Online Appendix for "Monitor Reputation and Transparency"*

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A Existence of Shirking Equilibrium

We now show that a solution to ODE (12) with boundary condition $V_s(1) = \frac{\alpha}{r}$ exists, using an approximation argument. The proof uses a "bounding box" which has finite upper and lower boundaries and whose right boundary is fixed below one. For any point on the boundary of this box, we can find an initial value so that the unique solution to the IVP hits this point. We then construct a sequence of boxes so that the right boundary approaches 1 and a corresponding sequence of solutions so that the value at the right boundary of the box converges to $\frac{\alpha}{r}$. To show that the limit actually satisfies $V_s(1) = \frac{\alpha}{r}$, we need to show that the sequence of solutions converges uniformly. For this we use the Arzelà-Ascoli Theorem, which we apply to a rescaled version of $V_s(x)$ that has a finite derivative.

The bounding box is for all $n \in \mathbb{N}$ given by

$$\begin{aligned} B_n &= \left\{(x,v) \in \mathbb{R}^2 | x \in [x_0, x_n], v \in \{-M, M\} \text{ if } x \in (x_0, x_n) \right. \\ &\quad \left. \text{and } v \in [-M, M] \text{ if } x \in \{x_0, x_n\}\right\} \end{aligned}$$

for some finite $M > \frac{\alpha}{r}$. Here, x_n is the right boundary of the box. We assume $\{x_n\}_{n=1}^{\infty}$ is an increasing sequence with $x_n \in (x_0, 1)$ for all n which converges to one as $n \to \infty$. Point 3 of Lemma 4 then implies that each point on B_n can be reached by some solution to the IVP, which we show below.

Corollary 1. *For each $(\hat{x}, \hat{v}) \in B_n$, there exists a v_{0n} such that the solution to the IVP with initial condition v_{0n} satisfies $V_s(\hat{x}, v_{0n}) = \hat{v}$.*

Proof. Picking $v_{0n} = -M$ ensures that $V_s(x_0) = -M$ and picking $v_{0n} = M$ ensures that $V_s(x_0) = M$. For any $v_{0n} \in (-M, M)$, either hits the upper or lower bounds or it hits the right boundary at x_n. Since $V_s(x)$ is continuous and monotone in v_{0n} by Point 3 of Lemma 4, the continuous mapping theorem implies that for any point $(\hat{x}, \hat{v}) \in B_n$, there exists an initial condition v_{0n} such that $V_s(\hat{x}) = \hat{v}$. ∎

We use this result to construct a sequence of solutions which satisfy a boundary condition at x_n. That condition will converge to $\frac{\alpha}{r}$. Since we are only interested in the properties of these solutions as x becomes large, we omit any dependence on the initial condition v_{0n} to save notation. We denote with $V_{sn}(x)$ the solution to Equation (12) which satisfies the boundary condition

$$V_{sn}(x_n) = \frac{\alpha}{r} - \kappa(1 - x_n) \tag{1}$$

for some fixed $\kappa > 0$. As $n \to \infty$, the derivative $V'_{sn}(x_n)$ becomes potentially unbounded, because x_n approaches one and the shirking ODE (12) has a singularity at $x = 1$. Therefore, we cannot use the Arzelà-Ascoli Theorem on V_{sn} directly. Instead, we study the transformation

$$g_n(x) = V_{sn}(x)(1-x),$$

which we extend to the entire interval $[x_0, 1]$ as follows:

$$\bar{g}_n(x) = \begin{cases} V_{sn}(x)(1-x) & \text{if} \quad x_0 \le x \le x_n \\ \frac{\alpha}{r}(1-x_n) - \kappa(1-x_n)^2 & \text{if} \quad x_n < x \le 1. \end{cases}$$

Lemma 1. *For all $n \in \mathbb{N}$, $\bar{g}_n(x)$ is uniformly bounded. It is also differentiable at all $x \in [x_0, 1]$ except at x_n and has a uniformly bounded derivative.*

