Online Appendix of *Learning by Choosing: Career Concerns with Observable Actions*

In this online appendix, we supplement technical details for the proof of Propositions 1 and 3.

Existence and Uniqueness of a Solution to Equation (10) and (11)

We have the following parameters: $\alpha > \beta > 1$, $0 < \mu < \bar{\mu}$, $\theta_H > \theta_L$, $\Delta := \theta_H - \theta_L$, and $x_m := (u_0 - \theta_L)/\Delta$ where $\theta_L + \mu < u_0 < \theta_H + \mu$ and

$$\bar{\mu} := \frac{\Delta}{4(\alpha^2-1)}\Big(2(2\alpha^2-\beta^2-1)x_m - (\alpha-1)(\beta^2+2\alpha+1) \\ + \sqrt{(2(\beta^2-1)x_m-(\alpha-1)(\beta^2+2\alpha+1))^2 - 8(\alpha-1)(\beta^2-1)(\alpha^2-\beta^2)x_m}\Big). \quad \text{(i)}$$

Equivalently, we write the inequality for x_m as

$$\frac{\mu}{\Delta} < x_m < \frac{\mu}{\Delta} + 1. \quad \text{(ii)}$$

We will first show that the following system of non-linear equation:

$$\left(\frac{\underline{x}}{\bar{x}}\right)^{\frac{\beta+1}{2}}\left(\frac{1-\underline{x}}{1-\bar{x}}\right)^{-\frac{\beta-1}{2}} = \frac{\Delta(\alpha+\beta)\left((\beta-1)x_m - (\beta+1-2x_m)\underline{x}\right)}{\mu\left((\alpha-1)(\beta-1)+2(\alpha+\beta)\bar{x}\right)} \quad \text{(iii)}$$

$$\left(\frac{\underline{x}}{\bar{x}}\right)^{-\frac{\beta-1}{2}}\left(\frac{1-\underline{x}}{1-\bar{x}}\right)^{\frac{\beta+1}{2}} = \frac{\Delta(\alpha-\beta)\left((\beta+1)x_m - (\beta-1+2x_m)\underline{x}\right)}{\mu\left((\alpha-1)(\beta+1)-2(\alpha-\beta)\bar{x}\right)} \quad \text{(iv)}$$

has a solution $(\underline{x}, \bar{x}) \in (0,1)^2$.

We note that (iii) defines a curve C_1 in $(0,1)^2$, and it can be rewritten as:

$$F_1(\bar{x};\alpha,\beta,\mu,\Delta,x_m) := \frac{\mu\left((\alpha-1)(\beta-1)+2(\alpha+\beta)\bar{x}\right)(1-\bar{x})^{\frac{\beta-1}{2}}}{\bar{x}^{\frac{\beta+1}{2}}} \\ = \frac{\Delta(\alpha+\beta)\left((\beta-1)x_m-(\beta+1-2x_m)\underline{x}\right)(1-\underline{x})^{\frac{\beta-1}{2}}}{\underline{x}^{\frac{\beta+1}{2}}} =: G_1(\underline{x};\alpha,\beta,\mu,\Delta,x_m). \quad \text{(v)}$$

By writing the LHS in the form of

$$\frac{\mu(\alpha-1)(\beta-1)(1-\bar{x})^{\frac{\beta-1}{2}}}{\bar{x}^{\frac{\beta+1}{2}}} + \frac{2\mu(\alpha+\beta)(1-\bar{x})^{\frac{\beta-1}{2}}}{\bar{x}^{\frac{\beta-1}{2}}},$$

we can see the LHS is an function monotonically decreasing in $\bar{x}$ going from $+\infty$ when $\bar{x}=0$ to 0 when $\bar{x}=1$. The RHS viewing as a function of $\underline{x}$ is greater than zero for all

$$0<\underline{x}<\min\left\{1,\underline{x}_1:=\frac{(\beta-1)x_m}{\beta+1-2x_m}\right\},$$

and so for any $\underline{x}$ in this domain, we can solve for a unique $\bar{x}$. We note that

$$(\beta-1)x_m-(\beta+1-2x_m)=-(1-x_m)(\beta+1);$$

therefore, $\underline{x}_1<1$ if $\mu/\Delta<x_m<1$ and $\underline{x}_1>1$ if $1<x_m<\min\{\mu/\Delta+1,(\beta+1)/2\}$. If $x_m>(\beta+1)/2$ then the RHS is also always positive.

Similarly, (iv) defines the curve C_2 in $(0,1)^2$, and it can be rewritten as:

$$\begin{aligned} F_2(\bar{x};\alpha,\beta,\mu,\Delta,x_m) &:= \frac{\mu\left((\alpha-1)(\beta+1)-2(\alpha-\beta)\bar{x}\right)\bar{x}^{\frac{\beta-1}{2}}}{(1-\bar{x})^{\frac{\beta+1}{2}}} \\ &= \frac{\Delta(\alpha-\beta)\left((\beta+1)x_m-(\beta-1+2x_m)\underline{x}\right)\underline{x}^{\frac{\beta-1}{2}}}{(1-\underline{x})^{\frac{\beta+1}{2}}} =: G_2(\underline{x};\alpha,\beta,\mu,\Delta,x_m). \end{aligned} \quad \text{(vi)}$$

