Online Appendix: Markets with Within-Type Adverse Selection

By Anh Nguyen and Teck Yong Tan*

I. Online Appendix: Extensions

A. *Three Quality Levels*

Consider the setup described in Section IV.A for $k = 3$. To streamline the exposition with the baseline model, let the three quality levels be L, M and H, with $v_L < v_M < v_H$ and $c_L < c_M < c_H$. Let $s_M = v_M - c_M$ and assume that $s_M > 0$. To shorten the notation slightly, let $\tau(1|\theta) = \tau_L(\theta)$ and $\tau(2|\theta) = \tau_M(\theta)$. Therefore, type-$\theta$ seller has $\tau_L(\theta)$ units of L, $\tau_M(\theta) - \tau_L(\theta)$ units of M, and $1 - \tau_M(\theta)$ units of H, and

$$
\begin{aligned}
C(q,\theta) =& qc_L + [q - \tau_L(\theta)]^+ (c_M - c_L) + [q - \tau_M(\theta)]^+ (c_H - c_M),\\
V(q,\theta) =& qv_L + [q - \tau_L(\theta)]^+ (v_M - v_L) + [q - \tau_M(\theta)]^+ (v_H - v_M).
\end{aligned}
$$

With an abuse of notation, let F denote the distribution of θ and f its density, assumed to be strictly positive over $\theta \in (0,1)$. To simplify the exposition, we also assume that τ_L and τ_H are both differentiable, and $\tau_L'(\theta), \tau_M'(\theta) > 0$ for all θ (i.e., single-crossing).

Our main program is program $(\mathcal{P})$, with the type λ replaced by θ. Note that

$$
\frac{\partial C(q,\theta)}{\partial \theta} = \mathcal{I}(q > \tau_L(\theta))\,\tau_L'(\theta)(c_M - c_L) + \mathcal{I}(q > \tau_M(\theta))\,\tau_M'(\theta)(c_H - c_M).
$$

Thus, by the envelope theorem,

$$
\begin{aligned}
U^S(\theta) = U^S(0) + (c_M - c_L)&\int_0^\theta \tau_L'(l)\,\mathcal{I}(q(l) > \tau_L(l))\,dl\\
&+ (c_H - c_M)\int_0^\theta \tau_M'(l)\,\mathcal{I}(q(l) > \tau_M(l))\,dl.
\end{aligned}
$$

Following the same argument as that for the baseline model, program $(\tilde{\mathcal{P}})$ in the

* Nguyen: Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh PA 15213. anhnguyen@cmu.edu. Tan: University of Nebraska-Lincoln, 730 N 14th St, Lincoln, NE 68588. ttan@unl.edu.

current setup is
(1)

$$\max_{q(\cdot),U^S(0)} \quad \int_0^1 S\left(q\left(\theta\right),\theta\right) f\left(\theta\right) d\theta - b \qquad \text{s.t.}$$

$$q\left(\cdot\right) \text{ is nondecreasing }, \quad U^S\left(0\right) \geq 0 \ , \quad \text{and} \quad \int_0^1 \psi^B\left(q\left(\theta\right),\theta\right) d\theta - U^S\left(0\right) = b$$

where

$$\begin{aligned} S\left(q,\theta\right) :=& V\left(q,\theta\right) - C\left(q,\theta\right), \\ \psi^S\left(q,\theta\right) :=& \left[\left(c_M - c_L\right)\tau_L'\left(\theta\right)\mathcal{I}\left(q > \tau_L\left(\theta\right)\right) + \left(c_H - c_M\right)\tau_M'\left(\theta\right)\mathcal{I}\left(q > \tau_M\left(\theta\right)\right)\right]\left[1 - F\left(\theta\right)\right], \\ \psi^B\left(q,\theta\right) :=& S\left(q,\theta\right) f\left(\theta\right) - \psi^S\left(q,\theta\right). \end{aligned}$$

A threshold schedule here is a schedule $q\left(\cdot\right)$ in which there exist θ_M and θ_H, with $\theta_M \leq \theta_H$, such that $q\left(\theta\right) = \tau_L\left(\theta\right)$ if $\theta < \theta_M$, $q\left(\theta\right) = \tau_M\left(\theta\right)$ if $\theta \in \left[\theta_M, \theta_H\right)$, and $q\left(\theta\right) = 1$ if $\theta \geq \theta_H$.

LEMMA 1: *If $q\left(\cdot\right)$ is a solution to program (1), it must satisfy the following properties:*

1) *$q\left(\theta\right) \geq \tau_L\left(\theta\right)$ for all θ.*
2) *If there exists an interval $X = \left[\underline{\theta}, \bar{\theta}\right]$ such that $q\left(\theta\right) > \tau_M\left(\theta\right)$ for all $\theta \in X$, then there is some $\hat{q}$ such that $q\left(\theta\right) = \hat{q}$ for all $\theta \in X$.*
3) *If there exists an interval $X = \left[\underline{\theta}, \bar{\theta}\right]$ such that $\tau_L\left(\theta\right) < q\left(\theta\right) \leq \tau_M\left(\theta\right)$ for all $\theta \in X$, then there is some $\hat{q}$ such that $q\left(\theta\right) = \min\left\{\tau_M\left(\theta\right), \hat{q}\right\}$ for all $\theta \in X$.*

PROOF:

Lemma 1 is the extension of Lemma 5 and follows from $S\left(q,\theta\right)$ and $\psi^B\left(q,\theta\right)$ being strictly increasing in q when $q < \tau_L\left(\theta\right)$, $q \in \left(\tau_L\left(\theta\right), \tau_M\left(\theta\right)\right)$ and $q \in \left(\tau_M\left(\theta\right), 1\right)$.

Next, let $q^*\left(\cdot\right)$ denote a solution to program (1). Let χ be the set of θ in which $q^*\left(\theta\right) \notin \left\{\tau_L\left(\theta\right), \tau_M\left(\theta\right), 1\right\}$. Note that if χ is empty, then q^* is a threshold schedule. We will show that χ is empty if the following condition holds:

CONDITION 1: *$f/\left[1-F\right]$ is nondecreasing, τ_L and τ_M are weakly concave, and $\frac{\tau_M(\theta)-\tau_L(\theta)}{\tau_L'(\theta)}\left(\frac{f(\theta)}{1-F(\theta)}\right)$ is nondecreasing in θ.*

The following is an example that satisfies Condition 1: F is log-concave (e.g., uniform distribution), and τ_M and τ_L are affine functions, with τ_M weakly steeper than τ_L.

We will prove that χ is empty under Condition 1 now. As before, without loss of generality, we assume that q^* is right-continuous. Let θ_1 be the infimum of χ. Henceforth, let $q^*\left(\theta_1\right)$ be denoted by q_1^*. Because of point 1 of Lemma 1 and that q^* is right-continuous, it must be the case that $q_1^* \in \left(\tau_L\left(\theta\right), 1\right)$. We break things down into two cases:

- Case 1: $\tau_L(\theta_1) < q_1^* \leq \tau_M(\theta_1)$.
- Case 2: $\tau_M(\theta_1) < q_1^* < 1$.

Consider Case 1 first. By point 3 of Lemma 1, there exists $\theta_2 < 1$ such $q^*(\theta) = \min\{\tau_M(\theta), \tau_L(\theta_2)\}$ for all $\theta \in [\theta_1, \theta_2]$ and there exists θ_3 such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. For some small $\varepsilon > 0$, define $\eta^1(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\theta_2+\eta^1(\varepsilon)} \tau_L'(\theta)\left[1-F(\theta)\right]d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)\left[1-F(\theta)\right]d\theta. \tag{2}$$

Pick the ε to be small enough such that $\theta_2 + \eta^1(\varepsilon) < \theta_3$ and $\tau_L\left(\theta_2 + \eta^1(\varepsilon)\right) < \tau_M(\theta_1 + \varepsilon)$. Let $\hat{q}_\varepsilon^1$ be the schedule in which

$$\hat{q}_\varepsilon^1(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_1, \theta_1+\varepsilon) \\ \min\left\{\tau_M(\theta), \tau_L\left(\theta_2+\eta^1(\varepsilon)\right)\right\} & \text{if } \theta \in [\theta_1+\varepsilon, \theta_2+\eta^1(\varepsilon)) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_2+\eta^1(\varepsilon), \theta_3) \end{cases}$$

Thus,

$$\begin{aligned}\delta_\varepsilon^1 &= \int_0^1 \left[S\left(\hat{q}_\varepsilon^1(\theta), \theta\right) - S\left(q^*(\theta), \theta\right)\right] f(\theta)\, d\theta \\ &= s_M \int_{\theta_1+\varepsilon}^{\theta_2+\eta^1(\varepsilon)} \left[\min\left\{\tau_M(\theta), \tau_L\left(\theta_2+\eta^1(\varepsilon)\right)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta \\ &- s_M \int_{\theta_1}^{\theta_2} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta\end{aligned}$$

By equation (2) and Lemma 2 below, we have $\delta_\varepsilon^1 > 0$. Equation (2) implies that $\int_0^1 \psi^S\left(\hat{q}_\varepsilon^1(\theta), \theta\right) d\theta = \int_0^1 \psi^S\left(q^*(\theta), \theta\right) d\theta$. This implies that $\int_0^1 \psi^B\left(\hat{q}_\varepsilon^1(\theta), \theta\right) d\theta - \int_0^1 \psi^B\left(q^*(\theta), \theta\right) d\theta = \delta_\varepsilon^1 > 0$. Thus, $\hat{q}_\varepsilon^1$ is feasible, but $\hat{q}_\varepsilon^1$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 1.

Next, consider Case 2. Let θ_2 be the largest θ in which $q^*(\theta) = q_1^*$. Lemma 1 implies that either $\tau_M(\theta_2) = q_1^*$ (Case 2A) or $\tau_L(\theta_2) = q_1^*$ (Case 2B).

