \min S(F_i^*)$ such that (OA.11) holds, a contradiction.

Now suppose $\min S(F_i^*) > \min S(F_j^*) = \underline{y}$. By the argument for the above case, we may assume $(\underline{y}, \min S(F_i^*)) \not\subseteq S(F_j^*)$. Let $e = \min\{\, y_j \in S(F_j^*) \mid y_j \geq \min S(F_i^*)) \,\}$. Then, for every $y_i \in (\underline{y}, e)$ it holds that $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)\lim_{\hat{e}\downarrow \underline{y}} \bar{Q}_i(\hat{e})$, whereas for every $y_i > e$, $(1-y_i)\bar{Q}_i(y_i) > (1-y_i)\lim_{\hat{e}\downarrow \underline{y}} \bar{Q}_i(\hat{e})$. By (i), we may assume that F_j^* has no atom at e, so that $(1-y_i)\bar{Q}_i(y_i)$ is continuous at e. It follows that there exist $e' < \min S(F_i^*)$ and $e'' > e \geq \min S(F_i^*)$ such that (OA.11) holds; a contradiction.

(iii) We first show $\max S(F_i^*) < 1$. By contradiction, suppose $\max S(F_i^*) = 1$. Thus, $[1-\max S(F_i^*)]\bar{Q}_i(\max S(F_i^*)) = 0$. Since both $\underline{y}$ and 1 are in $S(F_i^*)$, optimality (OA.10) implies that we can find $e \geq \underline{y}$ arbitrarily close to $\underline{y}$ such that for every $\lambda \in (0,1]$

$$
\begin{aligned}
&\lambda(1-e)\bar{Q}_i(e) + (1-\lambda)[1-\max S(F_i^*)]\bar{Q}_i(\max S(F_i^*)) \\
&= \lambda(1-e)\bar{Q}_i(e) \geq [1-\lambda e-(1-\lambda)]\bar{Q}_i(\lambda e+(1-\lambda)) \iff \bar{Q}_i(e) \geq \bar{Q}_i(\lambda e+1-\lambda).
\end{aligned}
$$

Since $\bar{Q}_i$ is nondecreasing, it follows that $\bar{Q}_i(y_i) = \bar{Q}_i(e)$ for all $y_i \in [e, 1)$. But since e can be chosen arbitrarily close to $\underline{y}$, this implies $S(F_j^*) \subseteq \{\underline{y}, 1\}$; a contradiction to our assumption in Lemma OA4.

It remains to show $\max S(F_i^*) = \max S(F_j^*)$. By contradiction, suppose $\max S(F_i^*) >$

$\max S(F_j^*)$. For every $y_i > \max S(F_j^*)$, we have $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)[\bar{p}_j(v^1) + \bar{p}_j(v^2)]$. By this linearity and $\max S(F_i^*) > \max S(F_j^*)$, we may assume that $S(F_i^*) \setminus [\underline{y}, \max S(F_j^*)]$ contains at least two elements. By optimality (OA.10), it follows that $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)[\bar{p}_j(v^1) + \bar{p}_j(v^2)]$ for almost every $y_i \in S(F_i^*)$. (Note that optimality (OA.10) requires that almost all points $(y_i, (1-y_i)\bar{Q}_i(y_i))$ with $y_i \in S(F_i^*)$ lie on a line, none lying above.) However, for every $e > \underline{y}$, F_i^* assigns positive probability to $[\underline{y}, e]$ by (ii), and $(1-y_i)\bar{Q}_i(y_i) < (1-y_i)[\bar{p}_i(v^1) + \bar{p}_j(v^2)]$ for y_i close to $\underline{y}$ by our assumption $S_j(F_j^*) \not\subseteq \{\underline{y}, 1\}$. Thus, we have a contradiction.