Note that

$$\begin{aligned} (\alpha-1)(\beta+1)-2(\alpha-\beta)\bar{x} &> (\alpha-1)(\beta+1)-2(\alpha-\beta) \\ &= (\alpha+1)(\beta-1)>0; \end{aligned}$$

therefore, the LHS is always positive, and in fact it is monotonically increasing in $\bar{x}$ going from 0 when $\bar{x}=0$ to $+\infty$ when $\bar{x}=1$, as can be seen by rewriting it as:

$$\frac{\mu((\alpha-1)(\beta+1)-2(\alpha-\beta))}{(1-\bar{x})^{\frac{\beta+1}{2}}}+\frac{2\mu(\alpha-\beta)\bar{x}^{\frac{\beta-1}{2}}}{(1-\bar{x})^{\frac{\beta-1}{2}}}.$$

The RHS viewing as a function of $\underline{x}$ is greater than zero for all

$$0<\underline{x}<\min\left\{1,\underline{x}_2:=\frac{(\beta+1)x_m}{\beta-1+2x_m}\right\},$$

and so for any $\underline{x}$ in this domain, we can solve for a unique $\bar{x}$. We note that

$$(\beta+1)x_m-(\beta-1+2x_m)=-(1-x_m)(\beta-1);$$

therefore, $\underline{x}_2 < 1$ if $\mu/\Delta < x_m < 1$ and $\underline{x}_2 > 1$ if $1 < x_m < \mu/\Delta + 1$.

We will consider two cases below.

Case I: $\mu/\Delta < x_m < 1$**:**

For both C_1, C_2 we can see that when $\underline{x} \to 0^+$, we also have $\bar{x} \to 0^+$. On the other hand, when $\underline{x} \to \underline{x}_1^- < 1$, we find $\bar{x} \to 1^-$ on C_1, while C_2 is continuous at $\underline{x} = \underline{x}_1$ with $0 < \bar{x} < 1$. If we can show that the initial slope $d\bar{x}/d\underline{x}$ near $(\underline{x}, \bar{x}) = (0,0)$ of C_2 is greater than C_1 then C_1 and C_2 must intercept at least once, hence we can conclude the existence of solution. We proceed as follows. Let $\delta\underline{x}, \delta\bar{x} > 0$ be small, then setting $(\underline{x}, \bar{x}) = (\delta\underline{x}, \delta\bar{x})$ and raising (v) to the power of $2/(\beta-1)$, expanding both sides keeping only the first order of $\delta\bar{x}, \delta\underline{x}$, we find:

$$\frac{d\bar{x}}{d\underline{x}}\bigg|_{(0,0)\in C_1} = \lim_{\delta\underline{x}\to 0} \frac{\delta\bar{x}}{\delta\underline{x}} = \left(\frac{\mu(\alpha-1)}{\Delta(\alpha+\beta)x_m}\right)^{\frac{2}{\beta+1}}. \tag{vii}$$

Doing the same for C_2, we find that

$$\frac{d\bar{x}}{d\underline{x}}\bigg|_{(0,0)\in C_2} = \lim_{\delta\underline{x}\to 0} \frac{\delta\bar{x}}{\delta\underline{x}} = \left(\frac{\Delta(\alpha-\beta)x_m}{\mu(\alpha-1)}\right)^{\frac{2}{\beta-1}}. \tag{viii}$$

Combining $0 < \mu < \bar{\mu}$ with (ii), we have that $\mu/\Delta < \min\{x_m, \bar{\mu}/\Delta\}$, or $\mu/(\Delta x_m) < \min\{1, \bar{\mu}/(\Delta x_m)\}$. On the other hand, we may check using (i) that for any fixed α, β, Δ; $\bar{\mu}/(\Delta x_m)$ is a monotonically decreasing function in $x_m \in (0,1)$. For example, we find that $d(\bar{\mu}/(\Delta x_m))/dx_m = 0$ has exactly one solution at $x_m = 0$, then it is straightforward to compute that $d(\bar{\mu}/(\Delta x_m))/dx_m|_{x_m=1} < 0$ and conclude that $d(\bar{\mu}/(\Delta x_m))/dx_m < 0$ for all $0 < x_m \leq 1$. Therefore,

$$\frac{\bar{\mu}}{\Delta x_m} < \lim_{x_m\to 0} \frac{\bar{\mu}}{\Delta x_m} = \frac{2(\alpha-\beta)(\alpha+\beta)}{(\alpha-1)(1+2\alpha+\beta^2)} < \frac{\alpha-\beta}{\alpha-1},$$

where the last inequality followed because

$$2(\alpha+\beta) - (1+2\alpha+\beta^2) = -1 + 2\beta - \beta^2 = -(\beta-1)^2 < 0.$$

It follows that

$$\frac{\mu}{\Delta x_m} < \min\left\{1, \frac{\bar{\mu}}{\Delta x_m}\right\} < \min\left\{1, \frac{\alpha-\beta}{\alpha-1}\right\} = \frac{\alpha-\beta}{\alpha-1},$$

and so,

$$\left.\frac{d\bar{x}}{d\underline{x}}\right|_{(0,0)\in C_2} = \left(\frac{\Delta(\alpha-\beta)x_m}{\mu(\alpha-1)}\right)^{\frac{2}{\beta-1}} > 1 > \left(\frac{\mu(\alpha-1)}{\Delta(\alpha-\beta)x_m}\right)^{\frac{2}{\beta+1}}$$
$$> \left(\frac{\mu(\alpha-1)}{\Delta(\alpha+\beta)x_m}\right)^{\frac{2}{\beta+1}} = \left.\frac{d\bar{x}}{d\underline{x}}\right|_{(0,0)\in C_1}.$$