Consider Case 2A first. Under Case 2A, there must exist θ_3 such that $q^*(\theta) = \tau_M(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. For some small $\varepsilon > 0$, define $\eta^{2A}(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\theta_2+\eta^{2A}(\varepsilon)} \tau_M'(\theta)\left[1-F(\theta)\right]d\theta = \int_{\theta_1}^{\theta_2} \tau_M'(\theta)\left[1-F(\theta)\right]d\theta. \tag{3}$$

Pick the ε to be small enough such that $\theta_2 + \eta^{2A}(\varepsilon) < \theta_3$. Let $\hat{q}_\varepsilon^{2A}$ be the schedule in which

$$\hat{q}_\varepsilon^{2A}(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_1, \theta_1 + \varepsilon) \\ \tau_M\left(\theta_2 + \eta^{2A}(\varepsilon)\right) & \text{if } \theta \in [\theta_1 + \varepsilon, \theta_2 + \eta^{2A}(\varepsilon)) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_2 + \eta^{2A}(\varepsilon), \theta_3) \end{cases}$$

Thus,

$$\begin{aligned}\delta_\varepsilon^{2A} =& \int_0^1 \left[S\left(\hat{q}_\varepsilon^{2A}(\theta), \theta\right) - S\left(q^*(\theta), \theta\right)\right] f(\theta)\, d\theta \\ =& s_H \left(\int_{\theta_1+\varepsilon}^{\theta_2+\eta^{2A}(\varepsilon)} \left[\tau_M\left(\theta_2 + \eta^{2A}(\varepsilon)\right) - \tau_M(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} \left[\tau_M(\theta_2) - \tau_M(\theta)\right] f(\theta)\, d\theta\right).\end{aligned}$$

By equation (3) and Lemma 3 below, we have $\delta_\varepsilon^{2A} > 0$. Equation (3) implies that $\int_0^1 \psi^S\left(\hat{q}_\varepsilon^{2A}(\theta), \theta\right) d\theta = \int_0^1 \psi^S\left(q^*(\theta), \theta\right) d\theta$. This implies that $\int_0^1 \psi^B\left(\hat{q}_\varepsilon^{2A}(\theta), \theta\right) d\theta - \int_0^1 \psi^B\left(q^*(\theta), \theta\right) d\theta = \delta_\varepsilon^{2A} > 0$. Thus, $\hat{q}_\varepsilon^{2A}$ is feasible, but $\hat{q}_\varepsilon^{2A}$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 2A.

Consider Case 2B next. Under Case 2B, there must exist θ_3 such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. Let $\tilde{\theta}$ be the type such that $\tau_M\left(\tilde{\theta}\right) = q^*$. Note that $\theta_1 < \tilde{\theta} < \theta_2$. For some small $\varepsilon > 0$, define $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\tilde{\theta}+\gamma_1(\varepsilon)} \tau_M'(\theta)\left[1 - F(\theta)\right] d\theta = \int_{\theta_1}^{\tilde{\theta}} \tau_M'(\theta)\left[1 - F(\theta)\right] d\theta \tag{4}$$

$$\int_{\tilde{\theta}+\gamma_1(\varepsilon)}^{\theta_2+\gamma_2(\varepsilon)} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta = \int_{\tilde{\theta}}^{\theta_2} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta \tag{5}$$

Pick the ε to be small enough such that $\theta_2 + \gamma_2(\varepsilon) < \theta_3$. Let $\tilde{\theta}'$ be the type such that $\tau_M\left(\tilde{\theta}'\right) = \tau_L\left(\theta_2 + \gamma_2(\varepsilon)\right)$. Let $\hat{q}_\varepsilon^{2B}$ be the schedule in which

$$\hat{q}_\varepsilon^{2B}(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_1, \theta_1 + \varepsilon) \\ \tau_M\left(\tilde{\theta} + \gamma_1(\varepsilon)\right) & \text{if } \theta \in [\theta_1 + \varepsilon, \tilde{\theta} + \gamma_1(\varepsilon)) \\ \tau_M(\theta) & \text{if } \theta \in [\tilde{\theta} + \gamma_1(\varepsilon), \tilde{\theta}') \\ \tau_L\left(\theta_2 + \gamma_2(\varepsilon)\right) & \text{if } \theta \in [\tilde{\theta}', \theta_2 + \gamma_2(\varepsilon)) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_2 + \gamma_2(\varepsilon), \theta_3) \end{cases}$$

Observe that equations (4) and (5) imply that $\int_0^1 \psi^S\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) d\theta = \int_0^1 \psi^S\left(q^*(\theta), \theta\right) d\theta$.

Moreover,

$$\delta_\varepsilon^{2B} = \int_0^1 \left[S\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) - S\left(q^*(\theta), \theta\right)\right] f(\theta)\, d\theta$$

(6)

$$= s_H \left(\int_{\theta_1+\varepsilon}^{\tilde{\theta}+\gamma_1(\varepsilon)} \left[\tau_M\left(\tilde{\theta}+\gamma_1(\varepsilon)\right) - \tau_M(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\tilde{\theta}} \left[\tau_M\left(\tilde{\theta}\right) - \tau_M(\theta)\right] f(\theta)\, d\theta \right)$$

(7)

$$+ s_M \Big(\int_{\tilde{\theta}+\gamma_1(\varepsilon)}^{\theta_2+\gamma_2(\varepsilon)} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2+\gamma_2(\varepsilon))\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta$$

(8)

$$- \int_{\tilde{\theta}}^{\theta_2} \left[\tau_L(\theta_2) - \tau_L(\theta)\right] f(\theta)\, d\theta \Big)$$

Equation (4) and Lemma 3 below jointly imply that line (6) is positive. Equation (5) and Lemma 2 below jointly imply that line (7) minus line (8) is positive. Therefore, $\delta_\varepsilon^{2B} > 0$. This implies that $\int_0^1 \psi^B\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) d\theta - \int_0^1 \psi^B\left(q^*(\theta), \theta\right) d\theta = \delta_\varepsilon^{2B} > 0$. Thus, $\hat{q}_\varepsilon^{2B}$ is feasible, but $\hat{q}_\varepsilon^{2B}$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 2B as well.

Since both Case 1 and Case 2 are not possible, χ must be an empty set. Therefore, q^* must be a threshold schedule. We summarize the argument above in the following proposition:

PROPOSITION 1: *Suppose that Condition 1 holds. If $q(\cdot)$ is a solution to program (1), $q(\cdot)$ must be a threshold schedule — i.e., there exist $\theta_M \le \theta_H$ such that $q(\theta) = \tau_L(\theta)$ if $\theta < \theta_M$, $q(\theta) = \tau_M(\theta)$ if $\theta \in [\theta_M, \theta_H)$, and $q(\theta) = 1$ if $\theta \ge \theta_H$.*

We conclude this subsection with the proofs of Lemmas 2 and 3.

LEMMA 2: *Let $\theta_1 < \theta_1' < \theta_2 < \theta_2'$. Under Condition 1, $\int_{\theta_1'}^{\theta_2'} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta$ implies that*

$$\int_{\theta_1'}^{\theta_2'} \left[\min\left\{\tau_M(\theta), \tau_L\left(\theta_2'\right)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta > 0.$$

PROOF:

For $x > \theta_1$, define $\phi(x)$ by $\int_x^{\phi(x)} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)\left[1 - F(\theta)\right] d\theta$.

This implies that $\phi'(x) = \frac{\tau_L'(x)[1-F(x)]}{\tau_L'(\phi(x))[1-F(\phi(x))]}$. Let

$$D(x) = \int_x^{\phi(x)} [\min\{\tau_M(\theta), \tau_L(\phi(x))\} - \tau_L(\theta)] f(\theta) d\theta - \int_{\theta_1}^{\theta_2} [\min\{\tau_M(\theta), \tau_L(\theta_2)\} - \tau_L(\theta)] f(\theta) d\theta.$$

Suppose first that $\tau_M(x) > \tau_L(\phi(x))$. This implies that

$$D(x) = \int_x^{\phi(x)} [\tau_L(\phi(x)) - \tau_L(\theta)] f(\theta) d\theta - \int_{\theta_1}^{\theta_2} [\tau_L(\theta_2) - \tau_L(\theta)] f(\theta) d\theta,$$

and

$$\begin{aligned} D'(x) &= [F(\phi(x)) - F(x)] \tau_L'(\phi(x)) \phi'(x) - [\tau_L(\phi(x)) - \tau_L(x)] f(x) \\ &= [F(\phi(x)) - F(x)] \frac{\tau_L'(x)[1-F(x)]}{[1-F(\phi(x))]} - [\tau_L(\phi(x)) - \tau_L(x)] f(x) \\ &\propto \frac{F(\phi(x)) - F(x)}{[1-F(\phi(x))][\phi(x) - x]} - \left[\frac{\tau_L(\phi(x)) - \tau_L(x)}{\phi(x) - x} \times \frac{1}{\tau_L'(x)}\right] \frac{f(x)}{1-F(x)} \\ &\geq \frac{F(\phi(x)) - F(x)}{[1-F(\phi(x))][\phi(x) - x]} - \frac{f(x)}{1-F(x)} \end{aligned}$$

where the inequality follows from $\tau_L'(x) \geq \frac{\tau_L(\phi(x)) - \tau_L(x)}{\phi(x) - x}$ because τ_L is concave. Using the same argument that establishes statement (A7) in the proof of Lemma 2, the last line is positive; thus, $D'(x) > 0$.

Next, suppose that $\tau_M(x) \leq \tau_L(\phi(x))$. Let $\hat{x} = \tau_M^{-1}(\tau_L(\phi(x)))$. Therefore,

$$D(x) = \int_x^{\hat{x}} [\tau_M(\theta) - \tau_L(\theta)] f(\theta) d\theta + \int_{\hat{x}}^{\phi(x)} [\tau_L(\phi(x)) - \tau_L(\theta)] f(\theta) d\theta - \int_{\theta_1}^{\theta_2} [\tau_L(\theta_2) - \tau_L(\theta)] f(\theta) d\theta$$

and

$$
\begin{aligned}
D'(x) &= -\left[\tau_M(x)-\tau_L(x)\right] f(x) + \left[F(\phi(x)) - F(\hat{x})\right] \tau_L'(\phi(x))\,\phi'(x) \\
&= \left[F(\phi(x)) - F(\hat{x})\right] \frac{\tau_L'(x)\left[1-F(x)\right]}{\left[1-F(\phi(x))\right]} - \left[\tau_M(x)-\tau_L(x)\right] f(x) \\
&\propto \frac{F(\phi(x)) - F(\hat{x})}{1-F(\phi(x))} - \frac{\tau_M(x)-\tau_L(x)}{\tau_L'(x)} \left(\frac{f(x)}{1-F(x)}\right) \\
&\geq \frac{F(\phi(x)) - F(\hat{x})}{1-F(\phi(x))} - \frac{\tau_M(\hat{x})-\tau_L(\hat{x})}{\tau_L'(\hat{x})} \left(\frac{f(\hat{x})}{1-F(\hat{x})}\right) \qquad (9) \\
&= \frac{F(\phi(x)) - F(\hat{x})}{1-F(\phi(x))} - \frac{\tau_L(\phi(x))-\tau_L(\hat{x})}{\tau_L'(\hat{x})} \left(\frac{f(\hat{x})}{1-F(\hat{x})}\right) \\
&\propto \frac{F(\phi(x)) - F(\hat{x})}{\left[1-F(\phi(x))\right]\left[\phi(x)-\hat{x}\right]} - \left[\frac{\tau_L(\phi(x))-\tau_L(\hat{x})}{\phi(x)-\hat{x}} \times \frac{1}{\tau_L'(\hat{x})}\right] \frac{f(\hat{x})}{1-F(\hat{x})} \\
&\geq \frac{F(\phi(x)) - F(\hat{x})}{\left[1-F(\phi(x))\right]\left[\phi(x)-\hat{x}\right]} - \frac{f(\hat{x})}{1-F(\hat{x})}
\end{aligned}
$$

The inequality in (9) is due to the last part of Condition 1. Therefore, as before, $D'(x) > 0$.