(iv) By contradiction, suppose there exist e', e''' where $\underline{y} < e' < e''' < \bar{y}$ such that $S(F_i^*)$ does not include (e', e''') but does include e'''. We may also assume $e''' \in S(F_j^*)$, for otherwise we get a contradiction analogously to the first part of the proof of (ii). For every $y_j \in (e', e''')$, $(1-y_j)\bar{Q}_j(y_j) = (1-y_j)\lim_{\hat{e}\downarrow e'} \bar{Q}_j(\hat{e})$, whereas for every $y_j > e'''$, $(1-y_j)\bar{Q}_j(y_j) > (1-y_j)\lim_{\hat{e}\downarrow e'} \bar{Q}_j(\hat{e})$. By (i), we may assume that F_i^* has no atom at e''', so that $(1-y_j)\bar{Q}_j(y_j)$ is continuous at e'''. It follows that there exist $e'' > e'''$ such that (OA.11) holds; a contradiction. $\square$

We can now show that the equilibrium is unique in the case of two bidders with identical priors over three possible valuations as considered in Proposition 4.

Proposition OA2. *Suppose $N = \{1, 2\}$ and $\bar{V}_i = \{v^1, v^2, v^3\}$ with $\bar{p}_i(v^k) = \rho^k > 0$ for $i \in N$ and $k \in \{1, 2, 3\}$. Let (b_1^*, b_2^*) be a Nash equilibrium of a disclosure game. Then, that disclosure game has no other Nash equilibrium.*

Proof. By Lemma OA3 and Lemma 1a), each b_i^* draws posterior p_i such that $V_i(p_i) = \{v^1\}$ with probability ρ^1. By (1), it follows that (almost) every other posterior p_i drawn by b_i^* has support $V_i(p_i) \subseteq \{v^2, v^3\}$, and we write $p_i(v^2) = 1 - p_i(v^3) = y_i$. As in Lemma OA4, we identify b_i^* with the distribution function F_i^* over $[0, 1]$ by which y_i is drawn. Since (b_1^*, b_2^*) is a Nash equilibrium, optimality (OA.10) holds.

Let $S(F_i^*)$ be the intersection of the support of F_i^* with $(0, 1]$. By Lemma OA3 and Lemma 1a), $\min S(F_i^*) \geq \underline{y}$ for both $i \in N$. We first show $S_i(F_i^*) \not\subseteq \{\underline{y}, 1\}$ for both $i \in N$, so that we can apply Lemma OA4. For both i, we can rule out $S(F_i^*) = \{1\}$: $S(F_i^*) = \{1\}$ means the support of F_i^* is $\{0, 1\}$ (perfect disclosure), resulting in payoff zero for bidder i, but i can obtain a strictly positive payoff by drawing any $y_i \in (\underline{y}, 1)$ with positive probability. By contradiction, suppose $S(F_j^*) = \{\underline{y}, 1\}$ or $S(F_j^*) = \{\underline{y}\}$. Note that the constraint in (OA.10) requires $\int_0^1 y_i \mathrm{d}F_i(y_i) = \rho^2/(1-\rho^1)$. We consider two cases. *Case 1:* $\rho^2/(1-\rho^1) \leq \underline{y}$. For every $y_i \in (0, \underline{y})$, we have $(1-y_i)\bar{Q}_i(y_i) = 0$, whereas for every $y_i \in (\underline{y}, 1]$, $(1-y_i)\bar{Q}_i(y_i) = (1-y_i)[\rho^1 + \int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j)] > 0$. By $\rho^2/(1-\rho^1) \leq \underline{y}$, it follows from optimality (OA.10) that $S(F_i^*) = \{\underline{y}\}$. But this is impossible in equilibrium since if both F_1^* and F_2^* have an atom at $\underline{y}$, then at least one bidder can obtain a strictly higher payoff through an ϵ-extension by Lemma 2b). *Case 2:* $\rho^2/(1-\rho^1) > \underline{y}$. In this case, the prior does not admit $S(F_j^*) = \{\underline{y}\}$, so $S(F_j^*) = \{\underline{y}, 1\}$. Optimality (OA.10) then implies that we can find $e > \underline{y}$ arbitrarily close to $\underline{y}$ such that

$$\begin{aligned} \forall \lambda \in (0,1):\ \lambda(1-e)\bar{Q}_j(e) + (1-\lambda)(1-1)\bar{Q}_j(1) \geq [1-\lambda e-(1-\lambda)]\bar{Q}_j(\lambda e+(1-\lambda)) \\ \implies \bar{Q}_j(e) \geq \bar{Q}_j(\lambda e + 1 - \lambda). \end{aligned}$$