Case II: $1 \leq x_m < \mu/\Delta + 1$**:**

As before, for both C_1, C_2 approaches the point $(0,0)$ as $\underline{x} \to 0^+$. Now, since $\underline{x}_1, \underline{x}_2 \notin (0,1)$ in this case, we have that both C_1, C_2 also approches the point $(1,1)$ as $\underline{x} \to 1^-$. The initial slope of C_1 and C_2 near the point $(0,0)$ are still given by (vii) and (viii), and since $\bar{\mu}/(\Delta x_m)$ is monotonically decreasing for $1 < x_m < \mu/\Delta + 1$ it remains true that $d\bar{x}/d\underline{x}|_{(0,0)\in C_2} > d\bar{x}/d\underline{x}|_{(0,0)\in C_1}$. We can compute the final slope near the point $(1,1)$ of both curves in a similar way by expanding (v) and (vi) to the first order of $1-\bar{x}$ and $1-\underline{x}$, we have

$$\left.\frac{d\bar{x}}{d\underline{x}}\right|_{(1,1)\in C_1} = \lim_{\underline{x}\to 1^-} \frac{1-\bar{x}}{1-\underline{x}} = \left(\frac{\Delta(\alpha+\beta)(x_m-1)}{\mu(\alpha+1)}\right)^{\frac{2}{\beta-1}}, \tag{ix}$$

and

$$\left.\frac{d\bar{x}}{d\underline{x}}\right|_{(1,1)\in C_2} = \lim_{\underline{x}\to 1^-} \frac{1-\bar{x}}{1-\underline{x}} = \left(\frac{\mu(\alpha+1)}{\Delta(\alpha-\beta)(x_m-1)}\right)^{\frac{2}{\beta+1}}. \tag{x}$$

But since $x_m - 1 < \mu/\Delta$, we then have

$$\left.\frac{d\bar{x}}{d\underline{x}}\right|_{(1,1)\in C_1} < \left(\frac{\alpha+\beta}{\alpha+1}\right)^{\frac{2}{\beta-1}} < \left(\frac{\alpha+1}{\alpha-\beta}\right)^{\frac{2}{\beta+1}} < \left.\frac{d\bar{x}}{d\underline{x}}\right|_{(1,1)\in C_2}.$$

Therefore, if we took $+\bar{x}$ to be an upward direction, then the curve C_2 approaches the point $(1,1)$ from below the curve C_1. But we already know that the curve C_2 was above the curve C_1 when leaving the point $(0,0)$, it must be the case that both curves intercept at some point, proving the existence of a solution $(\underline{x}, \bar{x}) \in (0,1)^2$.

Uniqueness and validity of the solution with $\underline{x} < \bar{x}$

In the following we argue that the solution $(\underline{x}, \bar{x}) \in (0,1)^2$ is unique and satisfies $\underline{x} < \bar{x}$. We can write (v) and (vi) as

$$\boldsymbol{R}(\underline{x}, \bar{x}, \boldsymbol{\theta}) = \begin{pmatrix} R_1(\underline{x}, \bar{x}; \boldsymbol{\theta}) \\ R_2(\underline{x}, \bar{x}; \boldsymbol{\theta}) \end{pmatrix} := \begin{pmatrix} F_1(\bar{x}; \boldsymbol{\theta}) - G_1(\underline{x}; \boldsymbol{\theta}) \\ F_2(\bar{x}; \boldsymbol{\theta}) - G_2(\underline{x}; \boldsymbol{\theta}) \end{pmatrix} = 0, \quad \boldsymbol{R} : (0,1)^2 \times \mathbb{R}^5 \to \mathbb{R}^2,$$

where we write $\boldsymbol{\theta} := (\alpha, \beta, \mu, \Delta, x_m) \in D$ for the rest of parameters and $D \subset \mathbb{R}^5$ denotes the domain where the parameters are valid (they satisfy all the required inequalities: $\alpha > \beta > 1$, $0 < \mu < \bar{\mu}, \Delta > 0, \mu/\Delta < x_m < \mu/\Delta + 1$). Clearly, D is open and path-connected. The relation $\boldsymbol{R}(\underline{x}, \bar{x}; \boldsymbol{\theta}) = 0$ defines a 5-dimensional surface C inside $(0,1)^2 \times \mathbb{R}^5$ and given a point $(\underline{x}, \bar{x}, \boldsymbol{\theta}) \in C$ the Implicit Function Theorem (IFT) tells us that if

$$0 \neq \begin{vmatrix} \frac{\partial R_1}{\partial \bar{x}} & \frac{\partial R_1}{\partial \underline{x}} \\ \frac{\partial R_2}{\partial \bar{x}} & \frac{\partial R_2}{\partial \underline{x}} \end{vmatrix} = \frac{\partial F_2}{\partial \bar{x}} \frac{\partial G_1}{\partial \underline{x}} - \frac{\partial F_1}{\partial \bar{x}} \frac{\partial G_2}{\partial \underline{x}}, \tag{xi}$$

