

Online Appendix for “Depreciating Licenses”

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SUPPLEMENTAL MATERIAL FOR SECTION II

A1. Proof of Proposition 1

First, conjecture that the value of being an owner at the start of any period, with value u_{St} , is

$$V(u_{St})$$

Let

$$V \equiv E_{u_{St} \sim F(\cdot)}[V(u_{St})]$$

be the ex-ante value of an owner, before she learns the realization of u_{St} .

INCUMBENT PROFIT. — The incumbent first chooses ψ_t , and pays cost $c(\psi_t)$. After ψ_t is determined, the value of the asset owner for keeping the good is:

$$\psi_t + u_{St} + \beta V$$

Likewise, the value of a buyer is:

$$\psi_t + u_{Bt} + \beta V$$

Since the auction is second-price, the dominant strategy for the buyer is to bid her true value.

The objective function of the incumbent is thus the following. If the incumbent bids above the buyer, she gets:

$$\underbrace{(\psi_t + u_{St} + \beta V)}_{\text{Continuation value}} - \underbrace{\tau(\psi_t + u_{Bt} + \beta V)}_{\text{Fee payment}}$$

i.e. she pays the buyer’s bid, keeps the asset and gets her continuation value. If the incumbent bids below the buyer’s value, she gets paid her bid for her share $(1 - \tau)$ of the asset, that is:

$$(1 - \tau)(\psi_t + m + \beta V)$$

In either case, the investment cost $c(\psi_t)$ is sunk. Hence, the incumbent's expected profit can be written as:

$$\Pi_t = \int_0^m [(\psi_t + u_{St} + \beta V) - \tau(\psi_t + u_{Bt} + \beta V)] dF(u_{Bt}) + \int_m^\infty (1 - \tau)(\psi_t + m + \beta V) dF(u_{Bt}) - c(\psi_t)$$

(A1)

$$\Pi_t = (1 - \tau)(\psi_t + \beta V) - c(\psi_t) + \int_0^m [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1 - \tau)m \int_m^\infty dF(u_{Bt})$$

PRICE SETTING. — Differentiating (A1) with respect to m , we have:

$$(A2) \quad \frac{\partial \Pi}{\partial m} = (u_{St} - m)f(m) + (1 - \tau)(1 - F(m))$$

Setting to 0 and rearranging, we have:

$$m - u_{St} = (1 - \tau) \frac{1 - F(m)}{f(m)}$$

This proves (5) of Proposition 1. Note that the derivative (A2) is monotonically decreasing in τ , hence (A1) has increasing differences in m and $-\tau$, hence the optimal m is monotonically decreasing in τ .

INVESTMENT. — Differentiating (A1) with respect to ψ_t , we have:

$$c'(\psi_t) = (1 - \tau)$$

This proves (4) of Proposition 1.

THE VALUE FUNCTION. — The incumbent's value function satisfies:

$$(A3) \quad V(u_{St}) = \Pi_t^* = (1 - \tau)(\psi_t^* + \beta V) + \int_0^{m^*(u_{St})} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1 - \tau)m^*(u_{St}) \int_{m^*(u_{St})}^\infty dF(u_{Bt}) - c(\psi_t^*)$$

The expected value V is thus the expectation of (A3) over u_{St} . In closed form, V is:

$$(A4) \quad V = \frac{(1-\tau)\psi_t^* - c(\psi_t^*)}{1-\beta(1-\tau)} + \frac{E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St})} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau)m^*(u_{St}) \int_{m^*(u_{St})}^{\infty} dF(u_{Bt}) \right]}{1-\beta(1-\tau)}$$

A2. Proof of Claims 1 and 2

These are special cases of Proposition 1, with $\tau = 0$ and $\tau = 1$ respectively.

A3. Proof of Proposition 2

In the following two subsections, we will show that marginal allocative welfare is:

$$(A5) \quad \int \left[(1-\tau) \frac{(f(m^*(u_{St}, \tau)))^2 h(m^*(u_{St}, \tau))}{(1-F(m^*(u_{St}, \tau)))(1-(1-\tau)h'(m^*(u_{St}, \tau)))} \right] dF(u_{St})$$

where

$$h(m) \equiv \frac{1-F(m)}{f(m)}$$

and marginal investment welfare from changing τ is:

$$(A6) \quad \frac{dW}{d\tau} = -\frac{\tau}{c''(\psi_t^*(\tau))}$$

Equating the sum of these to 0 and rearranging, we have (11).

Now, if F has continuous second derivatives, $h'(m^*(u_{St}, \tau))$ and $h(m^*(u_{St}, \tau))$ exist everywhere. Moreover, if $f(\cdot)$ is everywhere positive, then $h(\cdot)$ is nonzero everywhere, so (A5) is strictly positive at $\tau = 0$, if it exists. Now, (A6) is always 0 at $\tau = 0$, hence (11) cannot hold when $\tau = 0$: the marginal allocative value of raising τ is positive, whereas the marginal investment loss is 0.

Similarly, when $\tau = 1$, if c is strictly convex, (A6) is always negative, and (A5) is always 0. Hence, (11) cannot hold when $\tau = 1$: the marginal investment gain from lowering τ is positive, whereas the marginal allocative loss is 0. This proves that the first-order condition (11) cannot hold at $\tau = 1$ or $\tau = 0$, proving that the optimal τ must be interior.