Since $\bar{Q}_j$ is nondecreasing, it follows that $\bar{Q}_j(y_j) = \bar{Q}_j(e)$ for all $y_j \in [e, 1)$. Since e can be chosen arbitrarily close to $\underline{y}$, this implies $S(F_i^*) \cap (\underline{y}, 1) = \emptyset$. By Lemma 2b), it is impossible that also F_i^* has an atom at $\underline{y}$, and we already ruled out $S(F_i^*) = \{1\}$. Thus, $S(F_i^*) \cap [\underline{y}, 1] = \emptyset$; a contradiction to $\rho^2/(1-\rho^1) > \underline{y}$.

Since $S_i(F_i^*) \not\subseteq \{\underline{y}, 1\}$ for both $i \in N$, Lemma OA4 applies, by which there exists $\overline{y} \in (\underline{y}, 1)$ such that $S_i(F_i^*) = [\underline{y}, \overline{y}]$ and F_i^* has no atom in $(\underline{y}, \overline{y}]$ for both $i \in N$. Since $S(F_i^*) = [\underline{y}, \overline{y}]$, optimality (OA.10) implies that for both $i \in N$, $(1-y_i)\bar{Q}_i(y_i)$ is affine on $(\underline{y}, \overline{y}]$, that is, there exist $\psi_i, \xi_i \in \mathbb{R}$ such that on $(\underline{y}, \overline{y}]$

$$(1-y_i)\bar{Q}_i(y_i) = \psi_i + \xi_i y_i \iff \bar{Q}_i(y_i) = \frac{\psi_i + \xi_i y_i}{1-y_i}. \tag{OA.12}$$

The border conditions

$$\begin{aligned} \lim_{y_i \downarrow \underline{y}} \bar{Q}_i(y_i) &= \frac{\psi_i + \xi_i \underline{y}}{1-\underline{y}} = \rho^1 + (1-\rho^1)\int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j), \\ \lim_{y_i \uparrow \overline{y}} \bar{Q}_i(y_i) &= \frac{\psi_i + \xi_i \overline{y}}{1-\overline{y}} = \rho^1 + \rho^2, \end{aligned}$$

yield

$$\psi_i = \rho^1 - \rho^2 \frac{1-\overline{y}}{\overline{y}-\underline{y}}\underline{y} + (1-\rho^1)\int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j)\frac{1-\underline{y}}{\overline{y}-\underline{y}}\overline{y}, \tag{OA.13}$$

$$\xi_i = -\rho^1 + \rho^2 \frac{1-\overline{y}}{\overline{y}-\underline{y}} - (1-\rho^1)\int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j)\frac{1-\underline{y}}{\overline{y}-\underline{y}}. \tag{OA.14}$$

Note that for $y_i \geq \underline{y}$,

$$\bar{Q}_i(y_i) = \rho^1 + (1-\rho^1)\left[\int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j) + \int_{(\underline{y},y_i]} y_j \mathrm{d}F_j^*(y_j)\right]. \tag{OA.15}$$

By (OA.12), $\bar{Q}_i(y_i)$ is differentiable on $(\underline{y},\overline{y}]$. By (OA.15) and the Radon-Nikodym Theorem, this implies that F_j^* admits a density on $(\underline{y},\overline{y}]$, and we get

$$\int_{(\underline{y},y_i]} \frac{\mathrm{d}\bar{Q}_i(e)}{\mathrm{d}e}\frac{1}{e}\mathrm{d}e = (1-\rho^1)\int_{(\underline{y},y_i]} \mathrm{d}F_j^*(e).$$