For both cases, since $\lim_{x\downarrow\theta_1} D(x) = 0$, this implies that $D(x) > 0$ for $x > \theta_1$, which establishes the lemma.

LEMMA 3: *Let $\theta_1 < \theta_1' < \theta_2 < \theta_2'$. Under Condition 1, $\int_{\theta_1'}^{\theta_2'} \tau_M'(\theta)\left[1-F(\theta)\right] d\theta = \int_{\theta_1}^{\theta_2} \tau_M'(\theta)\left[1-F(\theta)\right] d\theta$ implies that*

$$
\int_{\theta_1'}^{\theta_2'} \left[\tau_M(\theta_2') - \tau_M(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} \left[\tau_M(\theta_2) - \tau_M(\theta)\right] f(\theta)\, d\theta > 0.
$$

The proof of Lemma 3 follows the same argument as that for Lemma 2 for the case of $\tau_M(x) > \tau_L(\phi(x))$; thus, we omit it.

B. Asymmetric Information on Endowment Size

We consider an extension wherein the size of the seller's endowment is also her private information, as described in Section IV.B of the main text.

Let $F(\cdot|n)$ denote the distribution of λ conditional on n and $J(\cdot)$ denote the distribution of n. Let their respective densities be $f(\cdot|n)$ and $j(\cdot)$. With a slight abuse of notations, let $(q(n,\lambda), t(n,\lambda))_{n\in[0,1],\lambda\in[0,n]}$ denote a direct mechanism, and let $U^B(n,\lambda) = V(q(n,\lambda),\lambda) - t(n,\lambda)$ and $U^S(n,\lambda) = t(n,\lambda) - C(q(n,\lambda),\lambda)$. Program $(\mathcal{P})$ in the current setup is

(10)

$$
\max_{q(\cdot),t(\cdot)} \int_0^1 \int_0^n U^S(n,\lambda) f(\lambda|n)\, j(n)\, d\lambda dn, \qquad \text{s.t.} \qquad (IC_S^e), (IR_S^e) \text{ and } (IR_B^e)
$$

where

(IC_S^e) $\quad U^S(n,\lambda) \geq t(n',\lambda') - C\left(q(n',\lambda'),\lambda\right) \quad \forall (n,\lambda),(n',\lambda'),$

(IR_S^e) $\quad U^S(n,\lambda) \geq 0 \quad \forall (n,\lambda)$

(IR_B^e)

$$\int_0^1 \int_0^n U^B(n,\lambda) f(\lambda|n) j(n) d\lambda dn \geq b$$

Because the type is two-dimensional, the type space does not have a complete order, which means that defining a monotonicity notion for the quantity schedule is not straightforward. The following is the appropriate monotonicity notion:

DEFINITION 1: $q(\cdot)$ *is* "monotonic" *if*

- *for any two types* (n',λ') *and* (n,λ) *in which* $\lambda' > \lambda$, *either* $q(n',\lambda') \geq q(n,\lambda)$ *or* $q(n,\lambda) > n' = q(n',\lambda')$.
- *for any two types* (n',λ) *and* (n,λ) *in which* $n' > n$, *either* $q(n',\lambda) = q(n,\lambda)$ *or* $q(n',\lambda) > n = q(n,\lambda)$.

In words, when $\lambda' > \lambda$, the type with λ' (or more Ls) must trade weakly more than the type with λ whenever the endowment of λ' permits. Therefore, if the lower λ trades more than the higher λ', it must imply that λ' trades her entire endowment (i.e., her endowment constraint binds). Next, if two types have the same λ, then they must trade the same quantity whenever their endowments permit. Therefore, if $q(n',\lambda) > q(n,\lambda)$, it must imply that type (n,λ) trades her entire endowment.

Similar to Lemma 1, the seller's truth-telling constraint (IC_S^e) can be replaced by the following two conditions:

$$U^S(n,\lambda) = U^S(n,0) + (c_H - c_L)\int_0^\lambda \mathcal{I}\left(q(n,l) > l\right) dl \quad \forall (n,\lambda), \tag{11}$$

(12) $\quad q(\cdot)$ is monotonic according to Definition 1.

From equation (11), for each n, constraint (IR_S^e) holds for all (n,λ) if it holds for $(n,0)$. Additionally, substituting in equation (11), the objective function of program (10) becomes

$$\int_0^1 U^S(0,n) j(n) dn + \int_0^1 (c_H - c_L)\int_0^n \mathcal{I}\left(q(\lambda) > \lambda\right)\left[1 - F(\lambda|n)\right] j(n) d\lambda dn,$$

and constraint (IR_B^e) becomes

$$\int_0^1 \int_0^n \psi^B\left(q(n,\lambda), n, \lambda\right) j(n) d\lambda dn - \int_0^1 U^S(0,n) j(n) dn \geq b,$$

where (with an abuse of notation)

$$\psi^B(q,n,\lambda) = S(q,\lambda) f(\lambda|n) - (c_H - c_L)\mathcal{I}(q>\lambda)[1-F(\lambda|n)].$$

This implies that constraint (IR_B^e) must bind. Therefore, program (10) becomes

(13)

$$\begin{array}{c} \max\limits_{q(\cdot),u_0} \quad \int_0^1 \int_0^n S(q(n,\lambda),\lambda) f(\lambda|n) j(n) d\lambda dn - b \qquad \text{s.t.} \\ q(\cdot) \text{ is monotonic }, \; u_0 \geq 0 \;, \quad \text{and} \quad \int_0^1 \int_0^n \psi^B(q(n,\lambda),n,\lambda) j(n) d\lambda dn - u_0 = b \end{array}.$$

LEMMA 4: *If $q(\cdot)$ is a solution to program (13), $q(\cdot)$ must satisfy the following two conditions:*

1) $q^*(n,\lambda) \geq \lambda$ for all (n,λ).

2) Let $\mathcal{X}$ be the set of λ such that $q(1,\lambda) > \lambda$.

a) If $\mathcal{X}$ is empty, then $q(n,\lambda) = \lambda$ for all (n,λ).

b) Suppose that $\mathcal{X}$ is nonempty. Let λ_1 denote the smallest λ in $\mathcal{X}$, and let λ_2 denote the smallest $\lambda \in (\lambda_1, 1]$ such that $q^*(1,\lambda) = \lambda$. It holds that for all $\lambda \in [\lambda_1, \lambda_2]$, $q(n,\lambda) = \min\{n, \lambda_2\}$.

PROOF:

$S(q,\lambda)$ is strictly increasing in q. When $q < \lambda$, $\psi^B(q,n,\lambda)$ is also strictly increasing in q. This explains point 1. Next, monotonicity of $q(\cdot)$ implies that for all λ, $q(1,\lambda) \geq q(n,\lambda)$. Point 2a hence follows. Finally, for Point 2b, since $\psi^B(q,n,\lambda)$ is also strictly increasing in q when $q > \lambda$, it must be the case the $q(1,\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1,\lambda_2]$. Point 2b then follows from the monotonicity of $q(\cdot)$.

Lemma 4 is the analog Lemma 5. Point 2b states that unless every type sells only their Ls, the optimal quantity schedule must feature some bunching, similar to the middle panel of Figure 1 for the baseline model. The difference is that because of the endowment constraint for some types, such bunching might not always be possible. When this happens, the endowment constraint for these types must bind.

The following is a sufficient condition for the solution quantity schedule to always be a threshold schedule:

CONDITION 2: *For all $\lambda' > \lambda$, $\frac{f(\lambda'|n\geq\lambda')}{1-F(\lambda'|n\geq\lambda')} - \frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} \geq \xi(\lambda',\lambda)$, where $\xi(x,\lambda) := -\frac{d}{dx}\log\int_x^1 [1-F(\lambda|n)] j(n) dn$.*

Note that $\xi(\lambda',\lambda)$ is always positive. Thus, Condition 2 requires the conditional hazard rate to be increasing sufficiently quickly (as opposed to only increasing).

The following is an example that satisfies Condition 2: $j(n) = 2n$ and $F(\lambda|n)$ is the uniform distribution over $[0, n]$.[1]

PROPOSITION 2: *Under Condition 2, if $q(\cdot)$ is a solution to program (13), then $q(\cdot)$ must be a threshold schedule — i.e., there exists x such that $q(n,\lambda) = \lambda$ if $\lambda \leq x$ and $q(n,\lambda) = n$ if $\lambda > x$.*

PROOF:

Let $q^*(\cdot)$ be an optimal schedule. Let λ_1 and λ_2 be as defined in Lemma 4. The lemma is proved by showing that $\lambda_2 = 1$. Suppose, for a contradiction, that $\lambda_2 < 1$. There must then exist $\lambda_3 > \lambda_2$ such that $q^*(1,\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Since $q^*(\cdot)$ is monotonic, this also implies that for any $n < 1$ and $\lambda \in [\lambda_2, \lambda_3]$, $q^*(n,\lambda) = \lambda$. Observe that

$$\begin{aligned}
&\int_0^1 \int_{\lambda_1}^{\lambda_3} S(q^*(n,\lambda),\lambda) f(\lambda|n) j(n)\, d\lambda dn \\
&= \int_{\lambda_3}^{n} \int_{\lambda_1}^{\lambda_3} S(q^*(n,\lambda),\lambda) f(\lambda|n) j(n)\, d\lambda dn \; + \; \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} S(q^*(n,\lambda),\lambda) f(\lambda|n) j(n)\, d\lambda dn \\
&= \int_0^1 \int_{\lambda_1}^{\lambda_3} \lambda s_L f(\lambda|n) j(n)\, d\lambda dn \\
&\quad + \int_{\lambda_2}^{1} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) s_H f(\lambda|n) j(n)\, d\lambda dn \; + \; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n - \lambda) s_H f(\lambda|n) j(n)\, d\lambda dn
\end{aligned}$$

For some small $\varepsilon > 0$ and $x \in [\lambda_1, \lambda_1 + \varepsilon]$, let $\phi(x)$ be such that

$$\begin{aligned}
&\int_{\phi(x)}^{1} \int_{x}^{\phi(x)} (\phi(x) - \lambda) s_H f(\lambda|n) j(n)\, d\lambda dn \; + \; \int_{x}^{\phi(x)} \int_{x}^{n} (n-\lambda) s_H f(\lambda|n) j(n)\, d\lambda dn \\
&= \int_{\lambda_2}^{1} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) s_H f(\lambda|n) j(n)\, d\lambda dn \; + \; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n-\lambda) s_H f(\lambda|n) j(n)\, d\lambda dn
\end{aligned} \tag{14}$$

We restrict ε to be small enough such that $\phi(\lambda_1 + \varepsilon) < \lambda_3$.