ALLOCATIVE WELFARE. — Allocative welfare, for type u_{St} , is:

$$\int_{m^*(u_{St}, \tau)}^{\infty} u_{Bt} dF(u_{Bt}) + u_{St} \int_0^{m^*(u_{St}, \tau)} dF(u_{Bt})$$

Note that we have:

$$\frac{dW(u_{St}, \tau)}{d\tau} = \frac{\partial W}{\partial m} \frac{\partial m}{\partial \tau}$$

Now,

$$\frac{\partial W}{\partial m} = (u_{St} - m) f(m)$$

Now, define the inverse hazard rate function $h(m)$ as:

$$h(m) \equiv \frac{1 - F(m)}{f(m)}$$

We can then write the markup FOC as:

$$m^*(u_{St}, \tau) - u_{St} - (1 - \tau) h(m^*(u_{St}, \tau)) = 0$$

Applying the implicit function theorem, we have:

$$(A7) \quad \frac{\partial m}{\partial \tau} = \frac{-h(m^*(u_{St}, \tau))}{1 - (1 - \tau) h'(m^*(u_{St}, \tau))}$$

Hence,

$$\frac{dW(u_{St}, \tau)}{d\tau} = (m^*(u_{St}, \tau) - u_{St}) \frac{f(m^*(u_{St}, \tau)) h(m^*(u_{St}, \tau))}{1 - (1 - \tau) h'(m^*(u_{St}, \tau))}$$

Substituting for $m - u_{St}$ using (5) and simplifying, we have:

$$\frac{dW(u_{St}, \tau)}{d\tau} = (1 - \tau) \frac{(1 - F(m^*(u_{St}, \tau))) h(m^*(u_{St}, \tau))}{(1 - (1 - \tau) h'(m^*(u_{St}, \tau)))}$$

The change in total allocative welfare is just the integral of this:

$$\frac{dW(\tau)}{d\tau} = \int \left[(1 - \tau) \frac{(1 - F(m^*(u_{St}, \tau))) h(m^*(u_{St}, \tau))}{(1 - (1 - \tau) h'(m^*(u_{St}, \tau)))} \right] dF(u_{St})$$

which is (A5).

INVESTMENT WELFARE. — Investment welfare is:

$$\psi_t^*(\tau) - c(\psi_t^*(\tau))$$

Differentiating, we have:

$$\frac{dW}{d\tau} = \frac{\partial W}{\partial \psi_t} \frac{\partial \psi_t^*}{\partial \tau}$$

Now,

$$\frac{\partial W}{\partial \psi_t} = 1 - c'(\psi_t)$$

And, applying the implicit function theorem to (4), we have:

$$\frac{\partial \psi_t}{\partial \tau} = -\frac{1}{c''(\psi_t)}$$

Hence,

$$(A8) \quad \frac{dW}{d\tau} = -\frac{1 - c'(\psi_t)}{c''(\psi_t)}$$

Since $c'(\psi_t) = 1 - \tau$, we have (A6).

SUPPLEMENTAL MATERIAL FOR SECTION III

B1. Proof of Claim 3

Since we have assumed away private values, buyers and license owners are identical. In any period, the license owner can choose to sell for $(1 - \tau)p$, or buy for τp . Buyers and the license owner agree on the continuation value in period t , which is:

$$V_t$$

The license owner is willing to keep the asset if:

$$V_t - \tau p_t \geq (1 - \tau)p_t$$

$$\implies p_t \leq V_t$$

The buyer is willing to buy if $p_t \leq V_t$. Thus, the unique market clearing price is $p_t = V_t$ in each period, which makes license owners indifferent between buying and selling in each period. In the first period, where there is no license owner, bidders will bid until $p_0 = V_0$.

Since license owners are always indifferent between buying and selling, and there is no uncertainty, to calculate a license owner's utility, we can simply assume a

single license owner purchases the license in period 0, and then keeps the good forever. Her expected utility is:

$$(B1) \quad \underbrace{\sum_{t=1}^{\infty} \beta^t E[\psi_t]}_{\text{Asset use value}} - \underbrace{p_0}_{\text{Initial license price}} - \underbrace{\sum_{t=1}^{\infty} \beta^t E[\tau_t p_t]}_{\text{License fee payments}}$$

The license owner must be indifferent between purchasing and not purchasing the license in period 0. Setting (B1) to 0 and rearranging, we have:

$$(B2) \quad \underbrace{p_0}_{\text{Initial license price}} + \underbrace{\sum_{t=1}^{\infty} \beta^t E[\tau_t p_t]}_{\text{License fee payments}} = \underbrace{\sum_{t=1}^{\infty} \beta^t E[\psi_t]}_{\text{Asset use value}}$$

The left hand side of (B2) is the expected net present value of the government's revenue, over the initial sale of the license and the future license fee payments. The right hand side is the expected net present value of the future common use values, ψ_t , of the asset; this is not affected by τ_t . Thus, (B2) shows that the net present value of the government's revenue does not depend on τ_t .