Filling in the values for ψ_i and ξ_i that we obtained in (OA.13) and (OA.14),

$$\begin{aligned}\int_{(\underline{y},y_i]} \mathrm{d}F_j^*(e) &= \frac{\psi_i+\xi_i}{1-\rho^1}\int_{(\underline{y},y_i]} \frac{1}{e(1-e)^2}\mathrm{d}e \\ &= \frac{\rho^2-(1-\rho^1)\int_{\{\underline{y}\}} y_j \mathrm{d}F_j^*(y_j)}{1-\rho^1}\frac{(1-\underline{y})(1-\overline{y})}{\overline{y}-\underline{y}}\int_{(\underline{y},y_i]}\frac{1}{e(1-e)^2}\mathrm{d}e. \end{aligned} \tag{OA.16}$$

Thus, for both $i \in N$, F_i^* is uniquely determined by $F_i^*(0)$, $\int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i)$, and $\overline{y}$.

We are left to show that $F_i^*(0)$, $\int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i)$, and $\overline{y}$ are uniquely determined. Observe first that if $\psi_i + \xi_i y_i < (1-y_i)\bar{Q}_i(y_i)$ for some y_i in $[0,\underline{y}]$ or $(\overline{y},1]$, then bidder i can obtain a strictly higher payoff by replacing F_i^* by a mean-preserving spread, contradicting optimality (OA.10). Hence, $\psi_i + \xi_i y_i \geq (1-y_i)\bar{Q}_i(y_i)$ for all $y_i \in [0,1]$. It follows that $\psi_i \geq 0$ for both $i \in N$. Moreover, if $\psi_i > 0$ then $F_i^*(0) = 0$. (Note that optimality (OA.10) requires that almost all points $(y_i,(1-y_i)\bar{Q}_i(y_i))$ with $y_i \in S(F_i^*)$ lie on a line, none lying above.)

We now show that $\int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i) = 0$ for both $i \in N$. Note first that we can have $\int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i) > 0$ for at most one i, for otherwise at least one bidder can get a strictly higher payoff through an ϵ-extension by Lemma 2b). By contradiction, suppose $\int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i) > 0$. Given that the bidders have the same prior, (OA.16) then requires $F_j^*(0) > 0$ for $j \neq i$, for otherwise F_j^* would strictly first-order stochastically dominate F_i^*, contradicting that F_i^*, F_j^* have the same mean. Since $F_j^*(0) > 0$, we have $\psi_j = 0$.

Using (OA.13), it then follows from $\int_{\{\underline{y}\}} \mathrm{d}F_j^*(y_j) = 0 < \int_{\{\underline{y}\}} \mathrm{d}F_i^*(y_i)$ that $\psi_i < 0$; a contradiction to our observation above that $\psi_1, \psi_2 \geq 0$.

Given that $\int_{\{\underline{y}\}} \mathrm{d}F_1^*(y_1) = \int_{\{\underline{y}\}} dF_2^*(y_2) = 0$, we have

$$\psi_1 = \psi_2 = \rho^1 - \rho^2 \frac{1-\overline{y}}{\overline{y}-\underline{y}} \underline{y},$$

and it remains to show that $F_i^*(0)$ and $\overline{y}$ are uniquely determined. Observe that ψ_1 is strictly increasing in $\overline{y}$, whereas (OA.16) is strictly decreasing in $\overline{y}$. It follows that the constraint in (OA.10) cannot be solved simultaneously by F_i^* with $\overline{y}$ such that $\psi_1 = 0$ and by F_i^* with $F_i^*(0) = 0$ and $\overline{y}$ such that $\psi_1 > 0$: the latter distribution function would strictly first-order stochastically dominate the former one, contradicting that they have the same mean. Finally, if $\overline{y}$ is the unique value that solves $\psi_1 = 0$, then $F_1^*(0), F_2^*(0)$ are equal and uniquely determined by the constraint in (OA.10) since the bidders have the same prior, and if $F_i^*(0) = 0$ for both $i \in N$ then $\overline{y}$ is uniquely determined by the constraint in (OA.10). $\square$

OA2 More Formal Definition of ϵ- and δ-Extensions

In this section, we give a more formal definition of the ϵ- and δ-extensions we introduced in Subsection IV.B. We first consider the process of replacing a signal structure with a more informative one in general, and we will call the more informative one an "extension". We then specialize the approach to ϵ- and δ-extensions.