Proof. $\bar{g}_n(x)$ is uniformly bounded because we have constructed the sequence $V_{sn}(x)$ so that for all $x \in [x_0, x_n]$, $V_{sn}(x)$ is inside the "bounding box", i.e. $V_{sn}(x) \in [-M, M]$. Since $g_n(x) = V_{sn}(x)(1-x)$, we must also have $g_n(x) \in [-M, M]$. From the definition of $\bar{g}_n(x)$ we can also see that it is uniformly bounded on $[x_n, 1]$ for all n.

To show the derivative is uniformly bounded whenever it exists, we only have to consider the derivatives on the intervals $[x_0, x_n]$.[1] We can substitute $g_n(x) = V_{sn}(x)(1-x)$ and $g_n'(x) = V_{sn}'(x)(1-x) - V_{sn}(x)$ into Equation (12) to obtain an ODE for $g_n(x)$. This ODE is

$$\left(r + \lambda(1-x)^2\right) g_n(x) = \alpha\left((1-x) - (1-x)^3\right) + \lambda x (1-x)^2 g_n'(x). \tag{2}$$

For any n, the derivative at x_n is bounded. To see this, we first solve for $V_{sn}'(x_n)$, using Equation (31) and the condition in Equation (1). This yields

$$V_{sn}'(x_n) = \frac{1}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right).$$

Therefore we have

$$g_n'(x_n) = \frac{1-x_n}{x_n}\left(\frac{\alpha}{\lambda} - \kappa + \left(\frac{\alpha}{r} - \frac{r\kappa}{\lambda}\right)\frac{1}{1-x_n}\right) - \frac{\alpha}{r} - \kappa(1-x_n). \tag{3}$$

As $n \to \infty$, this expression converges to $-\frac{r}{\lambda}\kappa$. This means that there exists a $\bar{K} > 0$ so that for all n, $|g_n'(x_n)| \le \bar{K}$. To see that $g_n'(x)$ must be bounded uniformly for all n and $x \le x_n$, we differentiate Equation (2) to obtain

$$\begin{aligned} 0 \quad = \quad & 2\lambda(1-x) g_n(x) + \alpha\left(3(1-x)^2 - 1\right) \\ & + \lambda x (1-x)^2 g_n''(x) - (r + \lambda(1-x)x) g_n'(x). \end{aligned}$$

[1] On $[x_n, 1]$, the result follows from inspecting the definition of $\bar{g}_n(x)$ above.

Suppose there exists an n and an $x_0 \leq x < x_n$ so that $|g_n'(x)| > K$. We choose K sufficiently large and larger than $\bar{K}$. Then, if $g_n'(x) > K$, the equation above immediately implies that $g_n''(x) > 0$, since $g_n(x)$ is uniformly bounded. But this means that $g_n'(x') > K$ for all $x' \geq x$. This is a contradiction, since we have just shown that $g_n'(x_n)$ is bounded by $\bar{K}$ for all n. Similarly, if $g_n'(x) < -K$, then $g_n''(x) < 0$, which again implies that $g_n'(x_n) < -\bar{K}$. ∎

We can now apply the Arzelà-Ascoli Theorem to the sequence of functions $\bar{g}_n(x)$. It establishes that there is a subsequence that converges to a continuous function $g^*(x)$. As we show below, we can take $g^*(x)$ to be continuously differentiable on $[x_0, 1]$ and to satisfy the ODE (2) on that interval without loss of generality.