then there exists an open set $U \times V \ni (\underline{x}, \bar{x}, \boldsymbol{\theta}), U \in (0,1)^2, V \in \mathbb{R}^5$ such that $C \cap U \times V$ is given by $(\underline{x}, \bar{x}) = g(\boldsymbol{\theta})$ for some continuously differentiable function $g : V \to U$. In fact, (xi) fails exactly at the point where C_1 touches C_2 with the same slope, and this condition can be written explicitly as:

$$\frac{d\bar{x}/d\underline{x}|_{(\underline{x}, \bar{x}, \boldsymbol{\theta}) \in C_1}}{d\bar{x}/d\underline{x}|_{\underline{x}, \bar{x}, \boldsymbol{\theta}) \in C_2}} = \frac{\partial G_1/\partial \underline{x}}{\partial F_1/\partial \bar{x}} \frac{\partial F_2/\partial \bar{x}}{\partial G_2/\partial \underline{x}} = \left(\frac{1-\underline{x}}{1-\bar{x}}\right)^{\beta} \left(\frac{\bar{x}}{\underline{x}}\right)^{\beta} \left(\frac{\alpha+\beta}{\alpha-\beta}\right) = 1. \tag{xii}$$

Suppose that (xii) is true, then dividing (iii) by (iv), we obtain:

$$\underline{x} = \frac{(\alpha+1)x_m\bar{x}}{(\alpha+2x_m-1)\bar{x} + (\alpha-1)(x_m-1)}. \tag{xiii}$$

Note that $\alpha + 2x_m - 1 > 0$ always. If $x_m \le 1$, then $\underline{x} > 0$ means that $\bar{x} > (\alpha-1)(1-x_m)/(\alpha + 2x_m - 1)$. But $\underline{x}$ is a decreasing function with $\bar{x}$ for $\bar{x} > (\alpha-1)(1-x_m)/(\alpha+2x_m-1)$, and so

$$\underline{x} \ge \lim_{\bar{x} \to 1^-} \underline{x} = \frac{(\alpha+1)x_m}{(\alpha+2x_m-1) + (\alpha-1)x_m - \alpha + 1} = 1.$$

Therefore, for all possible values of $\bar{x} \in (0,1)$, we have $\underline{x} \notin (0,1)$, so (xii) is impossible when $x_m \le 1$. If $x_m > 1$, let us substitute (xiii) back into (xii), we find that all $\underline{x}$ actually cancels out and we are left with:

$$\left(\frac{x_m(\alpha+1)}{(x_m-1)(\alpha-1)}\right)^{\beta}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)=1. \tag{xiv}$$

Note that this expression makes sense as we now have $x_m-1>0$, so its fractional power β is real. But $x_m<\frac{\mu}{\Delta}+1<\frac{\bar{\mu}}{\Delta}+1<\frac{\alpha^2-\beta^2}{\alpha^2-1}x_m+1$, where the last inequality can be seen by rewriting (i) as

$$\begin{aligned}\frac{\bar{\mu}}{\Delta}=\frac{\alpha^2-\beta^2}{\alpha^2-1}x_m-\frac{1}{4(\alpha^2-1)}\Big[&(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m\\&-\sqrt{((\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m)^2-8(\alpha-1)(\beta^2-1)(\alpha^2-\beta^2)x_m}\Big]\\&<\frac{\alpha^2-\beta^2}{\alpha^2-1}x_m.\end{aligned}$$

The square bracket term is always positive, because when $x_m=1$ we have

$$\begin{aligned}(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m&=(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)\\&>(\beta-1)(\beta^2+2\beta+1)-2(\beta^2-1)=(\beta-1)^2(\beta+1)>0.\end{aligned}$$

When $x_m>1$ increases, the term $(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m$ decreases toward zero, but the square–root term would becomes imaginary before it ever reaches zero. Note that the square–root term is smaller than $(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m$ in magnitude, so $(\alpha-1)(\beta^2+2\alpha+1)-2(\beta^2-1)x_m>0$ implies that the square bracket term is positive as claimed. If $x_m>1$ is sufficiently large, the square–root term will become real again, but we would also have $x_m>\frac{\bar{\mu}}{\Delta}+1$ as we can check that $\lim_{x_m\to+\infty}\left(\frac{\bar{\mu}}{\Delta}+1-x_m\right)=-\frac{\alpha-1}{2}<0$ and $\frac{\bar{\mu}}{\Delta}+1-x_m=0$ has no solution for $x_m>1$, so this case lies outside the valid parameters domain and can be ignored.

Using $x_m<\frac{\alpha^2-\beta^2}{\alpha^2-1}x_m+1$, or equivalently $\frac{x_m}{x_m-1}>\frac{\alpha^2-1}{\alpha^2-\beta^2}$, it follows that:

$$\left(\frac{x_m(\alpha+1)}{(x_m-1)(\alpha-1)}\right)^{\beta}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)>\left(\left(\frac{\alpha^2-1}{\alpha^2-\beta^2}\right)\left(\frac{\alpha+1}{\alpha-1}\right)\right)^{\beta}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)>1.$$

This is a contradiction to (xiv) and we conclude that it is not possible to satisfy condition (xii) for any $\boldsymbol{\theta}\in D$. From the existence of solution we have previously proven, it follows that the IFT can be applied over any given parameter $\boldsymbol{\theta}\in D$.