B2. Proof of Proposition 3

Prices in each auction are:

$$\begin{aligned} p &= \min[u_{Bt} + \psi_t + \beta V(\tau), m^*(u_{St}, \tau) + \psi_t + \beta V(\tau)] \\ &= \beta V(\tau) + \psi_t + \min[u_{Bt}, m^*(u_{St}, \tau)] \end{aligned}$$

Thus, total tax revenue is:

$$\frac{\tau}{1-\beta} p = \frac{\tau}{1-\beta} [\beta V(\tau) + \psi_t] + \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]]$$

We thus have:

$$\begin{aligned} R(\tau) &= V(\tau) + \frac{\tau E[p_t]}{1-\beta} \\ &= V(\tau) + \frac{\tau}{1-\beta} [\beta V(\tau) + \psi_t^*(\tau)] + \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]] \\ (B3) \quad R(\tau) &= \frac{(1-\beta(1-\tau))}{1-\beta} V(\tau) + \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]] + \frac{\tau \psi_t^*(\tau)}{1-\beta} \end{aligned}$$

Now, note that from (A4) of Appendix A, we have:

$$V(\tau) = \frac{(1-\tau)\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta(1-\tau)} + \frac{E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau) m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right]}{1-\beta(1-\tau)}$$

Hence,

$$\frac{(1-\beta(1-\tau))}{1-\beta} V(\tau) = \frac{(1-\tau)\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta} + \frac{E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau) m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right]}{1-\beta}$$

Hence,

$$R(\tau) = \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau) m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right] + \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]] + \frac{(1-\tau)\psi_t^*(\tau) + \tau\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta}$$

(B4)

$$R(\tau) = \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau) m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right] + \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]] + \frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta}$$

Now, note that we can write:

$$\begin{aligned} & \frac{\tau}{1-\beta} E_{u_{St} \sim F(\cdot)} [\min[u_{Bt}, m^*(u_{St}, \tau)]] = \\ & \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} \tau u_{Bt} dF(u_{Bt}) + \tau m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right] \end{aligned}$$

Hence, (B4) becomes:

$$(B5) \quad R(\tau) = \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1-\tau) m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^{\infty} dF(u_{Bt}) \right] + \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} \tau u_{Bt} dF(u_{Bt}) + \tau m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^{\infty} dF(u_{Bt}) \right] + \frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta}$$

Adding the two integrals, (B5) simplifies to:

$$R(\tau) = \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[u_{St} \int_0^{m^*(u_{St}, \tau)} dF(u_{Bt}) + m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^{\infty} dF(u_{Bt}) \right] + \frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta}$$

$$(B6) \quad R(\tau) = \underbrace{\frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} [u_{St} F(m^*(u_{St}, \tau)) + m^*(u_{St}, \tau) (1 - F(m^*(u_{St}, \tau)))]}_{Allocative} + \underbrace{\frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1-\beta}}_{Investment}$$

This is (14) of Proposition 3. We can now differentiate $R(\tau)$ by differentiating each of the allocative and investment terms with respect to τ . Beginning with the allocative term, we have:

$$\begin{aligned} & \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\frac{dm^*(u_{St}, \tau)}{d\tau} \frac{\partial}{\partial m^*} \left[u_{St} \int_0^{m^*(u_{St}, \tau)} dF(u_{Bt}) + m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^{\infty} dF(u_{Bt}) \right] \right] \\ &= \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\frac{dm^*(u_{St}, \tau)}{d\tau} [1 - F(m^*(u_{St}, \tau)) + (u_{St} - m^*) f(m^*(u_{St}, \tau))] \right] \\ (B7) \quad &= \frac{1}{1-\beta} E_{u_{St} \sim F(\cdot)} \left[\frac{dm^*(u_{St}, \tau)}{d\tau} (1 - F(m^*(u_{St}, \tau))) \left[1 - (m^* - u_{St}) \frac{f(m^*(u_{St}, \tau))}{(1 - F(m^*(u_{St}, \tau)))} \right] \right] \end{aligned}$$

Now, from rearranging (5) of Proposition 1, we have:

$$(B8) \quad \frac{(m^* - u_{St}) f(m^*(u_{St}, \tau))}{1 - F(m^*(u_{St}, \tau))} = 1 - \tau$$

Plugging (B8) into (1), we get:

$$(B9) \quad \frac{\tau}{1 - \beta} E_{u_{St} \sim F(\cdot)} \left[\frac{dm^*(u_{St}, \tau)}{d\tau} (1 - F(m^*(u_{St}, \tau))) \right]$$

Note that $\frac{dm^*(u_{St}, \tau)}{d\tau}$ is always weakly negative, and all other terms in (B9) are positive, hence (B9) is always weakly negative.

Now, the investment term in (B6) is:

$$\frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1 - \beta}$$

Note that $\psi_t^*(\tau) - c(\psi_t^*(\tau))$ is just investment welfare. This is monotonically decreasing in τ . From (A8) of Appendix A.A3, the derivative is:

$$(B10) \quad \frac{d}{d\tau} \frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1 - \beta} = - \frac{\tau}{(1 - \beta) c''(\psi_t)}$$

Expression (B10) is always negative. Combining (B9) and (B10), we get (15). Now, when c has continuous second derivative, c'' exists everywhere. When F has continuous and bounded second derivatives, from (A7), $\frac{\partial m}{\partial \tau}$ is finite and exists everywhere. Thus, all terms in $R'(\tau)$ exist. When $\tau = 0$, both terms are 0, so $R'(0) = 0$.

SUPPLEMENTARY MATERIAL FOR SECTION IV

C1. Numerical simulations

In Figures A.1, A.2, and A.3, we simulate outcomes as we vary, respectively, transactions costs, the number of buyers, and the time period before investments pay off.