Lemma 2. *There exists a subsequence of $\bar{g}_n(x)$ which converges uniformly to a function $g^*(x)$ which is continuously differentiable and satisfies Equation (2) on $[x_0, 1]$.*

Proof. From the previous Lemma and the Arzelà-Ascoli Theorem we know there exists a subsequence which converges to a continuous function $g^*(x)$. We now use a diagonalization procedure to show that there exists a subsequence such that $g^*(x)$ is continuously differentiable on $[x_0, 1)$. For a given n, the derivative $g_n'(x)$ satisfies

$$g_n'(x) = \frac{\left(r + \lambda (1-x)^2\right) g_n(x) - \alpha \left((1-x) - (1-x)^3\right)}{\lambda x (1-x)^2}$$

on some interval $[x_0, \bar{x}_1]$ for $\bar{x}_1 < x_n < 1$. Since the sequence g_n is equicontinuous on that interval and the right hand side of the above equation is continuous in both x and $g_n(x)$, $g_n'(x)$ is equicontinuous on that interval as well.[2] Thus, there exists a subsequence of g_n which converges to a limit that is continuously differentiable on $[x_0, \bar{x}_1]$. Proceeding iteratively, we then take a sequence of boundaries $\bar{x}_k$ which converges to one as $k \to \infty$. For each such k we can find a subsequence of g_n that converges to a continuously differentiable function. Thus, we can take the limit g^* to be continuously differentiable on $[x_0, 1)$ without loss of generality. Because of this, it also satisfies the ODE (2) on $[x_0, 1)$.

It remains to establish that g^* is continuously differentiable at $x = 1$. This follows from Equation (3) in the proof of the previous Lemma. We have

$$\lim_{n \to \infty} g^{*\prime}(x_n) = \lim_{n \to \infty} g_n'(x_n)$$

and Equation (3) shows that $\lim_{n \to \infty} g_n'(x_n) = -\frac{r\kappa}{\lambda}$. Thus, $g^{*\prime}(1)$ is finite. ∎

We now use the function g^* to show that our initial sequence of solutions $V_{sn}(x)$ converges to a limit that is continuous, solves the shirking ODE (12), and satisfies the boundary

[2] Note we are holding $\bar{x}_1$ fixed here.

condition $V_s(1) = \frac{\alpha}{r}$. To do this, we define the following function on the interval $[x_0, 1]$

$$V^*(x) = \frac{g^*(x)}{1-x}.$$

This function is continuously differentiable except perhaps at $x = 1$ and it satisfies the ODE (12), which can be seen by substituting it into Equation (2). We thus only have to show it satisfies the boundary condition at $x = 1$. If we let n_k denote the subsequence of n for which g_n converges to g^*, we have

$$V^*(x) = \lim_{k\to\infty} \frac{g_{n_k}(x)}{(1-x)} = \lim_{k\to\infty} V_{nk}(x).$$

Since for any n, $V_n(1) = \frac{\alpha}{r}$, we have

$$\begin{aligned}
\lim_{x\to 1} V^*(x) &= \lim_{x\to 1}\lim_{k\to\infty} V_{nk}(x) \\
&= \lim_{k\to\infty} V_{nk}(1) \\
&= \frac{\alpha}{r}.
\end{aligned}$$

This concludes our proof. We have shown that there exists a solution to the shirking ODE (12) on the interval $[x_0, 1]$ which satisfies the boundary condition $V_s(1) = \frac{\alpha}{r}$. Since any solution to the equation must satisfy $V(0) = 0$ (by Lemma 4, Point 1), we can extend this solution to the entire interval $[0, 1]$.

B Numerical Appendix

We use a finite difference approximation of the regulator value $W(x, V)$.[3] To improve speed and accuracy given the highly nonlinear domain, we use an unevenly spaced grid. Specifically, we start with evenly spaced grid in the x dimension with I elements, $X = (x_1, ..., x_I)$ and we denote a generic element x_i. Then, we compute a solution to the Hamilton-Jacobi equations defining the boundaries $\bar{V}(x)$ and $\underline{V}(x)$ (Equations (27) and (28)), using MATLAB's built in *bvp4c* function. Since using *bvp4c* does not guarantee that the grid is the same as the one we have defined, we linearly interpolate the solutions on the grid X. We denote the resulting values with $\widehat{\bar{V}}_i$ and $\widehat{\underline{V}}_i$, which are defined at each grid point $x_i \in X$.