Define schedule $\hat{q}_x$ as follows:

$$\hat{q}_x(n,\lambda) = \begin{cases} q^*(n,\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \\ \lambda & \text{, if } \lambda \in [\lambda_1, x) \\ \min\{n, \phi(x)\} & \text{, if } \lambda \in [x, \phi(x)) \\ \lambda & \text{, if } \lambda \in [\phi(x), \lambda_3) \end{cases}$$

[1] To be precise, this means that $f(\lambda|n) = \frac{1}{n}$ for $\lambda \in [0,n]$ and $f(\lambda|n) = 0$ for $\lambda > n$. It is readily verified that $\frac{f(\lambda|n \geq \lambda)}{1 - F(\lambda|n \geq \lambda)} = \frac{2}{1-\lambda}$ and $\xi(\lambda', \lambda) = \frac{2(\lambda' - \lambda)}{(1-\lambda')(1-\lambda) + (\lambda' - \lambda)(1-\lambda')}$; thus Condition 2 holds.

By construction,
(15)
$$\int_0^1 \int_0^n S\left(\hat{q}_x\left(n,\lambda\right),\lambda\right) f\left(\lambda|n\right) j\left(n\right) d\lambda dn = \int_0^1 \int_0^n S\left(q^*\left(n,\lambda\right),\lambda\right) f\left(\lambda|n\right) j\left(n\right) d\lambda dn.$$

Let
$$\psi^S\left(q,n,\lambda\right) = \mathcal{I}\left(q>\lambda\right)\left(c_H - c_L\right)\left[1 - F\left(\lambda|n\right)\right].$$
Therefore, $\psi^B\left(q,n,\lambda\right) = S\left(q,\lambda\right) f\left(\lambda|n\right) - \psi^S\left(q,n,\lambda\right)$. The difference in the buyer's expected utility between $\hat{q}_x\left(\cdot\right)$ and $q^*\left(\cdot\right)$ is

$$\begin{aligned}
D\left(x\right) =& \int_0^1 \int_0^n \left[\psi^B\left(\hat{q}_x\left(n,\lambda\right),n,\lambda\right) - \psi^B\left(q^*\left(n,\lambda\right),n,\lambda\right)\right] j\left(n\right) d\lambda dn \\
=& \int_0^1 \int_0^n \left[\psi^S\left(q^*\left(n,\lambda\right),n,\lambda\right) - \psi^S\left(\hat{q}_x\left(n,\lambda\right),n,\lambda\right)\right] j\left(n\right) d\lambda dn \\
=& \int_{\lambda_3}^1 \int_{\lambda_1}^{\lambda_3} \left[\psi^S\left(q^*\left(n,\lambda\right),n,\lambda\right) - \psi^S\left(\hat{q}_x\left(n,\lambda\right),n,\lambda\right)\right] j\left(n\right) d\lambda dn \\
&+ \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} \left[\psi^S\left(q^*\left(n,\lambda\right),n,\lambda\right) - \psi^S\left(\hat{q}_x\left(n,\lambda\right),n,\lambda\right)\right] j\left(n\right) d\lambda dn \\
=& \left(c_H - c_L\right)\left[\int_{\lambda_2}^1 \left(\int_{\lambda_1}^{\lambda_2} 1 - F\left(\lambda|n\right) d\lambda\right) h\left(n\right) dn + \int_{\lambda_1}^{\lambda_2} \left(\int_{\lambda_1}^{n} 1 - F\left(\lambda|n\right) d\lambda\right) j\left(n\right) dn\right] \\
&- \left(c_H - c_L\right)\left[\int_{\phi(x)}^1 \left(\int_x^{\phi(x)} 1 - F\left(\lambda|n\right) d\lambda\right) j\left(n\right) dn + \int_x^{\phi(x)} \left(\int_x^n 1 - F\left(\lambda|n\right) d\lambda\right) j\left(n\right) dn\right].
\end{aligned}$$

Differentiating $D\left(x\right)$ with respect to x, we have

$$D'\left(x\right) = \left[\int_x^1 \left[1 - F\left(x|n\right)\right] j\left(n\right) dn - \left(\int_{\phi(x)}^1 \left[1 - F\left(\phi\left(x\right)|n\right)\right] j\left(n\right) dn\right) \phi'\left(x\right)\right]\left(c_H - c_L\right)$$

From equation (14), we have

$$\begin{aligned}
\phi'\left(x\right) =& \frac{\int_{\phi(x)}^1 \left[\phi\left(x\right) - x\right] f\left(x|n\right) j\left(n\right) dn + \int_x^{\phi(x)} \left(n - x\right) f\left(x|n\right) j\left(n\right) dn}{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] j\left(n\right) dn} \\
>& \frac{\left[\phi\left(x\right) - x\right] \int_x^1 f\left(x|n\right) j\left(n\right) dn}{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] j\left(n\right) dn}
\end{aligned}$$

Therefore, we have $D'(x) > 0$ if

$$\frac{[\phi(x) - x] \int_x^1 f(x|n) j(n)\, dn}{\int_{\phi(x)}^1 [F(\phi(x)|n) - F(x|n)] j(n)\, dn} \le \frac{\int_x^1 [1 - F(x|n)] j(n)\, dn}{\left(\int_{\phi(x)}^1 [1 - F(\phi(x)|n)] j(n)\, dn\right)}$$

$$\Longleftrightarrow \frac{\int_x^1 f(x|n) j(n)\, dn}{\int_x^1 [1 - F(x|n)] j(n)\, dn} \le \frac{\int_{\phi(x)}^1 [F(\phi(x)|n) - F(x|n)] j(n)\, dn}{[\phi(x) - x] \left(\int_{\phi(x)}^1 [1 - F(\phi(x)|n)] j(n)\, dn\right)}$$

Fixing some λ, let $LHS = \frac{\int_\lambda^1 f(\lambda|n) j(n) dn}{\int_\lambda^1 [1-F(\lambda|n)] j(n) dn}$, and let $RHS(\lambda') = \frac{\int_{\lambda'}^1 [F(\lambda'|n) - F(\lambda|n)] j(n) dn}{(\lambda' - \lambda) \int_{\lambda'}^1 [1-F(\lambda'|n)] j(n) dn}$. By L'Hôpital's rule, $\lim_{\lambda' \downarrow \lambda} RHS(\lambda') = LHS$ and $\lim_{\lambda' \uparrow 1} RHS(\lambda') = \infty$. Suppose, for a contradiction, that there exists $\lambda' \in (\lambda, 1)$ such that $LHS > RHS(\lambda')$. This must imply that there exists $\hat{\lambda} \in (\lambda, 1)$ such that $LHS > RHS\left(\hat{\lambda}\right)$ and $\frac{d}{d\lambda'} RHS(\lambda') \Big|_{\lambda' = \hat{\lambda}} = 0$. By some algebra, $\frac{d}{d\lambda'} RHS(\lambda') \Big|_{\lambda' = \hat{\lambda}} = 0$ implies that

$$\begin{aligned}
\frac{\int_{\hat{\lambda}}^1 f\left(\hat{\lambda}|n\right) j(n)\, dn}{\int_{\hat{\lambda}}^1 \left[1 - F\left(\hat{\lambda}|n\right)\right] j(n)\, dn} &= RHS\left(\hat{\lambda}\right) \frac{\int_{\hat{\lambda}}^1 \left[1 - F\left(\hat{\lambda}|n\right)\right] j(n)\, dn}{\int_{\hat{\lambda}}^1 [1 - F(\lambda|n)] j(n)\, dn} + \frac{\left[1 - F\left(\lambda|\hat{\lambda}\right)\right] j\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^1 [1 - F(\lambda|n)] j(n)\, dn} \\
&< LHS + \frac{\left[1 - F\left(\lambda|\hat{\lambda}\right)\right] j\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^1 [1 - F(\lambda|n)] j(n)\, dn} \\
&= \frac{\int_\lambda^1 f(\lambda|n) j(n)\, dn}{\int_\lambda^1 [1 - F(\lambda|n)] j(n)\, dn} - \frac{d}{d\hat{\lambda}} \log\left(\int_{\hat{\lambda}}^1 [1 - F(\lambda|n)] j(n)\, dn\right),
\end{aligned}$$

where the inequality in the second line follows from $\lambda < \hat{\lambda}$ and $LHS > RHS\left(\hat{\lambda}\right)$. However, this contradicts Condition 2.[2] Therefore, it holds that $LHS \le RHS(\lambda')$, which implies that $D'(x) > 0$.

Since $D(\lambda_1) = 0$, there exists $x > \lambda_1$ such that $D(x) > 0$, thus implying that (16)

$$\int_0^1 \int_0^n \psi^B(\hat{q}_x(n, \lambda); n, \lambda) j(n)\, d\lambda dn > \int_0^1 \int_0^n \psi^B(q^*(n, \lambda); n, \lambda) j(n)\, d\lambda dn$$

This implies that $\hat{q}_x(\cdot)$ is also feasible, and from equation (15), $\hat{q}_x(\cdot)$ is also optimal. However, equation (16) implies that constraint (IR_B^e) does not bind, which is a contradiction.