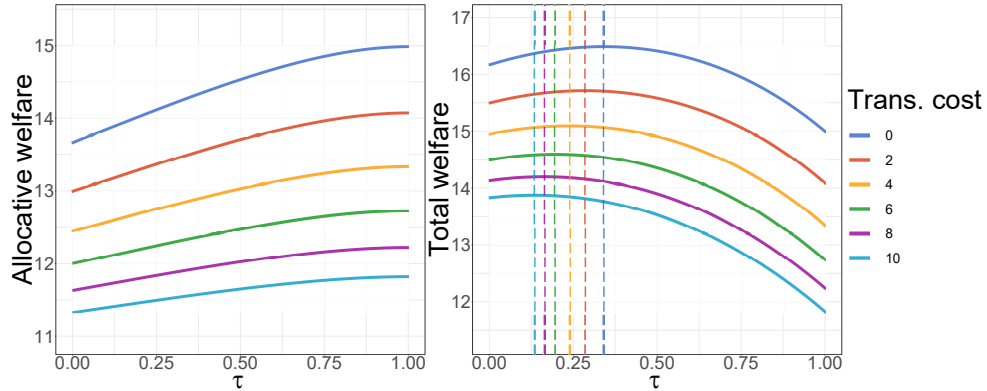
Figure A.1 varies transactions costs c . As we show in Subsection IV.A, this is equivalent to shifting buyers' values uniformly downwards by c , which for the exponential distribution, means shifting x_0 to $x_0 - c$. The left panel of Figure A.1 shows that, as we decrease transactions costs, allocative welfare decreases, as well as the allocative welfare gain from increasing τ . Intuitively, when transactions costs are high, buyers' values tend to be lower than sellers', so the welfare distortions from markups are quantitatively smaller. Allocative welfare is still

monotonically increasing in τ , but the gains are smaller the smaller x_0 is. The right panel shows that, taking investment welfare into account, the optimal τ tends to be smaller when transactions costs are high, though it is always interior.

Figure A.2 varies the number of buyers. The left panel shows that, as the number of buyers N increases, allocative efficiency tends to increase, since the highest bidder's private value tends to increase. As N gets very large, allocative welfare is also a flatter function of τ . Thus, when N is very large, the optimal τ approaches 0. This can be seen in the right panel: the optimal τ starts decreasing for large N . However, note that in this example, the optimal τ does not vary monotonically with N : the optimal τ is somewhat higher for $N = 2, 3, 4$ than it is with $N = 1$.

Figure A.3 shows results from varying χ , the time horizon on which investments pay off. The left panel shows that, when χ is large, τ decreases investment incentives much more rapidly. The right panel shows that this leads to a decrease in the optimal level of τ . However, the slope of investment welfare is always 0 when $\tau = 0$, so the optimal τ is always interior.

Figure A.1. : Transactions costs



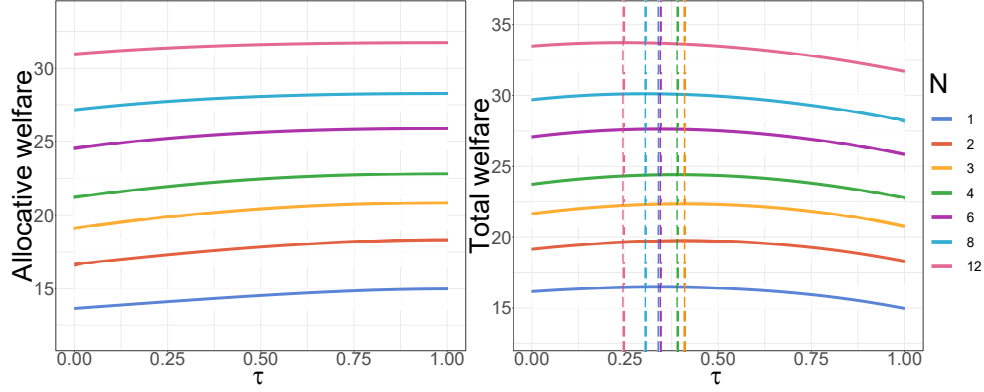
Notes. The left plot shows allocative welfare as a function of transactions cost c .

The right plot shows total welfare. Vertical dotted lines represent the welfare-maximizing values of τ , for different values of c . Throughout, values are exponential with minimum 0. The exponential rate parameter is $\lambda = 0.1$. The investment cost parameter is $\kappa = 10$.

C2. Proof of Claim 4

The payoff to the license owner of bidding p in the auction can be described as follows.