Suppose that there exists at least one $\boldsymbol{\theta}_0\in D$ such that the solution $(\underline{x},\bar{x})\in(0,1)^2$ to

(iii) and (iv) is unique and valid ($\underline{x} < \bar{x}$). We can check this is true, for example when $x_m < 1, \mu \to 0^+$ we have $\bar{x} \to 1^-, \underline{x} \to \underline{x}_1 = (\beta - 1)x_m/(\beta + 1 - 2x_m)$ and clearly no other solution is possible.[11] Now, given any other $\boldsymbol{\theta}_1 \in D$, we draw a path γ connecting $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$. If the solution is not unique at $\boldsymbol{\theta}_1$, i.e. there are more points on C above $\boldsymbol{\theta}_1$ than there are above $\boldsymbol{\theta}_0$, then there exists a solution $(\underline{x}', \bar{x}')$ which cannot be varied continuously to a solution above $\boldsymbol{\theta}_0$, contradicting the fact that we can cover γ with finitely many open sets where IFT applied [12]. Therefore, the solution at $\boldsymbol{\theta}_1$ is also unique. If the solution at $\boldsymbol{\theta}_1$ is not valid ($\underline{x} > \bar{x}$), then let us cover a compact set γ with finitely many open sets where IFT applied and vary the solution continuously to the solution above $\boldsymbol{\theta}_0$ which is known to satisfies $\underline{x} < \bar{x}$. At some point $\boldsymbol{\theta}^* \in \gamma \subset D$, we must have $\underline{x} = \bar{x}$, but it is straightforward to check that this implies $\mu = \bar{\mu}$, which is not a valid point in D, a contradiction.

Verification Theorem for the Optimal Control and Stopping Problem

Most verification theorems require C^2 candidate value functions which we do not have, because $\widehat{V}(x)$ is generally not C^2 at the stopping boundary, $x = \underline{x}$. It is well-known that in one dimension, Ito's formula works for functions that are C^1 everywhere and C^2 almost everywhere. Based on this observation, we are going to prove a verification theorem for our problem.

Consider any arbitrary admissible rules, $\widehat{\tau}$ and $\widehat{J}$, and the corresponding belief updating process,

$$\begin{aligned} d\widehat{x}_t &= \frac{\Delta}{\sigma(\widehat{J}_t)} \widehat{x}_t(1 - \widehat{x}_t) dW_t \text{ for } t \in [0, \widehat{\tau}], \\ \widehat{x}_0 &= x. \end{aligned}$$

Notice that by construction in Section 3.1, $\widehat{V}(x)$ is twice continuously differentiable for $x \in [0, 1]$ except for $x = \underline{x}$. That is, $\widehat{V}(x)$ is C^2 almost everywhere. This means that we can

[11] When $\mu \to 0^+$, we have from (iii) that either $\bar{x} \to 0^+$, or $\underline{x} \to 1^-$, or $\underline{x} \to \underline{x}_1^-$. With $\bar{x} \to 0^+$, the LHS of (iv) will be zero, so $\underline{x} \to \underline{x}_2^-$. But $\underline{x}_2 > \underline{x}_1$, so the RHS of (iii) will be negative, a contradiction. With $\underline{x} \to 1^-$, the RHS of (iii) will also be negative, because $\underline{x}_1 < 1$, a contradiction. This leaves us with just one possibility: $\underline{x} \to \underline{x}_1^-$, which implies $\bar{x} \to 1^-$. The slope ratio of C_1 and C_2 as in (xii) for any solutions $\underline{x} \to \underline{x}_1^-, \bar{x} \to 1^-$ must approach $+\infty$, hence there can only be exactly one such solution since C_1 and C_2 cannot intercept multiple times with this slope ratio.

[12] In other words, C_1 intercepts C_2 once given $\boldsymbol{\theta}_0$, but intercepts $2n+1$ times over $\boldsymbol{\theta}_1$, for some $n > 0$. Given that we have shown C_2 to leave $(0, 0)$ above C_1 and approach $(1, 1)$ from below C_1, as we deform both C_1 and C_2 smoothly from $\boldsymbol{\theta}_1$ to $\boldsymbol{\theta}_0$ there must be some point where C_1 only touches C_2 tangentially, and we have shown this to be impossible.

use Itô's formula to obtain

$$e^{-r\widehat{\tau}}\widehat{V}(\widehat{x}_{\widehat{\tau}}) = \widehat{V}(x) + \int_0^{\widehat{\tau}} \left[-re^{-rt}\widehat{V}(\widehat{x}_t) + \frac{1}{2}e^{-rt}\widehat{V}''(\widehat{x}_t)\frac{\Delta^2}{\sigma^2(\widehat{J}_t)}\widehat{x}_t^2(1-\widehat{x}_t)^2 \right] dt \\ + \int_0^{\widehat{\tau}} e^{-rt}\widehat{V}'(\widehat{x}_t)\frac{\Delta}{\sigma(\widehat{J}_t)}\widehat{x}_t(1-\widehat{x}_t)dW_t. \quad \text{(xv)}$$