[2]Note that $f(\lambda|n \ge \lambda) = \frac{\int_\lambda^1 f(\lambda|n) j(n) dn}{1 - J(\lambda)}$ and $1 - F(\lambda|n \ge \lambda) = \frac{\int_\lambda^1 j(n) dn - \int_\lambda^1 F(\lambda|n) j(n) dn}{1 - J(\lambda)} = \frac{\int_\lambda^1 [1 - F(\lambda|n)] j(n) dn}{1 - J(\lambda)}$. Therefore, $\frac{f(\lambda|n \ge \lambda)}{1 - F(\lambda|n \ge \lambda)} = \frac{\int_\lambda^1 f(\lambda|n) j(n) dn}{\int_\lambda^1 [1 - F(\lambda|n)] j(n) dn}$.

C. *Diminishing Marginal Utility*

This subsection provides the details for the extension described in Subsection IV.C. Our main program is still program $(\mathcal{P})$, but with V now defined in Subsection IV.C.

The argument to transform program $(\mathcal{P})$ to program $(\tilde{\mathcal{P}})$ considers only the seller's incentives, which is unchanged here; thus, the argument still applies here. However, note that in the current setup,

$$S(q,\lambda) = \begin{cases} \nu_L(q) - qc_L & \text{if } q \leq \lambda \\ \nu_L(\lambda) - \lambda c_L + \nu_H(q-\lambda) - (q-\lambda)c_H & \text{if } q > \lambda \end{cases}. \tag{17}$$

Let

$$\bar{s}_H(x) := \nu_H(x) - xc_H.$$

The condition required for the solution to program $(\tilde{\mathcal{P}})$ to be a threshold schedule is as follows:

CONDITION 3: *For any $z, x \in [0,1]$ such that $z > x$,*

$$\int_x^z \bar{s}'_H(z-\lambda) f(\lambda)\, d\lambda \geq \frac{\bar{s}_H(z-x)}{z-x}\left[F(z) - F(x)\right]. \tag{18}$$

Note that equation (18) can be written as

$$\int_x^z \left[\bar{s}'_H(z-\lambda) - \frac{\bar{s}_H(z-x)}{z-x}\right] f(\lambda)\, d\lambda \geq 0.$$

Since $\bar{s}_H$ is concave, by the mean value theorem, there exists $\bar{\lambda}$ such that $\bar{s}'_H(z-\lambda) > (=)\,[<]\, \frac{\bar{s}_H(z-x)}{z-x}$ if $\lambda > (=)\,[<]\,\bar{\lambda}$ — i.e., there are both positive and negative terms in the integrand. Thus, Condition 3 is a restriction on the curvature of $\bar{s}_H$ together with the distribution. The following is an example:

LEMMA 5: *Condition 3 always holds if F is the uniform distribution.*

PROOF:
When F is the uniform distribution, $F(z) - F(x) = z - x$; thus, the right-hand side of equation (18) is $\bar{s}_H(z-x)$. Since $f(\lambda) = 1$, the left-hand side of equation (18) is $\int_x^z \bar{s}'_H(z-\lambda)\, d\lambda$. By the fundamental theorem of calculus, this is equal to $\bar{s}_H(z-x)$.

PROPOSITION 3: *Under Condition 3, if $q(\cdot)$ is a solution to program $(\tilde{\mathcal{P}})$ in the current setup, then $q(\cdot)$ must be a threshold schedule, defined in Definition 1.*

PROOF:

Since $\nu_L'(x) > c_L$ and $\nu_H'(x) > c_H$ for all x, $S(q,\lambda)$ defined in equation (17) is still always strictly increasing in q. In turn, $\psi^B(q,\lambda)$ is also increasing in q when $q < \lambda$ and when $q > \lambda$. Thus, Lemma 5 in the proof of Proposition 1 still holds. Let λ_1, λ_2, λ_3, $\eta(\varepsilon)$ and $\hat{q}_\varepsilon$ be as defined in that proof. Following the exact arguments, we only have to show that δ_ε in equation (A6) is positive, where, over here,

$$\begin{aligned}\delta_\varepsilon &= \int_{\lambda_1}^{\lambda_3} S\left[(\hat{q}_\varepsilon(\lambda),\lambda) - S(q^*(\lambda),\lambda)\right] f(\lambda)\,d\lambda \\ &= \int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} \bar{s}_H(\lambda_2+\eta(\varepsilon)-\lambda) f(\lambda)\,d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{s}_H(\lambda_2-\lambda) f(\lambda)\,d\lambda\ .\end{aligned}$$

This is indeed the case from Lemma 6 below.

LEMMA 6: *Under Condition 3, when $f/(1-F)$ is nondecreasing, the following property holds: for any $\lambda_1 < \lambda_1' < \lambda_2 < \lambda_2'$, if $\int_{\lambda_1'}^{\lambda_2'} 1 - F(\lambda)\,d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\,d\lambda$, then*

$$\int_{\lambda_1'}^{\lambda_2'} \bar{s}_H\left(\lambda_2'-\lambda\right) f(\lambda)\,d\lambda > \int_{\lambda_1}^{\lambda_2} \bar{s}_H(\lambda_2-\lambda) f(\lambda)\,d\lambda. \tag{19}$$

PROOF:

Fix any $\lambda_1, \lambda_2 > 0$. For $x > \lambda_1$, define $\phi(x)$ to be such that $\int_x^{\phi(x)} 1 - F(\lambda)\,d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\,d\lambda$. ϕ is strictly increasing and (by the implicit function theorem) differentiable, with $\phi'(x) = \frac{1-F(x)}{1-F(\phi(x))}$. Let $\bar{x} = \phi^{-1}(1)$ and

$$\bar{D}(x) := \int_x^{\phi(x)} \bar{s}_H(\phi(x)-\lambda) f(\lambda)\,d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{s}_H(\lambda_2-\lambda) f(\lambda)\,d\lambda$$

Our goal is to show that $\bar{D}(x) > 0$ for all $x \in (\lambda_1, \bar{x}]$. Note that $\bar{D}(x)$ is also differentiable for $x \in (\lambda_1, \bar{x})$, with

$$\begin{aligned}\bar{D}'(x) &= \left(\int_x^{\phi(x)} \bar{s}_H'(\phi(x)-\lambda) f(\lambda)\,d\lambda\right) \frac{1-F(x)}{1-F(\phi(x))} - \bar{s}_H(\phi(x)-x) f(x) \\ &\propto \left[\frac{\int_x^{\phi(x)} \bar{s}_H'(\phi(x)-\lambda) f(\lambda)\,d\lambda}{\bar{s}_H(\phi(x)-x)}\right] \frac{1}{1-F(\phi(x))} - \frac{f(x)}{1-F(x)} \\ &\geq \left[\frac{F(\phi(x)) - F(x)}{\phi(x)-x}\right] \frac{1}{1-F(\phi(x))} - \frac{f(x)}{1-F(x)},\end{aligned}$$

where the inequality holds because of Condition 3. In turn, from the property in equation (A7) in Appendix A, $\bar{D}'(x) > 0$. Since $\bar{D}(x)$ is continuous for $x \in [\lambda_1, \bar{x}]$ and $\bar{D}(\lambda_1) = 0$, $\bar{D}'(x) > 0$ for all $x \in (\lambda_1, \bar{x})$ implies that $\bar{D}(x) > 0$ for all $x \in (\lambda_1, \bar{x}]$.

D. Stochastic Mechanism

We consider the use of stochastic mechanism in this subsection. Because the utility functions of both the seller and the buyer are linear in the transfers, it suffice to allow for stochasticity only in the quantity. A stochastic contract is a double (α, t), where t is still the transfer from the buyer to the seller, and α is the CDF of the quantity that the seller must supply to the buyer. The following are two important notations:

$$\bar{\alpha}(q) := 1 - \alpha(q)$$
$$\alpha^{\Delta}(q) := \alpha(q) - \sup_{x<q} \alpha(q)$$

$\bar{\alpha}(q)$ is the probability of having to supply more than q units under α. $\alpha^{\Delta}(q)$ denote the mass at q; thus, a deterministic contract consists of α where there is a q in which $\alpha^{\Delta}(q) = 1$.

Let

$$\bar{C}(\alpha, \lambda) = \int_0^1 C(q, \lambda)\, d\alpha(q)$$
$$\bar{V}(\alpha, \lambda) = \int_0^1 V(q, \lambda)\, d\alpha(q)$$
$$\bar{S}(\alpha, \lambda) = \bar{V}(\alpha, \lambda) - \bar{C}(\alpha, \lambda) = \int_0^1 S(q, \lambda)\, d\alpha(q)$$

where C and V are defined in equations (1) and (2). Thus, under a stochastic contract (α, t) between the buyer and the type-λ seller, the buyer's and the seller's expected utility are $\bar{V}(\alpha, \lambda) - t$ and $t - \bar{C}(\alpha, \lambda)$, respectively.

Let $\{\alpha(\cdot|\lambda), t(\lambda)\}_{\lambda \in [0,1]}$ denote a direct stochastic mechanism. Let $\bar{U}^B(\lambda) = \bar{V}(\alpha(\cdot|\lambda), \lambda) - t(\lambda)$ and $\bar{U}^S(\lambda) = t(\lambda) - \bar{C}(\alpha(\cdot|\lambda), \lambda)$. Our main program is $(\mathcal{P}_{stoch})$

$$\max_{\{\alpha(\cdot|\lambda), t(\lambda)\}_{\lambda \in [0,1]}} \int_0^1 \bar{U}^S(\lambda) f(\lambda)\, d\lambda, \qquad \text{s.t.} \qquad (\bar{IC}^S), (\bar{IR}^S) \text{ and } (\bar{IR}^B) \qquad ,$$

where

$$(\bar{IC}^S) \qquad \bar{U}^S(\lambda) \geq t(\lambda') - \bar{C}(\alpha(\cdot|\lambda'), \lambda) \qquad \forall \lambda, \lambda',$$

$$(\bar{IR}^S) \qquad \bar{U}^S(\lambda) \geq 0 \qquad \forall \lambda,$$

$$(\bar{IR}^B) \qquad \int_0^1 \bar{U}^B(\lambda) f(\lambda) \, d\lambda \geq b.$$

By the envelope theorem, constraint $(\bar{IC}^S)$ implies that

$$\frac{d\bar{U}^S(\lambda)}{d\lambda} = -\frac{\partial \bar{C}(\alpha(\cdot|\lambda), \lambda)}{\partial \lambda} = (c_H - c_L)\, \bar{\alpha}(\lambda|\lambda)$$

almost everywhere. Therefore,

$$(\bar{IC}^{S'}) \qquad \bar{U}^S(\lambda) = \bar{U}^S(0) + (c_H - c_L) \int_0^\lambda \bar{\alpha}(l|l)\, dl.$$

Consider the program

(20)

$$\max_{\{\alpha(\cdot|\lambda), t(\lambda)\}_{\lambda \in [0,1]}} \int_0^1 \bar{U}^S(\lambda) f(\lambda)\, d\lambda, \qquad \text{s.t.} \qquad (\bar{IC}^{S'}), (\bar{IR}^S) \text{ and } (\bar{IR}^B) \qquad ,$$

Program (20) is a relaxed version of program $(\mathcal{P}_{stoch})$ because it satisfies only a set of necessary conditions for constraint $(\bar{IC}^S)$. Thus, the value of program (20) is weakly higher than the value of program $(\mathcal{P}_{stoch})$. Say that a mechanism $\{\alpha(\cdot|\lambda)\}_{\lambda \in [0,1]}$ is deterministic if $\{\alpha(\cdot|\lambda), t(\lambda)\}$ is a deterministic contract for all λ. We will provide a condition under which program (20) has a solution mechanism that is deterministic and satisfies all the constraints of program $(\mathcal{P}_{stoch})$; thus, program $(\mathcal{P}_{stoch})$ also has a solution mechanism that is deterministic.