Similarly, we use *bvp4c* to compute the ODEs defining the monitor's boundary conditions $\bar{W}(x)$ and $\underline{W}(x)$ (Equations (29) and (30)) and we denote with $\widehat{\bar{W}}_i$ and $\widehat{\underline{W}}_i$ the linear

[3]See **?** for a characterization of convergence properties.

interpolation on X. We also obtain the derivatives $\bar{W}'(x)$ and $\underline{W}'(x)$ from *bvp4c*, which will be useful later, and we denote the linear interpolations on X as $\widehat{\bar{W}}'_i$ and $\widehat{\underline{W}}'_i$.

Next, we construct the grid in the V dimension. We fix a number J, and for each i, we define an evenly spaced grid with J elements, $V_i = (v_{i1}, ..., v_{iJ})$, with $v_{i1} = \widehat{\underline{V}}_i$ and $v_{iJ} = \widehat{\bar{V}}_i$. This grid choice has the following desirable property. As can be seen in Figure 2, the boundaries $\bar{V}(x)$ and $\underline{V}(x)$ become simultaneously very steep and very close together as x becomes small. Our grid features a smaller distance between elements in the V-dimension on that region, which improves accuracy where it is most needed.

To facilitate indexing, we define the $I \times J$ matrix of x-elements as

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_1 & ... & x_1 \\ x_2 & x_2 & ... & x_2 \\ \vdots & \vdots & \ddots & \vdots \\ x_I & x_I & ... & x_I \end{bmatrix}$$

and the $I \times J$ matrix of V-elements as

$$\boldsymbol{V} = \begin{bmatrix} v_{11} & v_{12} & ... & v_{1J} \\ v_{21} & v_{22} & ... & v_{2J} \\ \vdots & & \ddots & \vdots \\ v_{I1} & v_{I2} & ... & v_{IJ} \end{bmatrix}.$$

Thus, a generic element $v_{i,j}$ of $\boldsymbol{V}$ corresponds to the j'th element of the vector V_i. From here on, we use the short-hand notation $W_{i,j} = W(x_i, v_{i,j})$, $m_{i,j} = m(x_i, v_{i,j})$, etc.

We approximate the partial derivatives $W_x(x_i, v_{i,j})$ and $W_V(x_i, v_{i,j})$ using forward differences. The derivative in the V dimension is standard and given by

$$W_V(x_i, v_{i,j}) \approx \frac{W_{i,j+1} - W_{i,j}}{v_{i,j+1} - v_{i,j}} \equiv W_{V,i,j}. \tag{4}$$

Approximating the derivative in the x dimension faces two challenges. (1) For a given $(x_i, v_{i,j})$, the pair $(x_{i+1}, v_{i,j})$ may lie outside the domain. For example, we may have $v_{i,j} > \underline{V}(x_i)$, but $v_{i,j} < \underline{V}(x_{i+1})$. (2) For a given $(x_i, v_{i,j})$, $v_{i,j}$ may not be an element of V_{i+1}, i.e. the node $(x_{i+1}, v_{i,j})$ does not exist. Thus, we generally cannot compute the "naive" forward difference

$$W_x(x_i, v_{i,j}) \approx \frac{W(x_{i+1}, v_{i,j}) - W(x_i, v_{i,j})}{x_{i+1} - x_i}.$$

To solve the issue of derivatives at the boundaries, we replace the forward difference with the interpolated derivatives $\widehat{\overline{W}}'_i$ and $\widehat{\underline{W}}'_i$, respectively. That is,

$$W_{x,i,j} \equiv \widehat{\underline{W}}'_i \tag{5}$$

whenever $v_{i,j} < \widehat{\underline{V}}_{i+1}$ and

$$W_{x,i,j} \equiv \widehat{\overline{W}}'_i \tag{6}$$

whenever $v_{i,j} > \widehat{\overline{V}}_{i+1}$.