Consider any bidder $i \in N$. Let $\Delta\mathcal{P}_i$ be the set of all distributions on $\mathcal{P}_i$. For $p_i \in \mathcal{P}_i$, the subset of all distributions that average to p_i is

$$\hat{B}_i(p_i) = \left\{ b_i \in \Delta\mathcal{P}_i \;\middle|\; \int_{\mathcal{P}_i} p_i'(v_i) \mathrm{d}b_i(p_i') = p_i(v_i) \forall v_i \in \bar{V}_i \right\}.$$

For the prior $\bar{p}_i$, we introduced the notation $\hat{B}_i(\bar{p}_i) = B_i$ in Section I. Let $\mathcal{B}(\mathcal{P}_i)$ be the Borel σ-algebra on $\mathcal{P}_i$. An *extension kernel* is a function $g : \mathcal{P}_i \to \Delta\mathcal{P}_i$, $p_i \mapsto g_{p_i}$, satisfying

$$\forall p_i \in \mathcal{P}_i : g_{p_i} \in \hat{B}_i(p_i), \tag{OA.17}$$

$$\forall F \in \mathcal{B}(\mathcal{P}_i) : p_i \mapsto g_{p_i}(F) \text{ is measurable.} \tag{OA.18}$$

Thus, an extension kernel is a Markov kernel that satisfies (OA.17). A signal structure $b_i \in B_i$ and an extension kernel g define the *extension* $b_i' \in B_i$ given by $b_i'(F) = \int_{\mathcal{P}_i} g_{p_i}(F) \mathrm{d}b_i(p_i)$ for $F \in \mathcal{B}(\mathcal{P}_i)$.

In the following, we define ϵ- and δ-extensions via extension kernels.[4] That (OA.17) holds is clear from the discussion in Subsection IV.B. Therefore, we concentrate on (OA.18).

ϵ-extensions. For $v_i \in \bar{V}_i$, denote the set of posteriors with v_i as the highest possible valuation by

$$\mathcal{P}_i^{v_i} = \{\, p_i \in \mathcal{P}_i \mid v_i^{m_i} = v_i \} = \{\, p_i \in \mathcal{P}_i \mid p_i(v_i) > 0 \text{ and } p_i(v_i') = 0 \,\forall v_i' > v_i \,\}.$$

Since $\mathcal{P}_i^{v_i} \in \mathcal{B}(\mathcal{P}_i)$, the trace σ-algebra $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i) = \{\, \mathcal{P}_i^{v_i} \cap F \mid F \in \mathcal{B}(\mathcal{P}_i) \,\}$ is contained in $\mathcal{B}(\mathcal{P}_i)$. We now define the function $g : \mathcal{P}_i \to \Delta\mathcal{P}_i$ for ϵ-extensions. For p_i with $v_i^{m_i} = v_i$, g_{p_i} draws posterior p_i^ϵ with probability $1 - p_i(v_i)\epsilon$ and posterior p_i' with probability $p_i(v_i)\epsilon$, where $\epsilon \in (0,1)$, $V_i(p_i') = \{v_i\}$, $V_i(p_i^\epsilon) = V_i(p_i)$, and

$$p_i^\epsilon(v_i') = \begin{cases} \frac{1}{1-p_i(v_i)\epsilon} p_i(v_i') & \text{if } v_i' \in V_i(p_i) \setminus \{v_i\}, \\ \frac{1-\epsilon}{1-p_i(v_i)\epsilon} p_i(v_i) & \text{if } v_i' = v_i. \end{cases}$$