By the HJB equation (5), we have

$$\max\left\{ \mu(\widehat{J}_t) + \theta_L + \Delta x_t + \frac{1}{2}\widehat{V}''(\widehat{x}_t)\frac{\Delta^2}{\sigma^2(\widehat{J}_t)}\widehat{x}_t^2(1-\widehat{x}_t)^2 - r\widehat{V}(\widehat{x}_t), \right. \\ \left. u_0 - r\widehat{V}(\widehat{x}_t) \right\} \leq 0,$$

which further implies that

$$\int_0^{\widehat{\tau}} e^{-rt} \left[\mu(\widehat{J}_t) + \theta_L + \Delta x_t + \frac{1}{2}\widehat{V}''(\widehat{x}_t)\frac{\Delta^2}{\sigma^2(\widehat{J}_t)}\widehat{x}_t^2(1-\widehat{x}_t)^2 - r\widehat{V}(\widehat{x}_t) \right] dt \\ + \int_{\widehat{\tau}}^{\infty} e^{-rt} \left[u_0 - r\widehat{V}(\widehat{x}_t) \right] dt \leq 0.$$

By substituting equation (xv) into the inequality above, we have

$$\begin{aligned} \widehat{V}(x) \geq & e^{-r\widehat{\tau}}\widehat{V}(\widehat{x}_{\widehat{\tau}}) + \int_0^{\widehat{\tau}} re^{-rt}\widehat{V}(\widehat{x}_t)dt - \int_0^{\widehat{\tau}} \widehat{V}'(\widehat{x}_t)\frac{\Delta}{\sigma(\widehat{J}_t)}\widehat{x}_t(1-\widehat{x}_t)dW_t \\ & + \int_0^{\widehat{\tau}} e^{-rt} \left[\mu(\widehat{J}_t) + \theta_L + \Delta x_t - r\widehat{V}(\widehat{x}_t) \right] dt + \int_{\widehat{\tau}}^{\infty} e^{-rt} \left[u_0 - r\widehat{V}(\widehat{x}_t) \right] dt \\ = & \int_0^{\widehat{\tau}} e^{-rt} \left[\mu(\widehat{J}_t) + \theta_L + \Delta x_t \right] dt + \int_{\widehat{\tau}}^{\infty} e^{-rt}u_0 dt \\ & - \int_0^{\widehat{\tau}} \widehat{V}'(\widehat{x}_t)\frac{\Delta}{\sigma(\widehat{J}_t)}\widehat{x}_t(1-\widehat{x}_t)dW_t. \end{aligned}$$

To get the second equality above, we have used two observations. First, given $\widehat{\tau}$ is the stopping time, after which the social planner stops updating belief, we have $\widehat{x}_t = \widehat{x}_{\widehat{\tau}}$ for $\forall t \geq \widehat{\tau}$. Second, the transversality condition

$$\lim_{T\to\infty} \mathrm{E}[e^{-rT}\widehat{V}(x)] = 0,$$

is obviously satisfied because $\widehat{V}(x)$ is bounded for any $x \in [0,1]$. By taking expectation of the inequality above, we notice that the stochastic integral vanishes by Optional Stopping Theorem, and thus we have,

$$\widehat{V}(x) \geq \mathrm{E}\left[\int_0^{\widehat{\tau}} e^{-rt}\left[\mu(\widehat{J}_t) + \theta_L + \Delta x_t\right] dt + \int_{\widehat{\tau}}^{\infty} e^{-rt} u_0 dt\right].$$

Since $\widehat{\tau}$ and $\widehat{J}$ are arbitrary, this means that

$$\widehat{V}(x) \geq \sup_{\widehat{J}\in\mathscr{J},\widehat{\tau}\in\mathscr{T}} \mathrm{E}\left[\int_0^{\widehat{\tau}} e^{-rt}(\mu(\widehat{J}_t) + \theta_L + \Delta x_t) dt + \int_{\widehat{\tau}}^{\infty} e^{-rt} u_0 dt\right] = V(x).$$

To obtain the reverse inequality, we choose the specific control law $\widehat{\tau} = \tau^*$ and $\widehat{J}(\mathscr{F}_t) = J^*(x_t)$. Going through the exactly same calculations as above and using the fact that

$$\int_0^{\tau^*} e^{-rt}\left[\mu(J^*(x_t)) + \theta_L + \Delta x_t + \frac{1}{2}\widehat{V}''(x_t)\frac{\Delta^2}{\sigma^2(J^*(x_t))}x_t^2(1-x_t)^2 - r\widehat{V}(x_t)\right] dt \\ + \int_{\tau^*}^{\infty} e^{-rt}\left[u_0 - r\widehat{V}(x_t)\right] dt = 0,$$

we obtain the following

$$\widehat{V}(x) = \mathrm{E}\left[\int_0^{\tau^*} e^{-rt}(\mu(J^*(x_t)) + \theta_L + \Delta x_t) dt + \int_{\tau^*}^{\infty} e^{-rt} u_0 dt\right] \leq V(x),$$

where the second inequality above is by definition of $V(x)$. Therefore, we have proved the verification theorem: $\widehat{V}(x) = V(x)$.