Figure A.2. : Competition



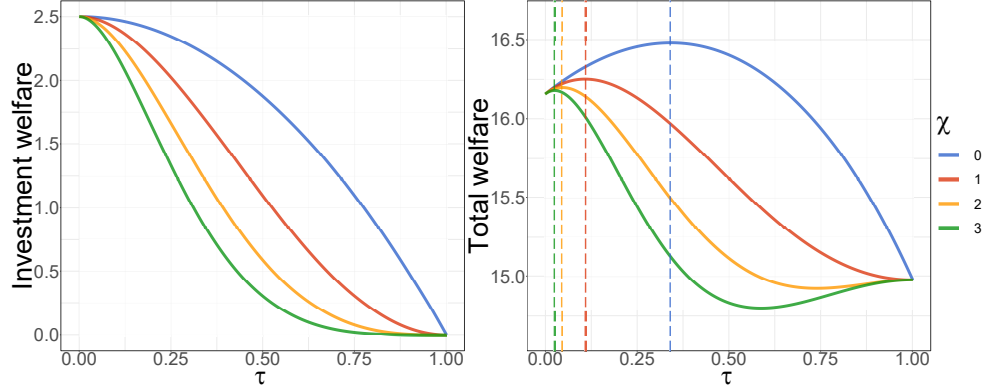
Notes. The left plot shows allocative welfare as a function of N , the number of buyers arriving each period. The right plot shows total welfare. Vertical dotted lines represent the welfare-maximizing values of τ , for different values of N . Throughout, sellers' and buyers' values are exponential with minimum $x_0 = 0$, and rate parameter $\lambda = 0.1$, and the investment cost parameter is $\kappa = 10$.

- With probability $N(1 - F)F^{N-1}$, the owner's bid is the second highest bid, so she sells her share $(1 - \tau)$ of the license at p , and gets $p(1 - \tau)$.
- With probability $1 - N(1 - F)F^{N-1} - F^N$, the owner's bid is lower than the second highest bid, so she sells her share $(1 - \tau)$ at the conditional expectation of second highest bid.
- With probability F^N , the owner keeps the asset, and pays τ times the conditional expectation of the highest bid.

We can thus write the incumbent's profit as:

$$(1 - \tau) \left[\int_p^\infty (b_2 - p) dF_2(b_2) + p(1 - F_1(p)) \right] + (\psi_t + \beta V + u_{it})(F_1(p)) - \tau \int_0^p (\psi_t + \beta V + b_1) dF_1(b_1)$$

Figure A.3. : Long-term investments



Notes. The left plot shows investment welfare as a function of χ , the number of periods that investments take to pay off. The right plot shows total welfare. Vertical dotted lines represent the welfare-maximizing values of τ , for different values of χ . Throughout, sellers' and buyers' values are exponential with minimum $x_0 = 0$, and rate parameter $\lambda = 0.1$. We set the investment cost parameter, κ , equal to $\frac{10}{2\beta^2\chi}$, which ensures that total attainable investment welfare does not vary with χ .

where F_1 and F_2 , respectively, are the distribution of the highest and second highest buyers' bids. In markup terms, this is:

$$(1 - \tau) (\beta V + \psi_t) + (1 - \tau) \int_m^\infty (b_2 - m) dF_2(b_2) + \\ (1 - \tau) m (1 - F_1(m)) + u_{it} F_1(m) - \tau \int_0^m b_1 dF_1(b_1)$$

Differentiate with respect to m , to get: and rearrange, to get:

$$(m - u_{it}) f_1(m) = (1 - \tau) (F_2(m) - F_1(m))$$

Now, the distributions of the first and second highest buyers' bids satisfy:

$$F_1 = F^N, \quad f_1 = NF^{N-1}f$$

$$F_2 = F^N + NF^{N-1}(1 - F)$$

Plugging in, we have:

$$(m - u_{it}) (NF^{N-1}(m) f(m)) = (1 - \tau) (NF^{N-1}(1 - F))$$

Rearranging, we have (19).

C3. Proof of Claim 5

An increase in ψ_t affects the value of any asset owner in period t . The social planner sets the marginal cost of investment in period t , $\frac{\partial c}{\partial \psi_{t+\chi}}$, equal to its discounted marginal value in period $t + \chi$. A unit of investment is always worth 1 in period $t + \chi$, so it is worth β^k in period- t dollars. This is (20).

To solve for equilibrium investment, let V_t be the expected value of an owner in any period t , before her private value is known. We have:

$$\begin{aligned}\Pi_t(u_{St}) &= \int_0^{m^*(u_{St}, \tau)} [(\psi_t + u_{St} + \beta V_{t+1}) - \tau(\psi_t + u_{Bt} + \beta V_{t+1})] dF(u_{Bt}) + \\ &\quad \int_{m^*(u_{St}, \tau)}^\infty (1 - \tau)(\psi_t + \beta V_{t+1} + m^*(u_{St}, \tau)) dF(u_{Bt}) \\ &= (1 - \tau)(\psi_t + \beta V_{t+1}) + \int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1 - \tau)m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt})\end{aligned}$$

Since the second piece is stationary, we have:

$$\begin{aligned}E[\Pi_t(u_{St})] &= E[(1 - \tau)(\psi_t + \beta V_{t+1})] + \\ E_{u_{St} \sim F(\cdot)} &\left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1 - \tau)m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right]\end{aligned}$$

Define

$$W = E_{u_{St} \sim F(\cdot)} \left[\int_0^{m^*(u_{St}, \tau)} [u_{St} - \tau u_{Bt}] dF(u_{Bt}) + (1 - \tau)m^*(u_{St}, \tau) \int_{m^*(u_{St}, \tau)}^\infty dF(u_{Bt}) \right]$$

Then we have:

$$(C1) \quad E[\Pi_t(u_{St})] = E[W + (1 - \tau)(\psi_t + \beta V_{t+1})]$$

Now, suppose during period t that $\psi_t, \dots, \psi_{t+\chi-1}$ have been determined by past license owners' investments. We can expand the recursion in (C1) to get:

$$E[\Pi_t(u_{St})] = E[W + (1 - \tau)(\psi_t + \beta(W + (1 - \tau)(\psi_{t+1} + \beta V_{t+2})))]$$