Next, we show that g satisfies (OA.18). For $F \in \mathcal{B}(\mathcal{P}_i)$, let $\hat{F}^{v_i} = \{\, p_i \in \mathcal{P}_i^{v_i} \mid p_i^\epsilon \in F \,\}$ and $\tilde{F}^{v_i} = \{\, p_i \in \mathcal{P}_i^{v_i} \mid p_i' \in F \,\}$. Then,

$$g_{p_i}(F) = \sum_{v_i \in \bar{V}_i} \left(\mathbf{1}_{\hat{F}^{v_i}}(p_i)[1 - p_i(v_i)\epsilon] + \mathbf{1}_{\tilde{F}^{v_i}}(p_i) p_i(v_i)\epsilon \right).$$

For any $v_i \in \bar{V}_i$, the restriction of the functions $p_i \mapsto p_i^\epsilon$ and $p_i \mapsto p_i'$, respectively, to $\mathcal{P}_i^{v_i}$ is continuous and hence measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. Since products and sums of measurable functions are measurable, it follows that the restriction of $p_i \mapsto g_{p_i}(F)$ to $\mathcal{P}_i^{v_i}$ is measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. That is, for any $a \in [0,1]$, $\{\, p_i \in \mathcal{P}_i^{v_i} \mid g_{p_i}(F) \geq a \,\} \in \mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. Since $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i) \subseteq \mathcal{B}(\mathcal{P}_i)$, this implies

$$\{\, p_i \in \mathcal{P}_i \mid g_{p_i}(F) \geq a \,\} = \bigcup_{v_i \in \bar{V}_i} \{\, p_i \in \mathcal{P}_i^{v_i} \mid g_{p_i}(F) \geq a \,\} \in \mathcal{B}(\mathcal{P}_i).$$

Hence, $p_i \mapsto g_{p_i}(F)$ is measurable, and so (OA.18) holds.

[4]The randomization over ϵ-extensions used in the proof of Claim A1 and the Δ-extensions specified in Subsection OA1.1 can be treated in a very similar manner.

δ-extensions. For $v_i \in \bar{V}_i$, denote the set of posteriors that satisfy (11) with $v_i^k = v_i$ by

$$\mathcal{P}_i^{v_i} = \bigcup_{v_i' \in \bar{V}_i : v_i' > v_i} (\{\, p_i \in \mathcal{P}_i \mid p_i(v_i) > 0 \text{ and } H_i(v_i, p_i) < 0 \} \cap \{\, p_i \in \mathcal{P}_i \mid p_i(v_i') = 0 \text{ or } H_i(v_i', p_i) \geq 0 \}).$$

Since ironed virtual valuations are continuous in posteriors (see Lemma 1b)), $\mathcal{P}_i^{v_i} \in \mathcal{B}(\mathcal{P}_i)$. Hence, the trace σ-algebra $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i) = \{\, \mathcal{P}_i^{v_i} \cap F \mid F \in \mathcal{B}(\mathcal{P}_i) \,\}$ is contained in $\mathcal{B}(\mathcal{P}_i)$.

In Subsection IV.B, the function $p_i \mapsto \delta(p_i)$ in the definition of δ-extensions was unspecified. Here, we define $p_i \mapsto \delta(p_i)$ such that $H_i(v_i^k, p_i^{\delta(p_i)}) = 0$ (cf. Lemma 3a)). Note that only these δ-extensions are used in the paper. For $v_i \in \bar{V}_i$, endow $\mathcal{P}_i^{v_i}$ with the trace σ-algebra $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. For $\xi \in [0,1]$, define $\xi \mapsto p_i^{\xi}$ by

$$p_i^{\xi}(v_i') = \begin{cases} \frac{1}{1-[1-P_i(v_i)]\xi} p_i(v_i') & \text{if } v_i' \leq v_i, \\ \frac{1-\xi}{1-[1-P_i(v_i)]\xi} p_i(v_i') & \text{if } v_i' > v_i. \end{cases}$$