To approximate derivatives inside the feasible domain, we compute the nearest neighbor to $v_{i,j}$ in the vector V_{i+1}. We denote with $n_+(i,j)$ its index, i.e. $v_{n_+(i,j)}$ is the nearest neighbor to $v_{i,j}$ in V_{i+1}. Then, we approximate the derivative as

$$W_x(x_i, v_{i,j}) \approx \frac{W_{i,n_+(i,j)} - W_{i,j}}{x_{i+1} - x_i} \equiv W_{x,i,j}. \tag{7}$$

The finite difference approximation to the regulator's Hamilton-Jacobi equation (26) is now defined as

$$\begin{aligned} rW_{i,j} \;=\;& \alpha\left(1 - m_{i,j}\right) + \left(\left(r + \gamma_{B,i,j} + \lambda m_{i,j}\right) v_{i,j} - \alpha\left(1 - m_{i,j}^2\right)\right) W_{V,i,j} \\ & + \left(\lambda x_i m_{i,j}^2 + x_i\left(1 - x_i\right)\left(\gamma_{B,i,j} - \gamma_{G,i,j}\right)\right) W_{x,i,j} \\ & - \left(\lambda m_{i,j}^2 + \gamma_{Bi,j}\left(1 - x_i\right)\right) W_{i,j} + \gamma_{G,i,j} x_i \left(\frac{\alpha}{r} - W_{i,j}\right). \end{aligned}$$

Here, $\gamma_{B,i,j}$ and $\gamma_{G,i,j}$ are the approximate optimal controls, which are given by

$$\gamma_{B,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad v_{i,j} W_{V,i,j} + x_i\left(1 - x_i\right) W_{x,i,j} - \left(1 - x_i\right) W_{i,j} \geq 0 \\ 0 & \text{if} \quad v_{i,j} W_{V,i,j} + x_i\left(1 - x_i\right) W_{x,i,j} - \left(1 - x_i\right) W_{i,j} < 0 \end{cases}$$

and

$$\gamma_{G,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad -x_i\left(1 - x_i\right) W_{x,i,j} + x_i\left(\frac{\alpha}{r} - W_{i,j}\right) \geq 0 \\ 0 & \text{if} \quad -x_i\left(1 - x_i\right) W_{x,i,j} + x_i\left(\frac{\alpha}{r} - W_{i,j}\right) < 0. \end{cases}$$

We use an explicit method to calculate a solution to the above equation. We start with an initial guess $W_{i,j}^0$, which is given by a linear interpolation between $\widehat{\overline{W}}_i$ and $\widehat{\underline{W}}_i$ at each i

and $v_{i,j}$.[4] Then, we update $W^n_{i,j}$, as

$$\begin{aligned} \frac{W^{n+1}_{i,j} - W^n_{i,j}}{\Delta} + rW_{i,j} \quad = \quad & \alpha\left(1 - m_{i,j}\right) \\ & + \left(\left(r + \gamma^n_{B,i,j} + \lambda m_{i,j}\right) v_{i,j} - \alpha\left(1 - m^2_{i,j}\right)\right) W^n_{V,i,j} \\ & + \left(\lambda x_i m^2_{i,j} + x_i\left(1 - x_i\right)\left(\gamma^n_{B,i,j} - \gamma^n_{G,i,j}\right)\right) W^n_{x,i,j} \\ & - \left(\lambda m^2_{i,j} + \gamma^n_{Bi,j}\left(1 - x_i\right)\right) W^n_{i,j} + \gamma^n_{G,i,j} x_i \left(\frac{\alpha}{r} - W^n_{i,j}\right). \end{aligned} \tag{8}$$