CONDITION 4: $\frac{(1-\lambda)f(\lambda)}{1-F(\lambda)}$ *is nondecreasing.*

PROPOSITION 4: *Under Condition (4), there is a solution mechanism to program ($\mathcal{P}_{stoch}$) that consists of an $\{\alpha(\cdot|\lambda)\}_{\lambda \in [0,1]}$ that takes the form of a deterministic threshold schedule — i.e., there exists a x such that $\alpha^\Delta(\lambda|\lambda) = 1$ for all $\lambda < x$ and $\alpha^\Delta(1|\lambda) = 1$ for all $\lambda \geq x$.*

We first provide some preliminary results to prove Proposition 4. First, constraint $(\bar{IC}^{S'})$ implies that $t(\lambda)$ must satisfy

$$(21) \qquad t(\lambda) = \bar{U}^S(0) + (c_H - c_L) \int_0^\lambda \bar{\alpha}(l|l)\, dl + \bar{C}(\alpha(\cdot|\lambda), \lambda)$$

Doing integration by parts, we have
(22)

$$\int_0^1 t(\lambda) f(\lambda) d\lambda = \bar{U}^S(0) + (c_H - c_L) \int_0^1 \bar{\alpha}(\lambda]|\lambda) [1 - F(\lambda)] d\lambda + \int_0^1 \bar{C}(\alpha(\cdot|\lambda), \lambda) f(\lambda) d\lambda.$$

Therefore,

$$\begin{aligned} \int_0^1 \bar{U}^B(\lambda) f(\lambda) d\lambda &= \int_0^1 \bar{S}(\alpha(\cdot|\lambda), \lambda) f(\lambda) d\lambda - (c_H - c_L) \int_0^1 \bar{\alpha}(\lambda|\lambda) [1 - F(\lambda)] d\lambda - \bar{U}^S(0). \\ &= \int_0^1 \bar{\psi}^B(\alpha(\cdot|\lambda), \lambda) d\lambda - \bar{U}^S(0), \end{aligned}$$

where

$$\bar{\psi}^B(\alpha, \lambda) = \bar{S}(\alpha, \lambda) f(\lambda) - (c_H - c_L) \bar{\alpha}(\lambda) [1 - F(\lambda)]. \tag{23}$$

Following the same argument as the one for Lemma 4, constraint ($\bar{IR}^B$) must bind, and we can transform program (20) to the following program:

$$(\tilde{\mathcal{P}}_{stoch}) \qquad \begin{array}{l} \max\limits_{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0)} \quad \int_0^1 \bar{S}(\alpha(\cdot|\lambda), \lambda) f(\lambda) d\lambda - b \qquad \text{s.t.} \\ \bar{U}^S(0) \geq 0 \ , \quad \text{and} \quad \underbrace{\int_0^1 \bar{\psi}^B(\alpha(\cdot|\lambda), \lambda) d\lambda - \bar{U}^S(0) = b}_{(\bar{IR}^{B'})} \end{array}$$

Thus, our objective is to show that there exists a solution $\left\{ \{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0) \right\}$ for program ($\tilde{\mathcal{P}}_{stoch}$) in which $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ takes the form of a deterministic threshold schedule. We note the following property, which should be obvious:

LEMMA 7: *For any $\left\{ \{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0) \right\}$, if there exists $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ that has the property that $\bar{\psi}^B(\hat{\alpha}(\cdot|\lambda), \lambda) \geq \bar{\psi}^B(\alpha(\cdot|\lambda), \lambda)$ and $\bar{S}(\hat{\alpha}(\cdot|\lambda), \lambda) > \bar{S}(\alpha(\cdot|\lambda), \lambda)$ for a set of λ that has a strictly positive measure, then $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$).*

Next, observe that

$$\begin{aligned} \bar{S}(\alpha, \lambda) &= \int_{q\in[0,\lambda]} s_L q d\alpha(q) + \int_{q\in(\lambda,1]} [\lambda s_L + (q - \lambda) s_H] d\alpha(q) \\ &= \left[\int_{q\in[0,1]} \min\{q, \lambda\} d\alpha(q) \right] s_L \ + \ \left[\int_{q\in(\lambda,1]} (q - \lambda) d\alpha(q) \right] s_H \end{aligned} \tag{24}$$

LEMMA 8: *If* $\left\{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ *is a solution to program* ($\tilde{\mathcal{P}}_{stoch}$)*, then for all* λ*,* $\alpha^\Delta(\lambda|\lambda)+\alpha^\Delta(1|\lambda)=1$.

PROOF:

Let $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be a solution to program ($\tilde{\mathcal{P}}_{stoch}$). Let $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be another mechanism where $\hat{\alpha}^\Delta(\lambda|\lambda)=\alpha^*(\lambda|\lambda)$ and $\hat{\alpha}^\Delta(1|\lambda)=\bar{\alpha}^*(\lambda|\lambda)$. Given the expression in equation (24), it is immediate that $\bar{S}(\hat{\alpha}(\cdot|\lambda),\lambda)\geq\bar{S}(\alpha^*(\cdot|\lambda),\lambda)$, with the inequality holding strictly if $\alpha^*(\cdot|\lambda)\neq\hat{\alpha}(\cdot|\lambda)$. Next, since $\bar{\hat{\alpha}}(\lambda\lambda)=\bar{\alpha}(\lambda|\lambda)$, from equation (23), $\bar{\psi}^B(\hat{\alpha}(\cdot|\lambda),\lambda)-\bar{\psi}^B(\alpha^*(\cdot|\lambda),\lambda)=\bar{S}(\hat{\alpha}(\cdot|\lambda),\lambda)-\bar{S}(\alpha^*(\cdot|\lambda),\lambda)$, which is positive from above. Therefore, if $\alpha^*(\cdot|\lambda)\neq\hat{\alpha}(\cdot|\lambda)$ for a positive measure of λ, by Lemma 7, $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$).

Henceforth, without loss of generality, we can restrict attention to only $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ with the property that $\alpha^\Delta(\lambda|\lambda)+\alpha^\Delta(1|\lambda)=1$.

LEMMA 9: *Under Condition (4), if* $\left\{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ *is a solution to program (20), then it must the case that for all* λ*,* $\alpha^\Delta(\lambda|\lambda)=1$ *or* $\alpha^\Delta(1|\lambda)=1$.

PROOF:

Suppose, for a contradiction, that the statement of the lemma does not hold — i.e., letting $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be a solution to program ($\tilde{\mathcal{P}}_{stoch}$), there exists an interval $[\lambda_1,\lambda_2]$ such that $\alpha^{*\Delta}(\lambda|\lambda),\alpha^{*\Delta}(1|\lambda)\in(0,1)$ for all λ. Let $\lambda^o=\frac{\lambda_1+\lambda_2}{2}$, and for $\lambda\in[\lambda_1,\lambda^o]$, let $\zeta(\lambda)=\lambda^o+\lambda-\lambda_1$. Therefore, $(\zeta(\lambda_1),\zeta(\lambda^o)]=(\lambda^o,\lambda_2]$ and $\zeta(\lambda)>\lambda$.

Let $k=\min_{\lambda\in[\lambda_1,\lambda^o]}\alpha^\Delta(\lambda|\lambda)$, and choose an $\varepsilon\in(0,k)$. For $\lambda\in(\lambda_1,\lambda^o)$, define $\eta(\lambda)$ as follows:

$$\begin{aligned}
& [1-F(\lambda)]\,\varepsilon=[1-F(\zeta(\lambda))]\,\eta(\lambda)\\
\Longrightarrow\qquad & \eta(\lambda)=\tfrac{[1-F(\lambda)]\varepsilon}{1-F(\zeta(\lambda))}.
\end{aligned}$$

We choose ε small enough such that $\eta(\lambda)<\min_{\lambda\in[\lambda^o,\lambda_2]}1-\alpha^{*\Delta}(1|\lambda)$.

Consider the schedule $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$, where $\hat{\alpha}(\cdot|\lambda)=\alpha^*(\cdot|\lambda)$ for all $\lambda\notin[\lambda_1,\lambda^o)\cup(\lambda^o,\lambda_2]$, and

$$\begin{aligned}
&\text{for }\lambda\in[\lambda_1,\lambda^o),\quad \hat{\alpha}(\lambda|\lambda)=\alpha^*(\lambda|\lambda)+\varepsilon\ ;\ \hat{\alpha}(1|\lambda)=\alpha^*(1|\lambda)-\varepsilon\\
&\text{for }\lambda\in(\lambda^o,\lambda_1],\quad \hat{\alpha}(\lambda|\lambda)=\alpha^*(\lambda|\lambda)-\eta\left(\zeta^{-1}(\lambda)\right)\ ;\ \hat{\alpha}(1|\lambda)=\alpha^*(1|\lambda)+\eta\left(\zeta^{-1}(\lambda)\right)
\end{aligned}$$

In words, a type $\lambda\in[\lambda_1,\lambda^o)$ is paired with a type $\zeta(\lambda)\in(\lambda^o,\lambda_2]$, where ζ is bijective. $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ is constructed as follows: for each $\lambda\in[\lambda_1,\lambda^o)$, there is an increase of probability ε for $\alpha^*(\lambda|\lambda)$ (and a decrease of ε for $\hat{\alpha}(\lambda|\lambda)$); and for its "paired" type $\zeta(\lambda)$, there is an increase of probability $\eta(\lambda)$ for $\alpha^*(1|\zeta(\lambda))$ (and a decrease of $\eta(\lambda)$ for $\hat{\alpha}(\zeta(\lambda)|\zeta(\lambda))$).