The function $(p_i, \xi) \mapsto H_i(v_i, p_i^{\xi})$ on $(p_i, \xi) \in \mathcal{P}_i^{v_i} \times [0,1]$ is continuous in each argument by the continuity of ironed virtual valuations. Hence, it is a Carathéodory function, which implies that the correspondence that assings to each $p_i \in \mathcal{P}_i^{v_i}$ the set $\{\, \xi \in [0,1] \mid H_i(v_i, p_i^{\xi}) = 0 \,\}$ admits a selector $p_i \mapsto \xi^*(p_i)$ that is measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$ (see Aliprantis and Border, 2006, Cor. 18.8, Thm. 18.13, Lem. 18.2). As shown in the proof of Lemma 3, for $v_i' > v_i$ the equality $H_i(v_i', p_i^{\xi}) = H_i(v_i', p_i)$ holds for any ξ with $H_i(v_i, p_i^{\xi}) = 0$, and thus also for $\xi = \xi^*(p_i)$.

We now define the function $g : \mathcal{P}_i \to \Delta\mathcal{P}_i$ for δ-extensions. If p_i does not satisfy (11), then g_{p_i} draws p_i with probability 1. If p_i satisfies (11) with $v_i^k = v_i$, then g_{p_i} draws posterior $p_i^{\delta(p_i)}$ with probability $1 - [1 - P_i(v_i)]\delta(p_i)$ and posterior p_i'' with probability $[1 - P_i(v_i)]\delta(p_i)$, where $\delta(p_i) = \xi^*(p_i)$, $V_i(p_i'') = \{\, v_i' \in V_i(p_i) \mid v_i' > v_i \,\}$,

$$p_i''(v_i') = \frac{p_i(v_i')}{1 - P_i(v_i)} \quad \forall v_i' \in V_i(p_i''),$$

$V_i(p_i^{\delta(p_i)}) = V_i(p_i)$, and

$$p_i^{\delta(p_i)}(v_i') = \begin{cases} \frac{1}{1-[1-P_i(v_i)]\delta(p_i)} p_i(v_i') & \text{if } v_i' \leq v_i, \\ \frac{1-\delta(p_i)}{1-[1-P_i(v_i)]\delta(p_i)} p_i(v_i') & \text{if } v_i' > v_i. \end{cases}$$

Next, we show that g satisfies (OA.18). For $F \in \mathcal{B}(\mathcal{P}_i)$, let $\hat{F}^{v_i} = \{\, p_i \in \mathcal{P}_i^{v_i} \mid p_i^{\delta(p_i)} \in F \,\}$ and $\tilde{F}^{v_i} = \{\, p_i \in \mathcal{P}_i^{v_i} \mid p_i'' \in F \,\}$. Then,

$$\begin{aligned} g_{p_i}(F) = \sum_{v_i \in \bar{V}_i} & (\mathbf{1}_{\hat{F}^{v_i}}(p_i)[1 - [1 - P_i(v_i)]\delta(p_i)] + \mathbf{1}_{\tilde{F}^{v_i}}(p_i)[1 - P_i(v_i)]\delta(p_i)) \\ & + \mathbf{1}_{F \setminus (\bigcup_{v_i \in \bar{V}_i} \mathcal{P}_i^{v_i})}(p_i). \end{aligned}$$

For any $v_i \in \bar{V}_i$, the restriction of the functions $p_i \mapsto P_i(v_i)$ and $p_i \mapsto p_i''$, respectively, to $\mathcal{P}_i^{v_i}$ is continuous and hence measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. The restriction of $p_i \mapsto p_i^{\delta(p_i)}$ to $\mathcal{P}_i^{v_i}$ is measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$ by the measurability of $p_i \mapsto \delta(p_i)$. It follows that the restriction of $p_i \mapsto g_{p_i}(F)$ to $\mathcal{P}_i^{v_i}$ is measurable with respect to $\mathcal{P}_i^{v_i} \cap \mathcal{B}(\mathcal{P}_i)$. Since furthermore $F \setminus (\bigcup_{v_i \in \bar{V}_i} \mathcal{P}_i^{v_i}) \in \mathcal{B}(\mathcal{P}_i)$, an analogous argument as for ϵ-extensions can now be used to show that the unrestricted function $p_i \mapsto g_{p_i}(F)$ is measurable with respect to $\mathcal{B}(\mathcal{P}_i)$. Hence, (OA.18) holds.

References

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