Here, Δ is the step size of the iteration and $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ are defined analogously as

$$\gamma^n_{B,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad v_{i,j} W^n_{V,i,j} + x_i\left(1 - x_i\right) W^n_{x,i,j} - \left(1 - x_i\right) W^n_{i,j} \geq 0 \\ 0 & \text{if} \quad v_{i,j} W^n_{V,i,j} + x_i\left(1 - x_i\right) W^n_{x,i,j} - \left(1 - x_i\right) W^n_{i,j} < 0 \end{cases} \tag{9}$$

and

$$\gamma^n_{G,i,j} = \begin{cases} \bar{\gamma} & \text{if} \quad -x_i\left(1 - x_i\right) W^n_{x,i,j} + x_i\left(\frac{\alpha}{r} - W^n_{i,j}\right) \geq 0 \\ 0 & \text{if} \quad -x_i\left(1 - x_i\right) W^n_{x,i,j} + x_i\left(\frac{\alpha}{r} - W^n_{i,j}\right) < 0. \end{cases} \tag{10}$$

The algorithm can be summarized as follows.

1. Start with guess $W^0_{i,j}$.
2. Compute $W_{x,i,j}$ and $W_{V,i,j}$ using Equations (4), (6), (5), and (7).
3. Compute $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ using Equations (9) and (10).
4. Compute $W^{n+1}_{i,j}$ using Equation (8).
5. Stop if the maximum distance

$$\max_{i,j} \left| W^{n+1}_{i,j} - W^n_{i,j} \right|$$

is below a specified tolerance, otherwise go to step 2.

Running this scheme on the entire x-domain $[0, 1]$ results in a number of problems. (1) As can be seen from Figure 2, the regulator is indifferent between any disclosure policy for any $x \geq x_h$.[5] This may lead the policies $\gamma^n_{B,i,j}$ and $\gamma^n_{G,i,j}$ to oscillate on the region $[x_h, 1]$, which may lead to convergence failures. (2) The law of motion for beliefs may have an endogenous

[4] That is,

$$W^0_{i,j} = \underline{\widehat{W}}_i + \left(v_{i,j} - v_{i,1}\right) \frac{\widehat{\overline{W}}_i - \underline{\widehat{W}}_i}{v_{i,J} - v_{i,1}}.$$

[5] Note that this is consistent with our previous qualitative results in Section 4.

interior singularity at high x values. To see this, consider again the shirking region $[x_h, 1]$. On this region, $m(x, V) = 1 - x$, independently of V or the regulator's disclosure policy. Then, we can calculate the law of x_t as

$$\begin{aligned}\frac{dx_t}{dt} &= \lambda x_t (1 - x_t)^2 + (\gamma_{Bt} - \gamma_{Gt}) x_t (1 - x_t) \\ &= x_t (1 - x_t) (\lambda (1 - x_t) + \gamma_{Bt} - \gamma_{Gt}).\end{aligned}$$

When x_t is sufficiently large and $\gamma_{Bt} = 0$, the dx_t/dt may change sign depending on whether $\gamma_{Gt} = 0$ and $\gamma_{Gt} > 0$. (3) The law for V_t may change sign depending on whether $\gamma_{Bt} > 0$ or $\gamma_{Bt} = 0$ for high values of x.

To avoid these issues, we use our theoretical characterization to restrict the problem as follows. For $x > x_h$, we know that the monitor shirks irrespective of the disclosure policy and that the regulator's value is linear (see also Figure 2). Thus, without loss of generality, the optimal disclosure policy at any $x_t > x_h$ is given by $\gamma_{Bt} = \gamma_{Gt} = 0$. The regulator's boundary values coincide at x_h, i.e. $\bar{W}(x_h) = \underline{W}(x_h)$. We can now restrict the grid of x-values to $[0, x_h]$ and use the boundary condition $W(x_h, V) = \bar{W}(x_h)$ for any $V \in [\underline{V}(x_h), \bar{V}(x_h)]$. With this modification, and sufficiently small step size Δ, and a sufficiently fine grid X, the explicit scheme converges monotonically.