Results for the Proof of Proposition 3

Proof of Lemma 3

Proof. For $x \in (\overline{x}_W, 1)$, note that each of the two expressions (20), (21) have their right-hand sides increase continuously and monotonically with D. Moreover, as we limit D to zero, both expressions (20), (21) yield the same value $rU(x) = w(x)$. Thus, $U(x, D, \overline{x}_W)$ increases continuously and monotonically with D, even as D goes from $D < 0$ to $D > 0$, for any given value $x \in (\overline{x}_W, 1)$. Moreover, the derivatives of right-hand sides of (20), (21) decrease continuously and monotonically with D, and so does the value $\frac{\partial}{\partial x}U(x, D, \overline{x}_W) = U'_{D,\overline{x}_W}(x)$, for any given $x \in (\overline{x}_W, 1)$. Respectively, from expressions (16),(17) the same applies to $x = \overline{x}_W$: The value of $U(x = \overline{x}_W, D, \overline{x}_W)$ (respectively, of $\frac{\partial}{\partial x}U(x = \overline{x}_W, D, \overline{x}_W)$) increases (respectively, decreases) continuously and monotonically with D. In Lemma 2, we established that, for $x \in (0, \overline{x}_W)$, function $U(x)$ satisfies a second-order ODE of the form $U''(x) = H(x, U(x))$, with $H(.,.)$ satisfying Lipshitz conditions on an interval $[\varepsilon, \overline{x}_W]$, for any $\varepsilon > 0$; with boundary conditions given by (16)-(17) at point $x = \overline{x}_W$. Respectively, since the values $U(x = \overline{x}_W, D, \overline{x}_W), \frac{\partial}{\partial x}U(x = \overline{x}_W, D, \overline{x}_W) = U'(\overline{x}_W)$ change continuously with D, function $U(x, D, \overline{x}_W)$ as a solution to the ODE $U''(x) = H(x, U(x))$, would change continuously with D as well, for any value of $x \in (0, \overline{x}_W)$. Monotonicity of $U(x, D, \overline{x}_W)$ with respect to D, for $x \in (0, \overline{x}_W)$, follows from Lemma 11. Namely, consider two values $D_1 < D_2$. If functions $U_{D_1,\overline{x}_W}(x)$, $U_{D_2,\overline{x}_W}(x)$ both satisfy the same differential equation (either the one in (18) or the one in (19)) on $(0, \overline{x}_W)$, then we use the result of Lemma 11 for condition (xvii) to show that the difference $U_{D_2,\overline{x}_W}(x) - U_{D_1,\overline{x}_W}(x)$ increases, and $U'_{D_2,\overline{x}_W}(x) - U'_{D_1,\overline{x}_W}(x)$ decreases, as x decreases. If at some point function $U_{D_1,\overline{x}_W}(x)$ would satisfy (19) while $U_{D_2,\overline{x}_W}(x)$ would satisfy (18), then $U''_{D_1,\overline{x}_W}(x) \leq 0$, $U''_{D_2,\overline{x}_W}(x) \geq 0$ and hence as x decreases, the difference $U_{D_2,\overline{x}_W}(x) - U_{D_1,\overline{x}_W}(x)$ increases. Thus, for two values $D_1 < D_2$, one has $U(x, D_1, \overline{x}_W) < U(x, D_2, \overline{x}_W)$, for $x \in (0, \overline{x}_W)$. □

Proof of Lemma 4

Proof. If the wage cutoff increases from $\overline{x}_W$ to $\overline{x}'_W > \overline{x}_W$, the wage function $w(x)$ would decrease between $\overline{x}_W$ and $\overline{x}'_W$ (as seen from (14)). Note that expressions (20), (21) are solutions to equations (18), (19), respectively. Equations (18), (19) are equivalent to $U''(x) = H(x, U(x))$, where $H(x, U(x)) \equiv \frac{2\sigma_j^2[rU(x) - w(x)]}{\Delta^2 x^2 (1-x)^2}$, where $j = I$ if $rU(x) \geq w(x)$, and $j = P$ if $rU(x) < w(x)$. Note that function $H(x, U(x))$ decreases with $w(x)$. That is, for any $x \in (\overline{x}_W, \overline{x}'_W)$, we have $U_{D,\overline{x}_W}(x) < U_{D,\overline{x}'_W}(x)$, $U'_{D,\overline{x}_W}(x) > U'_{D,\overline{x}'_W}(x)$. Indeed, we need to compare solutions to two ODEs with the same boundary conditions (values $U(x), U'(x)$) at $x = \overline{x}'_W$, but different values for function $U''(x) = H(x, U(x))$. After the change in cutoff, the value of $H(x, U)$ increased for all $x \in (\overline{x}_W, \overline{x}'_W)$, and hence, within a small interval of values $x \in (\overline{x}'_W - \varepsilon, \overline{x}'_W)$, the value $U(x)$ increased while the value $U'(x)$ decreased. Hence, applying the mean value theorem (similar to proof of Lemma 11) we get that the value of $U(x)$ increases (respectively, the value of $U'(x)$ decreases) for any $x \in (\overline{x}_W, \overline{x}'_W)$.

Finally, for $x < \overline{x}_W$, we can apply Lemma 11 with respect to function $V_1(x) = U_{D,\overline{x}'_W}(x)$, $V_2(x) = U_{D,\overline{x}_W}(x)$, and $\hat{x} = \overline{x}_W$ to show that after the change in a cutoff, $U(x)$ would increase for all $x < \overline{x}_W$: $U_{D,\overline{x}_W}(x) < U_{D,\overline{x}'_W}(x)$.