Expanding further to χ periods in the future,

$$E[\Pi_t(u_{St})] = E\left[\left[\sum_{\tilde{t}=t}^{t+\chi} (1-\tau)^{\tilde{t}-t} \beta^{\tilde{t}-t} W\right] + \left[\sum_{\tilde{t}=t}^{t+\chi} (1-\tau)^{\tilde{t}-t+1} \beta^{\tilde{t}-t} \psi_{\tilde{t}}\right] + (1-\tau)^{\chi+1} \beta^{\chi+1} V_{t+\chi+1}\right]$$

Hence differentiating with respect to $\psi_{t+\chi}$, we have:

$$\frac{\partial c}{\partial \psi_{t+\chi}} = \beta^\chi (1-\tau)^{\chi+1}$$

This proves Claim 5.

C4. Proof of Claim 6

First, we wish to calculate the price of the license once ψ is known, $p(\psi)$. Since all agents are identical, the auction price must make the license owner in each period indifferent between holding the asset and selling the asset. This means:

$$\underbrace{p(1-\tau)}_{\text{Sale revenue}} = \underbrace{\psi}_{\text{Use value}} - \underbrace{\tau p}_{\text{Fee payment}} + \underbrace{\beta(1-\tau)p}_{\text{Next-period sale revenue}}$$

Rearranging, we have:

$$(C2) \quad p(\psi) = \frac{\psi}{1 - \beta(1-\tau)}$$

Since agents are indifferent between holding and selling, and there is no uncertainty after ψ is realized, agents' utility from owning the license is equal to their revenue from selling the license, $(1-\tau)p(\psi)$. The price in the initial auction has no uncertainty, so the variance in the utility of an agent who buys the license in the first period is thus simply the variance of $(1-\tau)p(\psi)$ over uncertainty in ψ , that is, the variance of:

$$\frac{(1-\tau)\psi}{1 - \beta(1-\tau)}$$

This is (23).

Now, note that agents' expected utility for owning the asset, after ψ is known, must be equal to its price, since the price makes agents indifferent between owning the asset and selling it in each period. The price in the initial auction has no uncertainty, so the variance in the utility of an agent who buyers the license in the first period is thus simply the variance of $p(\psi)$ over uncertainty in ψ . This is

(23).

Now, conditional on ψ , the administrator's fee revenue is:

$$(C3) \quad \frac{\tau p(\psi)}{1 - \beta} = \frac{\tau \psi}{(1 - \beta)(1 - \beta(1 - \tau))}$$

The price in the initial auction has no uncertainty, so the variance of the administrator's revenue is just the variance of (C3). This is (24).

SUPPLEMENTARY MATERIAL FOR SECTION V

D1. Self-assessment mechanisms

In this appendix, we analyze self-assessment mechanisms. We assume the same preference and investment structure as the baseline model. However, suppose that the incumbent license owner must announce some price p . She pays τp to the government regardless of what the buyer bids. If the buyer's bid is higher than the incumbent's price, the incumbent is paid p , and the buyer takes the asset. This is a “self-assessed tax” mechanism, in the sense that the incumbent must pay taxes based on her self-assessed price p , which is also a binding reserve price for buyers.

The following claim characterizes license owners' optimal price-setting and investment decisions under this mechanism.

Claim 1. The license owner bids:

$$V + \psi_t + m$$

where V is the continuation value, which is common to buyers and the license owner, and the optimal markup satisfies:

$$(D1) \quad m - u_{St} = \frac{(1 - F(m)) - \tau}{f(m)}$$

Investment satisfies:

$$(D2) \quad c'(\psi_t) = 1 - \tau$$

Intuitively, Claim 1 says that, for the same τ , from (D2), investment incentives are identical to those in Proposition 1, the baseline mechanism. However, the incumbent's optimal markup is different. From (D1), a incumbent with type u_{St}

sets m equal to u_{St} , so allocative efficiency is achieved, when we have:

$$\tau = 1 - F(u_{St})$$

that is, when τ is equal to the probability that the incumbent sells the asset. This has a number of implications.

For any given incumbent type, full allocative efficiency can be achieved with a τ which is strictly lower than 1. Intuitively, this is because the self-assessed tax mechanism produces stronger incentives to set lower prices than the second-price mechanism. Under the second-price mechanism, the incumbent's price announcement only indirectly affects the taxes she pays if she wins, by increasing the set of buyer bids that she wins over; under the self-assessment mechanism, incumbents always pay the price they announce, so they have stronger incentives to announce low prices. As a result, for a single incumbent type, the optimal τ is more efficient, because it achieves full efficiency at the cost of sacrificing less investment welfare.

The self-assessment mechanism has two weaknesses relative to the second-price mechanism we discuss in the main text. The optimal τ differs for different incumbents: any given fixed τ will be too high for some incumbents and too low for others. Since the license designer must commit to a sequence of τ 's when she allocates the license, it is not possible to adjust τ depending on the realized incumbent types.¹ Relatedly, with the self-assessed tax mechanism, it is possible for the administrator to overshoot the optimal τ , and actually decrease allocative efficiency. If τ is higher than $1 - F(m)$, license owners will announce markups m below their private values u_{St} , and will sell too often, relative to the social optimum.