Note that $\bar{\alpha}^*(\lambda|\lambda) = \alpha^{*\Delta}(1|\lambda)$ and $\bar{\hat{\alpha}}(\lambda|\lambda) = \hat{\alpha}^{\Delta}(1|\lambda)$. Therefore,

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} \bar{\hat{\alpha}}(1|\lambda)[1-F(\lambda)]\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}(1|\lambda)[1-F(\lambda)]\,d\lambda + \int_{\lambda^o}^{\lambda_2} \hat{\alpha}^{\Delta}(1|\lambda)[1-F(\lambda)]\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}(1|\lambda)[1-F(\lambda)]\,d\lambda + \int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}\left(1|\zeta^{-1}(\lambda)\right)[1-F(\zeta(\lambda))]\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} \left(\alpha^{*\Delta}(1|\lambda)-\varepsilon\right)[1-F(\lambda)] + \left[\alpha^{*\Delta}(1|\zeta(\lambda))+\eta(\lambda)\right][1-F(\zeta(\lambda))]\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} \left[\alpha^{*\Delta}(1|\lambda)+\alpha^{*\Delta}(1|\zeta(\lambda))\right][1-F(\lambda)]\,d\lambda + \int_{\lambda_1}^{\lambda^o} [[1-F(\lambda)]\varepsilon + [1-F(\zeta(\lambda))]\eta(\lambda)][1-F(\lambda)]\,d\lambda \\
&= \int_{\lambda_1}^{\lambda_2} \alpha^{*\Delta}(1|\lambda)[1-F(\lambda)]\,d\lambda.
\end{aligned}$$

This implies that

(25)
$$\int_{\lambda_1}^{\lambda_2} \bar{\psi}^B(\hat{\alpha}(\cdot|\lambda),\lambda)\,d\lambda - \bar{\psi}^B(\alpha^*(\cdot|\lambda),\lambda)\,d\lambda = \int_{\lambda_1}^{\lambda_2} \left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\,d\lambda.$$

Next, similar to above,

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\hat{\alpha}^{\Delta}(1|\lambda)\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} (1-\lambda) f(\lambda)\hat{\alpha}^{\Delta}(1|\lambda) + (1-\zeta(\lambda)) f(\zeta(\lambda))\hat{\alpha}^{\Delta}(1|\zeta(\lambda))\,d\lambda \\
&= \int_{\lambda_1}^{\lambda^o} (1-\lambda) f(\lambda)\left[\hat{\alpha}^{\Delta}(1|\lambda)-\varepsilon\right] + \left[(1-\zeta(\lambda)) f(\zeta(\lambda))\left[\hat{\alpha}^{\Delta}(1|\zeta(\lambda))+\eta(\lambda)\right]\right]d\lambda \\
&= \int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\alpha^{*\Delta}(1|\lambda)\,d\lambda
\end{aligned}$$

(26)
$$+\int_{\lambda_1}^{\lambda^o} (1-\zeta(\lambda)) f(\zeta(\lambda))\eta(\lambda) - (1-\lambda) f(\lambda)\varepsilon\ d\lambda$$

Observe that

$$\begin{aligned}(1-\zeta(\lambda)) f(\zeta(\lambda)) \eta(\lambda) &= \frac{(1-\zeta(\lambda)) f(\zeta(\lambda))}{1-F(\zeta(\lambda))}[1-F(\lambda)]\varepsilon \\ &> \frac{(1-\lambda) f(\lambda)}{1-F(\lambda)}[1-F(\lambda)]\varepsilon \;=\; (1-\lambda) f(\lambda)\varepsilon,\end{aligned}$$

where the inequality is because of Condition (4). Therefore, the line in (26) is strictly positive, meaning that

$$\int_{\lambda_1}^{\lambda_2}(1-\lambda) f(\lambda)\hat{\alpha}^{\Delta}(1|\lambda)\, d\lambda > \int_{\lambda_1}^{\lambda_2}(1-\lambda) f(\lambda)\alpha^{*\Delta}(1|\lambda)\, d\lambda.$$

This implies that

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_2}\left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\, d\lambda \\ &= \int_{\lambda_1}^{\lambda_2}(1-\lambda)\left[\hat{\alpha}^{\Delta}(1|\lambda) - \alpha^{*\Delta}(1|\lambda)\right] s_H f(\lambda)\, d\lambda \;>\; 0.\end{aligned}$$

Therefore, from equation (25), $\int_{\lambda_1}^{\lambda_2}\bar{\psi}^B(\hat{\alpha}(\cdot|\lambda),\lambda)\, d\lambda - \bar{\psi}^B(\alpha^*(\cdot|\lambda),\lambda)\, d\lambda > 0$. By Lemma 7, $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program $(\tilde{\mathcal{P}}_{stoch})$, which is a contradiction.

Henceforth, without loss of generality, we can restrict attention to only $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ with the property that either $\alpha^{\Delta}(\lambda|\lambda)=1$ or $\alpha^{\Delta}(1|\lambda)=1$.

LEMMA 10: *Under Condition (4), if* $\left\{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ *is a solution to program* $(\tilde{\mathcal{P}}_{stoch})$*, there must exist* x *such that* $\alpha^{\Delta}(\lambda|\lambda)=1$ *for all* $\lambda < x$ *and* $\alpha^{\Delta}(1|\lambda)=1$ *for all* $\lambda \geq x$*.*

PROOF:

Let $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be a solution to program $(\tilde{\mathcal{P}}_{stoch})$. Suppose, for a contradiction, that the statement of the lemma does not hold. This implies that there exist $\lambda_1 < \lambda_2 < \lambda_3 < 1$ such that $\alpha^{*\Delta}(1|\lambda)=1$ for all $\lambda\in[\lambda_1,\lambda_2)$ but $\alpha^{*\Delta}(\lambda|\lambda)=1$ for all $\lambda\in[\lambda_2,\lambda_3]$. For $x\in[\lambda_1,\lambda_1+\varepsilon]$, define $\phi(x)$ by

$$\int_{x}^{\phi(x)} 1-F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1-F(\lambda)\, d\lambda. \tag{27}$$

Pick ε such that $\phi(\lambda_1+\varepsilon)=\lambda_3$. Equation (27) implies that $\phi'(x) = \frac{1-F(x)}{1-F(\phi(x))}$.

Let

$$\begin{aligned}
D(x) &:= \int_x^{\phi(x)} (1-\lambda) f(\lambda)\, d\lambda \\
\implies D'(x) &= [1-\phi(x)] f(\phi(x)) \phi'(x) - (1-x) f(x) \\
&= [1-\phi(x)] f(\phi(x)) \frac{1-F(x)}{1-F(\phi(x))} - (1-x) f(x) \\
&\propto \frac{[1-\phi(x)] f(\phi(x))}{1-F(\phi(x))} - \frac{(1-x) f(x)}{1-F(x)} \quad \geq \quad 0,
\end{aligned}$$

where the inequality is due to Condition (4) and is strict if $x > \lambda_1$. Thus,

$$\int_{\lambda_1+\varepsilon}^{\lambda_3} (1-\lambda) f(\lambda)\, d\lambda = D(\lambda_1+\varepsilon) > D(\lambda_1) = \int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\, d\lambda. \tag{28}$$

Consider the schedule $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ where $\hat{\alpha}(\cdot|\lambda) = \alpha^*(\cdot|\lambda)$ for all $\lambda \notin [\lambda_1, \lambda_3]$, and $\hat{\alpha}^{\Delta}(\lambda|\lambda) = 1$ for all $\lambda \in [\lambda, \lambda_1+\varepsilon)$, and $\hat{\alpha}^{\Delta}(1|\lambda) = 1$ for all $\lambda \in [\lambda_1+\varepsilon, \lambda_3]$. Observe that

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_3} \left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\, d\lambda \\
=& \left[\int_{\lambda_1+\varepsilon}^{\lambda_3} (1-\lambda) f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\, d\lambda\right] s_H > 0,
\end{aligned}$$

where the inequality is from equation (28). Additionally,

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_3} \left(\bar{\hat{\alpha}}(\lambda) - \bar{\alpha}^*(\lambda)\right) [1-F(\lambda)] f(\lambda)\, d\lambda \\
=& \int_{\lambda_1+\varepsilon}^{\lambda_3} \bar{\hat{\alpha}}^{\Delta}(\lambda) [1-F(\lambda)]\, d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{\alpha}^{*\Delta}(\lambda) [1-F(\lambda)]\, d\lambda.
\end{aligned}$$

Thus, $\int_{\lambda_1}^{\lambda_3} \bar{\psi}^B(\hat{\alpha}(\cdot|\lambda)) - \bar{\psi}^B(\alpha^*(\cdot|\lambda))\, d\lambda = \int_{\lambda_1}^{\lambda_3} \left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\, d\lambda > 0$. By Lemma 7, $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program $(\tilde{\mathcal{P}}_{stoch})$, which is a contradiction.

Finally, Proposition 4 is a corollary of Lemma 10.

II. Online Appendix: Additional Results

A. On the Constrained Pareto Frontier

This subsection studies the curvature of the Pareto frontier. From Proposition 2, if the solution is of class A1, the frontier is $\hat{S}(0) - b$, which has a constant slope of -1. If the solution is of class A2 , the frontier is $\hat{\psi}^S(\lambda^*(b))$.