Continuity follows from standard arguments - that the solution to a second-order differential equation, that satisfies Lipshitz conditions, depends continuously on boundary conditions. $\square$

Proof of Lemma 5

Proof. Let's look at how function $U(x)$ behaves for different values of D. For a value of $M > 0$ big enough, if $D > M$, then, on interval $x \in (\overline{x}_W, 1)$, the solution to (20) would lie above u_0/r, moreover, at $x = \overline{x}_W$, the value $U(\overline{x}_W)$ would lie above $w(\overline{x}_W)/r$, while $U'(\overline{x}_W)$ would be negative. Respectively, the value of $U(x)$ for $x < \overline{x}_W$, would satisfy (18)-(19) with boundary conditions at $x = \overline{x}_W$, that is, $U''(x) > 0$, $U'(x) < 0$ and $U(x) > u_0/r$ for all x.

At the same time, for a value of $N > 0$ large enough, if $D < -N$, then from (21), the graph of $U(x)$ would be lower than u_0/r for some $x \in (\overline{x}_W, 1)$. Thus, due to Lemma 3, as we change D continuously from $-N$ to M, we can find an upper bound value $D = \hat{D}$: For all $D \leq \hat{D}$, the graph of $U(x)$ will have common points with a horizontal line u_0/r, while for $D > \hat{D}$, there will be no such common points. More precisely, at $D = \hat{D}$, the graph of $U(x)$ will touch the horizontal line u_0/r at a unique point $x = \underline{x}_W$, that is, $U(\underline{x}_W) = u_0/r$, $U'(\underline{x}_W) = 0$, and for $x \neq \underline{x}_W$, $U(x) > u_0/r$. Otherwise, if $U'(\underline{x}_W) \neq 0$, this contradicts the definition of $\hat{D}$.

$\square$

Technical Lemma—Monotonicity

Lemma 11. *Assume two functions $V_1(x)$ and $V_2(x)$ such that, for some task $k \in \{P, I\}$, they both satisfy the differential equation $rV_i(x) = w(x) + \Delta x + \frac{\Delta^2}{2\sigma_k^2}x^2(1-x)^2V_i''(x)$, on a certain interval $[x_0, x_1]$ of beliefs, with $w(x)$ equal to productivity of one of the tasks (not necessarily equal k). Assume that at some belief $\hat{x} \in [x_0, x_1]$ one gets the following relations:*

$$V_1(\hat{x}) \geq V_2(\hat{x}) \text{ and } V_1'(\hat{x}) \geq V_2'(\hat{x}), \tag{xvi}$$

with at least one inequality being strict. Then, both relations in (xvi) are satisfied as strict inequalities for all $x \in (\hat{x}, x_1]$.

Similarly, suppose at some belief $\hat{x}' \in [x_0, x_1]$ one gets the following relations:

$$V_1(\hat{x}') \geq V_2(\hat{x}') \text{ and } V_1'(\hat{x}') \leq V_2'(\hat{x}'), \tag{xvii}$$

with at least one inequality being strict. Then, both relations in (xvii) are satisfied as strict inequalities for all $x \in [x_0, \hat{x}')$.

Proof. The proof is written regarding (xvi) assuming both inequalities are strict, and is similar for (xvii) and/or one of inequalities being weak. The proof is done by contradiction and uses the mean value theorem. Assume there exists $\widetilde{x} \in (\hat{x}, x_1]$ such that the first inequality in (xvi) is violated: $V_1(\widetilde{x}) \leq V_2(\widetilde{x})$. Denote by x_2 the lowest value of $x \in (\hat{x}, \widetilde{x}]$ such that $V_1(x) \leq V_2(x)$ (such value of x_2 is well defined since both functions $V_1(x), V_2(x)$ are twice continuously differentiable). Since $V_1(\hat{x}) > V_2(\hat{x})$ and $V_1(x_2) \leq V_2(x_2)$, by the mean value theorem, there exists $x_3 \in (\hat{x}, x_2)$ such that $V_1'(x_3) < V_2'(x_3)$. Since $V_1'(\hat{x}) > V_2'(\hat{x})$ and $V_1'(x_3) < V_2'(x_3)$, by the mean value theorem, there exists $x_4 \in (\hat{x}, x_3)$ such that $V_1''(x_4) < V_2''(x_4)$. However, since both functions satisfy the same differential equation in the statement of the Lemma, the inequality $V_1''(x_4) < V_2''(x_4)$ means $V_1(x_4) < V_2(x_4)$, which contradicts the definition of x_2, thus contradicting the initial assumption $V_1(\widetilde{x}) < V_2(\widetilde{x})$.

Similarly, assume the second inequality of (xvi) is violated at $\widetilde{x}$: $V_1'(\widetilde{x}) \leq V_2'(\widetilde{x})$. Since $V_1'(\hat{x}) > V_2'(\hat{x})$, by the mean value theorem, there exists $x_5 \in (\hat{x}, \widetilde{x})$ such that $V_1''(x_5) < V_2''(x_5)$. Thus, one gets that $V_1(x_5) < V_2(x_5)$, which is impossible by the argument from the previous paragraph. □