PROOF OF CLAIM 1. — Under the self-assessment mechanism, if the incumbent bids above the buyer, she gets:

$$\underbrace{(\psi_t + u_{St} + \beta V)}_{\text{Continuation value}} - \underbrace{\tau (\psi_t + m + \beta V)}_{\text{Fee payment}}$$

That is, the incumbent pays the buyer's bid, keeps the asset and gets her continuation value. If the incumbent bids below the buyer's value, she gets paid her bid for her share $(1 - \tau)$ of the asset, that is:

$$(1 - \tau) (\psi_t + m + \beta V)$$

¹Any mechanism which attempted to elicit license owners' types and use them to set τ would generate further incentive problems; license owners who recognize their type announcements affect τ 's will have further incentives to distort their types.

In either case, the investment cost $c(\psi_t)$ is sunk. Hence, the incumbent's expected profit can be written as:

$$\Pi_t = \int_0^m [(\psi_t + u_{St} + \beta V) - \tau(\psi_t + m + \beta V)] dF(u_{Bt}) + \int_m^\infty (1 - \tau)(\psi_t + m + \beta V) dF(u_{Bt}) - c(\psi_t)$$

$$(D3) \quad \Pi_t = (1 - \tau)(\psi_t + \beta V) - c(\psi_t) + \int_0^m u_{St} dF(u_{Bt}) + m \int_m^\infty dF(u_{Bt}) - \tau m$$

Differentiating with respect to m , we have:

$$(u_{St} - m) f(m) + (1 - F(m)) - \tau = 0$$

Rearranging, we have (D1). Differentiating (D3) with respect to ψ_t and rearranging, we have (D2).

D2. The Vickrey-Clarke-Groves mechanism

Suppose the administrator allocates the asset by running a Vickrey-Clarke-Groves (VCG) mechanism in each period. Suppose the common value of the asset is ψ_t . Suppose the seller's private value is u_{St} , and the buyer's is u_{Bt} . Let V represent the stationary value of being a seller at the start of any period (which will generally be different under the VCG mechanism than in the baseline model). Under the VCG mechanism, if $u_{Bt} > u_{St}$, then the buyer's payment to the administrator increases by the seller's value,

$$(D4) \quad u_{St} + \psi_t + \beta V$$

and the buyer receives the license. If $u_{St} > u_{Bt}$, then the seller's net payment to the administrator increases by the buyer's value,

$$(D5) \quad u_{Bt} + \psi_t + \beta V$$

and the seller receives the license. Now, we impose individual rationality period-by-period for the seller: the seller must achieve higher utility than she would get from refusing to participate in period t , then participating from $t + 1$ onwards, in any period. This implies that the seller cannot make any payment to the administrator if she keeps the license: payments can only be made if the license is actually traded. Thus, in order for the seller's net payment to the administrator to be $u_{Bt} + \psi_t + \beta V$ if she keeps the license, the seller must get paid $u_{Bt} + \psi_t + \beta V$

if she sells to the buyer. Intuitively, the VCG mechanism with two agents is a second-price auction for the buyer – the buyer must pay the seller’s true value to keep the asset. It is a first-price auction for the seller: if she sells to the buyer, she is paid the buyer’s true value for the asset.

Under this mechanism, the expected utility for a seller with type u_{St} in period t is:

$$\begin{aligned} E_{u_{Bt}} \max[u_{Bt} + \psi_t + \beta V, u_{St} + \psi_t + \beta V] \\ = \psi_t + \beta V + E_{u_{Bt}} \max[u_{Bt}, u_{St}] \end{aligned}$$

Hence, the seller’s utility increases one-for-one with increases in ψ_t . The seller’s optimal investment decision is thus:

$$c'(\psi_t) = 1$$

which is socially efficient. Thus, the VCG mechanism achieves both allocative and investment efficiency.

However, the VCG mechanism is not budget-balanced, period by period. In fact, it is guaranteed to make a budget deficit for the administrator, whenever trade occurs. When trade does not occur, there are no payment. Trade occurs whenever $u_{Bt} > u_{St}$, and the buyer pays (D4), whereas the seller is paid (D5). Thus, whenever trade occurs, the administrator must subsidize the difference between the buyer and sellers’ values, $u_{Bt} - u_{St}$.

We note that, when considering the upfront revenue V that the administrator makes from the initial sale of the license, the VCG mechanism may still be budget-balanced ex ante for the administrator: the present value of the upfront revenue may be enough to cover the costs of future auctions. However, administrators may not be able to commit to a mechanism which requires repeated, and in principle unbounded, subsidies throughout the life of the mechanism.

Another weakness of the VCG mechanism is that it is vulnerable to collusion: the buyer and seller can collude to extract subsidy revenue from the increases. If the buyer raises her bid, her payment does not change, but the subsidy revenue the seller receives from the administrator increases. This collusion strategy is also possible when there are more bidders. Suppose, for example, that there are multiple buyers, and suppose at least two buyers have values higher than the seller’s value. The VCG mechanism then requires that the highest valued buyer pays the second highest valued buyer’s value, whereas the seller is paid the highest buyer’s value. This implies that market participants can collude to extract revenue from the administrator: if the highest-valued buyer raises her bid, this increases the payment the administrator makes to the seller, without changing the price that the highest-valued buyer pays. In comparison, the depreciating license mechanism always creates positive revenue for the government, every time an auction is run.