LEMMA 11: *Suppose that the density f is differentiable over $(0,1)$. For all $b \in \left(E[v] - c_H, \bar{b}\right)$, $\hat{\psi}^S(\lambda^*(b))$ is twice differentiable with respect to b, and $\frac{d^2}{db^2}\hat{\psi}^S(\lambda^*(b)) \leq (\geq) 0$ if $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}$ is increasing (decreasing) at $\lambda = \lambda^*(b)$.*

Observe that $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}$ is constant for the uniform distribution. This explains why the slope of the Pareto frontier of the A2 allocations under the uniform distribution is a constant. PROOF:

Let $\mathcal{V}^S(b) = \hat{\psi}^S(\lambda^*(b))$. Twice differentiating $\mathcal{V}^S(b)$, we have

$$\begin{aligned}
\mathcal{V}^{S'}(b) &= -(c_H - c_L)\left[1 - F(\lambda^*(b))\right]\lambda^{*'}(b) \\
\mathcal{V}^{S''}(b) &\propto f(\lambda^*(b))\left[\lambda^{*'}(b)\right]^2 - \left[1 - F(\lambda^*(b))\right]\lambda^{*''}(b)
\end{aligned}$$

$$\Longrightarrow \quad \mathcal{V}^{S''}(b) \leq 0 \quad \Longleftrightarrow \quad \frac{f(\lambda^*(b))}{1 - F(\lambda^*(b))} \leq \frac{\lambda^{*''}(b)}{\left[\lambda^{*'}(b)\right]^2} \tag{29}$$

Let $\hat{\psi}^B(x) = \hat{S}(x) - \hat{\psi}^S(x)$. Doing total differentiation twice on $\hat{\psi}^B(\lambda^*(b)) = b$, we have $\frac{\lambda^{*''}(b)}{\left[\lambda^{*'}(b)\right]^2} = -\frac{\hat{\psi}^{B''}(\lambda^*(b))}{\hat{\psi}^{B'}(\lambda^*(b))}$, where

$$\begin{aligned}
\hat{\psi}^{B'}(\lambda) &= -(1-\lambda)s_H f(\lambda) + (c_H - c_L)\left[1 - F(\lambda)\right] \\
\hat{\psi}^{B''}(\lambda) &= f(\lambda)\left[s_H - c_H + c_L\right] - (1-\lambda)s_H f'(\lambda)
\end{aligned}$$

Therefore, equation (29) holds if and only if

$$\frac{f(\lambda^*(b))}{1 - F(\lambda^*(b))} \leq -\frac{\hat{\psi}^{B''}(\lambda^*(b))}{\hat{\psi}^{B'}(\lambda^*(b))}$$

$\hat{\psi}^{B'}(\lambda^*(b))$ must be positive; if not, there exists $x < \lambda^*(b)$ in which $\hat{\psi}^B(x) = b$, which contradicts the definition of $\lambda^*(b)$. When $\hat{\psi}^{B'}(\lambda) = -(1-\lambda)s_H f(\lambda) +$

$(c_H - c_L)[1 - F(\lambda)] \geq 0$,

$$\begin{aligned}
& & \tfrac{f(\lambda)}{1-F(\lambda)} \leq -\tfrac{f(\lambda)[s_H - c_H + c_L] - (1-\lambda)s_H f'(\lambda)}{-(1-\lambda)s_H f(\lambda) + (c_H - c_L)[1-F(\lambda)]} \\
&\Longleftrightarrow & f(\lambda)[1-F(\lambda)] \leq [1-F(\lambda)](1-\lambda) f'(\lambda) + (1-\lambda)[f(\lambda)]^2 \\
&\Longleftrightarrow & \tfrac{1}{1-\lambda} \leq \tfrac{f'(\lambda)}{f(\lambda)} + \tfrac{f(\lambda)}{1-F(\lambda)} \\
&\Longleftrightarrow & -\tfrac{d}{d\lambda}\log(1-\lambda) \leq \tfrac{d}{d\lambda}\log f(\lambda) - \tfrac{d}{d\lambda}\log(1-F(\lambda)) \\
&\Longleftrightarrow & \tfrac{d}{d\lambda}\log\left[\tfrac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0 \\
&\Longleftrightarrow & \left[\tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}\right] \times \tfrac{d}{d\lambda}\left[\tfrac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0
\end{aligned}$$

Since $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)} > 0$, the last line is equivalent to $\frac{d}{d\lambda}\left[\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0$.

B. The Monopsonist's Screening Problem

The mechanism implementing the buyer-optimal SB allocation is also a solution to the problem of a monopsonist who can offer a menu of trade contracts to screen the seller. However, although the mechanism that attains the buyer-optimal SB allocation is unique, the solution to the monopsonist's problem is not necessarily unique. This is because it is possible that the monopsonist can obtain the buyer-optimal second best utility ($\bar{b}$) while giving the seller a lower expected utility than what she gets under the buyer-optimal SB allocation.

In this section, we specifically study the monopsonist's screening problem in our model. We fully characterize the set of optimal screening mechanisms (which includes the mechanism in Proposition 2 for $b = \bar{b}$) and show that the quantity schedules of *all* the optimal screening mechanisms are still always threshold schedules. Moreover, this property holds even if $f/(1-F)$ is not monotonic.

Henceforth, assume that F still admits a density f, but $f/(1-F)$ is not necessarily increasing. The monopsonist's problem is

$$\max_{q(\cdot),t(\cdot)} \int_0^1 U^B(\lambda) f(\lambda)\, d\lambda, \qquad \text{s.t.} \qquad (IC^S) \text{ and } (IR^S).$$

Using a similar argument to Lemma 1, the problem becomes

$$\max_{\text{nondecreasing } q(\cdot)} \int_0^1 \psi^B(q(\lambda), \lambda)\, d\lambda. \tag{30}$$

LEMMA 12: *If $q^*(\cdot)$ is a solution to program (30), $q^*(\cdot)$ must be a threshold schedule.*

PROOF:

Suppose that $q^*(\cdot)$ is a solution to program (30). It is straightforward to observe that the properties in Lemma 5 hold. Therefore, we only have to prove that $\lambda_2 = 1$. Suppose, for a contradiction, that $\lambda_2 < 1$. There must then exist $\lambda_3 > \lambda_2$ such $q^*(\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Pick some $\varepsilon > 0$ such that $\varepsilon < \max\{\lambda_3 - \lambda_2, \lambda_2 - \lambda_1\}$. For any $x \in [\lambda_2 - \varepsilon, \lambda_2 + \varepsilon]$, define the schedule $\tilde{q}_x(\cdot)$ as follows:

$$\tilde{q}_x(\lambda) = \begin{cases} x & \text{, if } \lambda \in [\lambda_1, x) \\ \lambda & \text{, if } \lambda \in [x, \lambda_3) \\ q^*(\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \end{cases} \tag{31}$$

Note that $\tilde{q}_x(\cdot)$ is also a nondecreasing schedule. Next, let $\xi(x) := \int_0^1 \psi^B(\tilde{q}_x(\lambda), \lambda) f(\lambda)\, d\lambda$ be the objective value under schedule $\tilde{q}_x(\cdot)$. Therefore,

$$\begin{aligned} \xi(x) = &\int_{\lambda \notin [\lambda_1, \lambda_3)} \psi^B(q^*(\lambda), \lambda) f(\lambda)\, d\lambda + + \int_x^{\lambda_3} \underbrace{\lambda s_L}_{\psi^B(\lambda,\lambda)} f(\lambda)\, d\lambda \\ &+ \int_{\lambda_1}^x \underbrace{\left(x s_L + (x - \lambda)(s_H - s_L) - (c_H - c_L) \frac{1 - F(\lambda)}{f(\lambda)} \right)}_{\psi^B(x,\lambda)} f(\lambda)\, d\lambda \end{aligned}$$

The first two derivatives of $\xi(x)$ are

$$\begin{aligned} \xi'(x) &= -(c_H - c_L)[1 - F(x)] + [F(x) - F(\lambda_1)]\, s_H. \\ \xi''(x) &= f(x)[(c_H - c_L) + s_H] > 0. \end{aligned}$$

Since $\tilde{q}_{\lambda_2}(\cdot)$ is the optimal schedule $q^*(\cdot)$, $x = \lambda_2$ must be a local maximizer. However $\xi''(\lambda_2)$ is strictly positive — contradiction. Therefore, λ_2 must be 1.

By Lemma 12, we can restrict our search for the optimal quantity schedule to threshold schedules. Under the threshold-x quantity schedule, the value of the objective function is

$$\begin{aligned} \hat{\psi}^B(x) &:= \int_0^x \psi^B(\lambda, \lambda) f(\lambda)\, d\lambda + \int_x^1 \psi^B(1, \lambda) f(\lambda)\, d\lambda \\ &= \int_0^1 \lambda s_L f(\lambda)\, d\lambda \ + \ \int_x^1 \left[(1 - \lambda) s_H - (c_H - c_L) \frac{1 - F(\lambda)}{f(\lambda)} \right] f(\lambda)\, d\lambda. \end{aligned} \tag{32}$$

Let $R(\lambda) := \frac{(1-\lambda) f(\lambda)}{1 - F(\lambda)}$.

PROPOSITION 5: *The monopsonist's optimal menus of contracts is*

$$\left(q^*\left(\lambda\right), t^*\left(\lambda\right)\right) = \begin{cases} \left(\lambda, \lambda c_L\right) & , \ \forall \lambda < \lambda^B \\ \left(1, C\left(1, \lambda^B\right)\right) & , \ \forall \lambda \geq \lambda^B \end{cases},$$

where $\lambda^B \in \arg\max_x \hat{\psi}^B\left(x\right)$. *The buyer's equilibrium expected utility is* $\hat{\psi}^B\left(\lambda^B\right)$. *If* $\lambda^B \neq 0, 1$, *then* λ^B *must satisfy* $R\left(\lambda^B\right) = \frac{c_H - c_L}{s_H}$.

- *If* $R\left(\lambda\right)$ *is strictly decreasing in* λ *for all* $\lambda \in (0, 1)$, *then* λ^B *is either* 0 *or* 1.
- *If* $R\left(\lambda\right)$ *is strictly increasing in* λ *for all* $\lambda \in (0, 1)$, *then* λ^B *is unique, and the optimal schedule is the pointwise optimal schedule — i.e.,* $q^*\left(\lambda\right) = \arg\max_q \psi^B\left(q, \lambda\right)$ *for all* λ.

The first-order condition of $\max_x \hat{\psi}^B\left(x\right)$ is $R\left(\lambda\right) = \frac{c_H - c_L}{s_H}$; thus, this is a necessary condition for an interior optimal threshold. If $R\left(\lambda\right)$ is decreasing, $\hat{\psi}^B\left(x\right)$ is quasiconvex, hence leading to a corner solution. In contrast, if $R\left(\lambda\right)$ is increasing, the pointwise optimum of $\psi^B\left(\cdot, \lambda\right)$ is nondecreasing, which means that it is the unique optimum.