NUMERICAL SIMULATIONS

E1. Details of numerical simulations

In this appendix, we calculate results for the noncentered exponential distribution, which are used in all numerical simulations: Figures 1, 2, A.1, A.2, and A.3. Suppose buyers' values u_{Bt} are drawn from the noncentered exponential distribution, with minimum x_0 :

$$F(u_{Bt}) = 1 - e^{-\lambda(u_{Bt}-x_B)}, \quad f(u_{Bt}) = \lambda e^{-\lambda(u_{Bt}-x_B)}$$

Where x_0 is the minimum of u_{Bt} . Sellers' values are also noncentered exponential, with a possibly different minimum value x_S .

Allocative welfare. We will solve for markups and welfare conditional on u_{St} . Applying (7) of Proposition 1, markups are:

$$(E1) \quad m^*(u_{St}, \tau) = u_{St} + (1 - \tau) \frac{e^{-\lambda(x-x_B)}}{\lambda e^{-\lambda(x-x_B)}}$$

$$m^*(u_{St}, \tau) = \max \left[u_{St} + \frac{(1 - \tau)}{\lambda}, x_B \right]$$

where, it is never optimal to set $m(u_{St})$ lower than x_B , the minimum of buyers' values. Allocative welfare for a seller with type u_{St} is thus:

$$(E2) \quad W(u_{St}, \tau) = u_{St} F(m^*(u_{St}, \tau)) + E[u_{Bt} \mid u_{Bt} > m^*(u_{St}, \tau)] (1 - F(m^*(u_{St}, \tau)))$$

Now, for exponential distributions,

$$E[u_{Bt} \mid u_{Bt} > m^*(u_{St}, \tau)] = m^*(u_{St}, \tau) + \frac{1}{\lambda}$$

Hence, plugging in expressions for F , (E2) becomes:

$$(E3) \quad W(u_{St}, \tau) = u_{St} \left(1 - e^{-\lambda(m^*(u_{St}, \tau) - x_B)} \right) + \left(m^*(u_{St}, \tau) + \frac{1}{\lambda} \right) e^{-\lambda(m^*(u_{St}, \tau) - x_B)}$$

We can compute expected welfare by numerically integrating (E3) over u_{St} .

Investment. From (4) of Proposition 1, agents invest until:

$$\frac{\psi_t}{\kappa} = 1 - \tau$$

$$\implies \psi_t^*(\tau) = (1 - \tau) \kappa$$

Investment welfare is:

$$\begin{aligned} \psi_t^*(\tau) - \frac{\psi_t^*(\tau)^2}{2\kappa} &= (1 - \tau) \kappa - \frac{((1 - \tau) \kappa)^2}{2\kappa} \\ &= (1 - \tau^2) \frac{\kappa}{2} \end{aligned}$$

Revenue. We can calculate revenue using expression (14) of Proposition 3. Revenue for type u_{St} is:

$$(E4) \quad \frac{1}{1 - \beta} \left(u_{St} \left(1 - e^{-\lambda(m^*(u_{St}, \tau) - x_B)} \right) + m^*(u_{St}, \tau) e^{-\lambda(m^*(u_{St}, \tau) - x_B)} \right) + \frac{\psi_t^*(\tau) - c(\psi_t^*(\tau))}{1 - \beta}$$

We compute expected revenue by numerically integrating (E4) over u_{St} .

Transaction costs. If buyers must pay some transaction cost c , then their private values are effectively drawn from a noncentered exponential distribution with minimum $x_0 - c$. All calculations are otherwise unchanged.

Long-term investment. If investment takes χ periods to pay off, the first-order condition is (21). Agents invest until:

$$\begin{aligned} \frac{\psi_t^*(\tau)}{\kappa} &= \beta^\chi (1 - \tau)^{\chi+1} \\ \implies \psi_t^*(\tau) &= \beta^\chi (1 - \tau)^{\chi+1} \kappa \end{aligned}$$

Investment welfare generated in each period is:

$$\begin{aligned} &\beta^\chi \psi - \frac{\psi^2}{2\kappa} \\ (E5) \quad &= \beta^{2\chi} (1 - \tau)^{\chi+1} \kappa \left(1 - \frac{(1 - \tau)^{\chi+1}}{2} \right) \end{aligned}$$

In the simulations of Figure A.3, for each χ , we set

$$\kappa = \frac{10}{2\beta^{2\chi}}$$

The result is that total investment welfare – (E5) when $\tau = 0$ – does not vary as we change χ .

Auctions. Suppose there are N buyers. Let $F_1(\cdot)$ denote the CDF of the

highest buyer's value. This has distribution:

$$F_1(u_{Bt}) = F^N(u_{Bt}) = \left(1 - e^{-\lambda(u_{Bt} - x_B)}\right)^N$$

From Claim 4, sellers' optimal markups are still described by (E1). To calculate welfare, note that the license is transferred to the highest-valued buyer if her value is higher than $m^*(u_{St}, \tau)$, and kept by the seller otherwise. Hence, welfare for a seller of type u_{St} is:

$$(E6) \quad W(u_{St}, \tau) = u_{St}F_1(m^*(u_{St}, \tau)) + \int_{m^*(u_{St}, \tau)}^{\infty} u_{Bt}dF_1(u_{Bt})$$

For the simulations in Figure A.2, we calculate (E6) numerically for each u_{St} , and then integrate over u_{St} .